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Logic Density in Gabor Frame Structures

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Abstract:

The Gabor system G(g,1+ε,ε-1)={e^2iπ(ε-1)nt g(t-(1+ε)m):m,n∈Z}is examined with respect to its frame property, uunder reasonable oversampling conditions, that is, when (ε+1,ε-1∈Q).

An appropriate "rational" counterpart of the Ron-Shen Gramian is developed, establishing that for every peculiar pane function (g), The structure: G(g,ε+1,ε-1) fails to produce a frame if ε^2=(2n-1)/(n-1).

A particular focus is placed on the initial Hermite function, h_1=te^(-πt^2).

Key word: Gabor system, rational, window function, initial Hermite function.

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1. Introduction

One of the primary concerns with Gabor analysis is the following. Determine the set of lattice parameters
$(\varepsilon > 1)$ for the Gabor system:

$$\mathcal{G}(g, \varepsilon+1, \varepsilon-1) = \left\{ e^{2i\pi(\varepsilon-1)nt} g(t-(\varepsilon+1)m) : m, n \in \mathbb{Z} \right\};$$

forms a frame in $L^2(\mathbb{R})$ given a window function $g \in L^2(\mathbb{R})$: A frame for $L^2(\mathbb{R})$is given by:

$$\mathcal{F}(g) := \{(\varepsilon+1, \varepsilon-1) \in \mathbb{R}_+^2 : \mathcal{G}(g, \varepsilon+1, \varepsilon-1)\ \}.$$

We evaluate some of the available information on the structure set $\mathcal{F}(g)$, which comes after [1] and [2] (in a more condensed format). In more relaxed circumstances ,if $g \in$ Feichtinger algebra M^1,

the set $\mathcal{F}(g)$ is open in $\mathbb{R}_+^2$ and includes the region around the origin. If $g \in$ Feichtinger algebra M^1, the open set $\mathcal{F}(g)$ in $\mathbb{R}_+^2$includes the region around the origin. In addition, $\mathcal{F}(g) \subset \prod_+ := \left\{(\varepsilon+1, \varepsilon-1) \in \mathbb{R}_+^2 : \varepsilon < \sqrt{2}\right\}$ for $g \in M^1$, according to fundamental density theorems [3,4,5] and an adaptation of the uncertainly principle [6, 7]. Only a small number of functions for which $\mathcal{F} = \prod_+$have been identified up until quite recently. The hyperbolic secant $g(t) = (e^t + e^{-t})^{-1}$, while the Gaussian

$g(t) = e^{-\pi t^2}$ [8,9,10], and the one- and two-sided exponential functions $g(t) = e^{-i}\mathrm{i}_{\mathbb{R}_+}$ (here $\varepsilon = \sqrt{2}$)also produces a frame) and $g(t) = e^{-\pi t^2}$) [11,12] are included in the list, along with their shifts and Fourier transformations. In [2], a significant discovery was made when the authors established that any fully positive function of finite type had this feature, this results in an infinite family of functions where $\mathcal{F}(g) = \prod_+$.

However, [13] shows that even for a "simple" function like the characteristic function $g = 1_I$ of an interval,

the set $\mathcal{F}(g)$ may have a somewhat complex shape. When g is concentrated in both time and frequency and $\mathcal{F}(g) \neq \prod_{+}$, there are certain instances. First, we want to get the Hermite function $h_1(t) = te^{-\pi t2}$. This conclusion is motivated by the uncertainty principle, which states that h_1 lowers the Heisenberg uncertainty of all functions orthogonal to the Gaussian, as well as present studies on vector-valued Gabor frames [14]. The paper is organized as following. Basic information from Gabor analysis and notation are presented in the next section. We examine the scenario of logical oversampling, where $\varepsilon^2 = \frac{p}{q} + 1 \in \mathbb{Q}$ .The union of hyperbolas is not contained in the set $\mathcal{F}(g)$ for any odd function $g \in M^1$ (especially h_1):

$$\varepsilon^2 = \frac{2n-1}{n-1} \quad \Rightarrow \quad (\varepsilon+1, \varepsilon-1) \notin \mathcal{F}(g), \quad n = 1,2, \dots. \qquad (1)$$

The vector-valued Zak transform, which encodes the frame operator as matrix multiplication in a space of vector-valued functions, is examined in [15] and [4, ch. 8] to lay the groundwork for the proof. notice the following part, we factorize the vector-valued Zak transform matrix and extract a component that is a rational version of the well-known Ron-Shen Gramian (see [16] and [4]). We hypothesize that the only limitation on the set $\mathcal{F}(h_1)$ is condition (1).

$$\mathcal{F}(h_1) = \left\{(\varepsilon+1, \varepsilon-1) \in \prod\nolimits_{+} : \varepsilon^2 \neq \frac{2n-1}{n-1}, n = 1,2, \dots . \right\}.$$

Regretfully, we are unable to fully verify this conjecture. We provide numerical verification for a larger collection of points and show it analytically only for a subset of $\prod_{+}$. The rational equivalent of the Gramain provided in section 4 serves as the foundation for these creations.

2.Preliminaries

The fundamental information from the Gabor analysis is reviewed in this part and will be applied later. For a more thorough explanation and the background information on the topic, we direct the reader to [4].

We investigate the lattice $\wedge = (\varepsilon+1)\mathbb{Z} \times (\varepsilon-1)\mathbb{Z}$, the Gabor system has an area function g and a value

$\varepsilon > 1$.

$$\mathcal{G}(g, \varepsilon+1, \varepsilon-1) = \{\pi_\lambda : \lambda \in \wedge\}$$

The standard time-frequency shift is indicated by the symbol:

$\pi_\lambda : g \to e^{2\pi ibt} g(t-a)$ for $\lambda = (a, b)$. The definition of the Gabor frame operator $S_{g.\wedge} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is:

$$\sum_i S_{g,\wedge} f_i(t) = \sum_i \Big(\sum_{\lambda \in \wedge} \langle f_i, \pi_\lambda g \rangle_{L^2(\mathbb{R})} \pi_\lambda g(t)\Big), \qquad f_i \in L^2(\mathbb{R})$$

The operation $\mathcal{G}(g,\wedge)$ is bounded if and only if the Gabor frame operator is invertible and g is in the modulation space $M^1(\mathbb{R})$, as defined later in this section. In this case, the operator $S_{g,\wedge}$ can be written as a multiplication operator in a vector-valued function space.
Assume that for reasonably prime $p, q \in \mathbb{N}$, $\varepsilon^2 = \frac{p}{q} + 1$. Examine the following: the dimension of vector-valued function $\mathcal{H}_{\varepsilon+1,p} = L^2\left(Q_{\varepsilon+1,p}, \mathbb{C}^p\right)$,and the rectangle $Q_{\varepsilon+1,p} = [0, \varepsilon + 1/p) \times [0, [0, 1/\varepsilon + 1)$. Recall that a function $f_i \in L^2(\mathbb{R})$has a Zak transform defined as:

$$\sum_i \mathcal{Z}_{\varepsilon+1} f_i(x, w_i) = \sum_i \sum_{n\in\mathbb{Z}} f_i\,(x - (\varepsilon + 1)n)e^{2\pi i n(\varepsilon+1)w_i}. \qquad (2)$$

We study the vector-valued Zak transform $\vec{\mathcal{Z}}_{\varepsilon+1}: L^2(\mathbb{R}) \to \mathcal{H}_{\varepsilon+1,p}$ defind as follows, in accordance with [4,ch.8]:

$$\vec{\mathcal{Z}}_{\varepsilon+1} \sum_i f_i(x, w_i) = \mathcal{Z}_{\varepsilon+1} \sum_i \left(\left(x + \frac{\varepsilon + 1}{p} r, w_i\right)\right)_{r=1}^{p}, \qquad (x, w_i) \in Q_{\varepsilon+1,p}.$$

The unitary mapping between $L^2(\mathbb{R})$ and $\mathcal{H}_{\varepsilon+1,p}$ is the vector-valued Zak transform, subject to normalization. Additionally, note:

$$\sum_i A_r^s(x, w_i) = (\varepsilon + 1)(\sum_i \sum_{j=0}^{q-1} \overline{\mathcal{Z}_{\varepsilon+1} g\left(x + \frac{\varepsilon + 1}{p} s, w_\imath - (\varepsilon - 1)\jmath\right)}\, \mathcal{Z}_{\varepsilon+1} g\left(x + \frac{\varepsilon + 1}{p} s, w_i - (\varepsilon - 1)j\right) e^{2\pi i j(r-s)/q}),$$

As an example, take into consideration the $p \times p$ matrix function $\mathcal{A}(x, w_i)$=$\sum_i((A_r^s(x, w_i))_{r,s=0}^{p-1})$, where $(x, w_i) \in Q_{\varepsilon+1,p}$.

Theorem (1.1): Zibulski and Zeevi. We obtain :
$\vec{\mathcal{Z}}_{\varepsilon+1} \sum_i((S_{g,\varepsilon+1,(\varepsilon-1)} f_i)(x, w_i)) = \sum_i(\mathcal{A}(x, w_i)\vec{\mathcal{Z}}_{(\varepsilon+1)} f_i(x, w_i))$, for nearly all

$(x, w_i) \in Q_{\varepsilon+1,p}$, based on the aforementioned assumptions. We always assume that g is a member of the modulation space $M^1(\mathbb{R})$,in the following. We merely restate the definition here, and readers are directed to [4] for a more thorough presentation. Examine the immediate Fourier transform, where f_i acts as the aperture function, given a functions f_i in the Schwartz class $S(\mathbb{R})$

$$\sum_i (V_{f_i} g(x, w_i)) = \lim_{b\to-\infty} (\sum_i (\int_b^\infty g(t)\, f_i(x - t) e^{2\pi i w_i t} dt)).$$

Definition (1.2) : The modulation space $M^1(\mathbb{R})$ includes functions $g \in L^2(\mathbb{R})$ with

$$\sum_i (\int_{\mathbb{R}} \int_{\mathbb{R}} \left|V_{f_i} g(x, w_i)\right| dx dw_i) < \infty$$

For certain (or all) non-trivial functions, $f_i$is in $\mathcal{S}(\mathbb{R})$. The following statement is the result of Theorem (1.1) [1].

Corollary (1.3) : The Gabor set $\mathcal{G}(g,\wedge)$ is a structure in $L^2(\mathbb{R})$ only if:

$$det\sum_i \mathcal{A}(x,w_i) \neq 0 \quad ,(x,w_i) \in Q_{\varepsilon+1,p}. \tag{3}$$

To make this condition more understandable, we factorize matrix $\mathcal{A}$ A. Consider column vectors:

$$\sum_i X^j(x,w_i) = \sum_i \left(X_r^j(x,w_i)\right)_{r=0}^{p-1}, \qquad j=0,1,\dots\dots,q-1$$

In which:

$$\sum_i X_r^j(x,w_i) = \mathcal{Z}_{(\varepsilon+1)}g\sum_i\left(t+\frac{(\varepsilon+1)r}{p},w_i-(\varepsilon-1)j\right)e^{2\pi ijr/q}, \tag{4}$$

Including $p\times q$ matrix:

$$\sum_i Q(t,w_i) = \left(X^j\right)_{j=0}^{q-1} = \sum_i\left(\mathcal{Z}_{(\varepsilon+1)}g\left(t+\frac{(\varepsilon+1)r}{p},w_i-(\varepsilon-1)j\right)e^{2\pi ijr/q}\right)_{r=0,j=0}^{p-1,q-1}$$

Clearly

$$\sum_i \mathcal{A}(x,w_i) = \sum_i Q(x,w_i)Q^T(x,w_i).$$

Here, as always,Q^TT represents the adjoint matrix of Q.

$$\vec{\mathcal{Z}}_{(\varepsilon+1)}\sum_i \mathcal{A}(x,w_i)f_i(x,w_i) = \sum_i(\sum_{j=0}^{q-1}\langle X^j(x,w_i),\vec{\mathcal{Z}}_{(\varepsilon+1)}f_i(x,w_i)\rangle\langle X^j(x,w_i),\vec{\mathcal{Z}}_{(\varepsilon+1)}f_i(x,w_i)\rangle),$$

For each $(x,w_i)\in Q_{\varepsilon+1,p}$, the $X^j(x,w_i), j=0,1,\dots\dots.q-1$, must span the entire $\mathbb{C}^p$,to meet condition (3).

Corollary (1.4): Let $\varepsilon^2 = \frac{p}{q}+1 \in \mathbb{Q}$, and $g\in M^1(\mathbb{R})$. If $\mathcal{G}(g,\wedge)$ is a farm in $L^2(\mathbb{R})$, rank $\sum_i Q(x,w_i) =$ must be equal to p for any $(x,w_i))$ in $Q_{\varepsilon+1,p}$. In the following section, we apply this requirement to investigate Gabor systems formed by odd functions.

3. The Odd Functions that is Generate Gabor Frames

The next part includes what comes next.

Theorem (3.1): Assume g is an odd function in $M^1(\mathbb{R})$) and $\varepsilon^2 = \frac{2n-1}{n}$, where

$n=1,2,\dots\dots$,Then, $\mathcal{G}(g,\wedge)$ does not constitute a frame in $L^2(\mathbb{R})$.

Proof: It is sufficient to establish that

$$rank\, Q(0,0) < 2n-1 \tag{5}$$

The conclusion follows from Theorem (1.1) [2]. Equation (5) is calculated from the findings that for odd open spaces, the elements of the matrices $Q(0,0)$have additional symmetries. We consider $\varepsilon =$

0, this may always be accomplished by appropriately scaling g. We assume $\varepsilon = 0$, which can be done by appropriately scaling g.

Lemma (3.2): Let g be an odd function in $M^1(\mathbb{R})$with $\varepsilon = 0,\ \varepsilon = \frac{p}{q} + 1 \in \mathbb{Q}$ and $M_s^j = M_s^j(0,0)$, The functions such as $M_s^j(x, w_i)$ are defined in [4]. Then:

$$M_s^j = M_{p-s}^{q-j}, \quad s = 0,1, \dots\dots p-1, \quad j = 0,1, \dots\dots q-1 \tag{6}$$

The explanation of the Zak transformation, along with the fact that g is odd, makes the lemma simple to prove. We $q - p = 1$. Assume q is an even number; $q = 2k + 2,\ p = 2k + 1$. Q(0,0) denotes the $(2k + 1) \times (2k + 2)$ matrix. Additional relations must exist for the zero row, zero column, and $(k + 1) - th(k + 1)$ -th column of $Q(0,0)$. In particular:

$$X_0^0 = 0, \qquad X_0^{k+1} = 0, \qquad X_0^j = -X_0^{q-j} \qquad \text{-zero row;} \tag{7}$$

$$X_s^0 = -X_{p-s}^0, \qquad s = 1, \dots\dots, p-1 \qquad \text{-zero column;} \tag{8}$$

$$X_s^{k+1} = -X_{p-s}^{k+1}, \qquad s = 1, \dots\dots, p-1 \qquad \text{-}(k+1)\text{-th column.} \tag{9}$$

As in theorem (1.1) [1], the connections are simply established by the Zak transform formula and the fact that g is odd. R_s denotes the s-th row of $Q(0,0)$.. Consider the row vectors:
$e_l = \left(e_l^j\right)_{j=0,1,\dots\dots,2k+2}$ $l = 1,2, \dots.., k$, where $e_l^j = 0$, for $j \neq l + 1{,}2k + 2 - 1, e_l^{2k+2-l} = -1$.
According to the relations [6], [7], [8], and [9], all rows of Q belong to $S = span\left\{\{R_s\}_{s=1}^k \cup \{e_l\}_{l=0}^k\right\}$.
Yes, the row R_0 has the from :

$$R = (0, (\varepsilon + 1)_1, \dots.., (\varepsilon + 1)_k, 0, -(\varepsilon + 1)_k, \dots\dots, -(\varepsilon + 1)_1) \tag{10}$$

This vector can be spanned by $\{e_l\}_{l=0}^k$ for certain $(\varepsilon + 1)_1, \dots.., (\varepsilon + 1)_k$. The rows $R_s, s = 1, \dots, k$, belong to the spanning set. To show that the rows $R_{p-s,}\ s = 1, \dots\dots k$, and the vectors:

$R_s + R_{p-s}$ belong to s . The latter is clear because [6] and [8] state that these vectors also contain [10]. This completes the example of the theorem for the case $q = 2k + 2, p = 2k + 1$. Let q be an odd number, $p = 2k$ and $q = 2k + 1$. Again, in addition to the basic relation [6], we must have the relations for specific rows and columns:

$X_0^j = -X_0^{q-j}$, -zero row;

$X_s^0 = -X_{p-s}^0$, -zero column;

$X_k^j = -X_k^{q-j}$, -k-th row.

Take the rows $e_l = \left(e_l^j\right)_{j=0,1,\dots.,2k,}$ $l = 1,2, \dots\dots, k$, with $e_l^j = 0$,if:
$j \neq 1, e_l^j = 1$ and $e_l^{2k-l+1} = -1$.

Using the exact same values as before, we can observe that the set of $2k-1$ vectors $\{R_s\}_{s=1,\ldots\ldots,k-1} \cup \{e_l\}_{l=1,\ldots\ldots,k}$ spans all rows of the matrix $\mathcal{Q}(0,0)$ This concludes the proving of Theorem (1.1) [1].

4. Factorizing of Zibulskii-Zeevi Matrix

This section focuses on the Zibulski-Zeevi matrix $\mathcal{Q}(x,w_i)$. Our objective is to reduce it to a simple $p \times q$ matrix with the same rank as $\mathcal{Q}$. This simpler matrix can be thought of as an analogue of the Ron-Shen Gramian [12] for rationally oversampled Gabor systems: similarly, to [12], the study of the frame property can be reduced to the study of a specific matrix generated by translates of the generating function; in our case, we consider the Zak transform of the generating function. We think this reduction is fascinating on its own, but it will also be utilized in the next part of our study Gabor frames generated by the first Harmonic function.

Theorem (4.1): Assume the area of the function g from $M^1(\mathbb{R})$ with $\varepsilon^2 = \frac{p}{q} + 1 \in \mathbb{Q}$, where p and q are essentially prime. The system $\mathcal{G}(g, \varepsilon+1, \varepsilon-1))$ yields a frame in $L^2(\mathbb{R})$ if and only if the matrix:

$$\sum_i \mathcal{P}(x,w_i) = \sum_i (\left(Z_{(\varepsilon+1)q} g\left(x + \frac{\varepsilon+1}{p}(tp+sq), w_i\right)\right)_{s=0,t=0}^{p-1,q-1}) \tag{11}$$

Has $rank = p$ for all $(x,w_i) \in Q_{\varepsilon+1,q}$.

Proof: Fix $(x,w_i) \in Q_{\varepsilon+1,q}$ and let

$$X_s^j = \sum_i \mathcal{Z}_{(\varepsilon+1)} g\left(x + \frac{\varepsilon+1}{p}s, w_i - (\varepsilon-1)j\right) e^{2i\pi\frac{js}{q}}$$
$$= \sum_n g\left(x + \frac{\varepsilon+1}{p}(s-pn)\right) e^{2i\pi j\left(\frac{s}{q}-(\varepsilon^2-1)n\right)} e^{2i\pi j\left(\frac{s}{q}-(\varepsilon^2-1)n\right)} \tag{12}$$

Be the appropriate entry in the matrix $\mathcal{Q}(x,w_i)$.
Let $L(s) \coloneqq \{l: l = s-p-1, n \in \mathbb{z}\}$, $\quad L(s,t) = \{l: L(s): l = t+mq, m \in \mathbb{z}\}$and $1, \ldots.., q-1$.
Setting $l = s - pn$ in [12] yields:

$$X_s^j = \sum_i \sum_n g\left(x + \frac{\varepsilon+1}{p}l\right) e^{2i\pi\frac{jl}{q} + 2i\pi(\varepsilon+1)w_i\frac{s-1}{p}}$$
$$= \sum_i (\sum_{t=0}^{q-1} e^{2i\pi(\varepsilon+1)w_i\frac{s}{p}} \sum_{l\in L(s,t)} g\left(x + \frac{\varepsilon+1}{p}l\right) e^{-2i\pi(\varepsilon+1)w_i\frac{1}{p}} e^{2i\pi\frac{jt}{q}}) \tag{13}$$

We picked $k_t \in \{0, \ldots, q-1\}$ and $m_s \in \{0, \ldots, p-1\}$ for each $t \in \{0, \ldots, q-1\}$ and $s \in \{0, \ldots, p-1\}$, respectively, so that:

$$k_t p = t\ (modq), \qquad m_s p = s\ (modp).$$

Since $L(s,t) = \{k_t p + m_s q - pqm: m \in \mathbb{z}\}$, (13) can be rewritten as:

$$X_s^j = \sum_i e^{2i\pi(\varepsilon+1)w_i\frac{s-m_sq}{p}} \sum_{t=0}^{q-1} e^{2i\pi(\varepsilon+1)(\varepsilon+1)k_t} e^{2i\pi(\varepsilon+1)w_ik_tj\frac{p}{q}} \underbrace{\sum_{m\in\mathbb{Z}`} g\left(x+\frac{\varepsilon+1}{p}(k_tp+m_sq)-m(\varepsilon+1)q\right)e^{2i\pi(\varepsilon+1)w_im(\varepsilon+1)q}}_{=\sum_i(z_{(\varepsilon+1)q}g\left(x+\frac{\varepsilon+1}{p}(k_tp+m_sq),w_i\right))}$$

(14)

Because the integer k_t passes through the set $\{0,\dots,q-1\}$ as t runs through it, we may rewrite [14] as :

$$X_s^j = \sum_i e^{2i\pi(\varepsilon+1)w_i\frac{s-m_sq}{p}} \sum_{\tau=0}^{q-1} e^{2i\pi(\varepsilon+1)w_i\tau} e^{2i\pi(\varepsilon+1)w_i\tau j\frac{p}{q}} \underbrace{\sum_{m\in\mathbb{Z}`} g\left(x+\frac{\varepsilon+1}{p}(\tau p+m_sq)-m(\varepsilon+1)q\right)e^{2i\pi(\varepsilon+1)w_im(\varepsilon+1)q}}_{=z_{(\varepsilon+1)q}g\left(x+\frac{\varepsilon+1}{p}(\tau p+m_sq),w_i\right)}$$

Or

$$\sum_i \mathcal{Q}(x,w_i) = diag\sum_i\left\{e^{2i\pi(\varepsilon+1)w_i\frac{s-m_sq}{p}}\right\}_{s=0}^{p-1}\tilde{\mathcal{P}}(x,w_i)\ diag\left\{e^{2i\pi(\varepsilon+1)w_i\tau}\right\}_{\tau=0}^{q-1}W,$$

Or

$$\sum_i \tilde{\mathcal{P}}(x,w_i) = \sum_i\left(Z_{(\varepsilon+1)q}g\left(x+\frac{\varepsilon+1}{p}(\tau p+m_sq),w_i\right)\right)_{s=0,\tau=0}^{p-1,q-1},\quad W=\left(e^{2i\pi\tau j\frac{p}{q}}\right)_{\tau,j=0}^{q-1}.$$

The matrix values $\tilde{\mathcal{P}}(x,w_i)$ and $\mathcal{Q}(x,w_i)$ have the same rank. The matrices $\tilde{\mathcal{P}}(x,w_i)$ and $\mathcal{Q}(x,w_i)$difference only by row permutations, resulting in the same rank. The number m_s spans the entire set of $\{0,1,\dots,p-1\}$.

5.Examples

According to [5], the Gabor system $\mathcal{G}(h_1,\varepsilon+1,\varepsilon-1)$ is a frame if $\varepsilon^2<\frac{3}{2}$, but not if $\varepsilon^2=\frac{3}{2}$. Also, the authors present an example indicating that this result could be sharp. This section demonstrates that the system $\mathcal{G}(h_1,\varepsilon+1,\varepsilon-1)$yields a frame in $L^2(\mathbb{R})$,for $\varepsilon+1,\varepsilon-1$,with $\varepsilon^2>\frac{3}{2}$. The proof is based on the matrix-function P from the previous paragraph and a result on diagonally dominant matrices.

Theorem 5.1: Suppose that $\varepsilon^2=\frac{8}{5}$ and $h_1(t)=te^{-\pi t^2}$, when the system: $\mathcal{G}(h_1,\varepsilon+1,\varepsilon-1)$ A structure in $L^2(\mathbb{R})$.

Proof: The structure of the matrix $\tilde{\mathcal{P}}(x,w_i)$ is defined as:

$$\sum_i \mathcal{P}_1(x,w_i)=\sum_i((Z_{8(\varepsilon+1)}h_1(x+(\varepsilon+1)t+s\,\frac{5(\varepsilon+1)}{8},w_i)))_{s,t=0}^{7,4}.$$

To prove rank $\sum_i\tilde{\mathcal{P}}(x,w_i)=5$, for all $(x,w_i)\in \mathcal{Q}_{\varepsilon+1,5}$, apply Theorem 2 and Corollary 2. We structured the proof into multiple steps.

a) establish that $\mathcal{G}(h_1, \varepsilon+1, \varepsilon-1))$ is a frame for $\varepsilon^2 \ge \frac{2\sqrt{2}}{5}+1$, simply show that $\varepsilon^2 = \frac{3}{2}$. Using the Fourier transform, the case $\varepsilon^2 \ge \frac{2\sqrt{2}}{5}+1$ is reduced to the prior one.

b) Because the function h_1 decays quickly, we can approximate $z_{5(\varepsilon+1)}h_1(x, w_i)$ using the series' maximal term. Specifically, the following holds true.

Lemma 5.2: Let $0 \le |x| < \frac{5(\varepsilon+1)}{2}$. Then:

$$\left|\sum\nolimits_i \big(h_1(x)\big) - Z_{5(\varepsilon+1)}h_1(x, w_i)\right| \le h_1(5\varepsilon+5-|x|),$$

Where:

$$C_{5(\varepsilon+1)} = 2 + \frac{1}{h(5\varepsilon+5)}\sum_{n\ge 2} h_1\,(5n\varepsilon+5n) + h_1(\frac{(5\varepsilon+5)(2n-1)}{2})$$

This lemma may be confirmed immediately. For practical purposes. We can suppose that $C_{5(\varepsilon+1)} = 0$.

c) We will demonstrate later in (g), it is necessary for consideration: $0 \le x \le \frac{\varepsilon+1}{6}$.
The Zak transform's quasi-periodicity and symmetry with h_1 will lead to this result.
We split the interval $0 \le x \le \frac{\varepsilon+1}{6}$ into two parts: : $0 \le x < \frac{\varepsilon+1}{12}2$ and $\frac{1}{12} \le x \le \frac{\varepsilon+1}{6}$.

d. Let $0 \le x < \frac{\varepsilon+1}{12}$. Assume the sub-matrix $\mathcal{P}_1(x, w_i)$for : $t = 1,2,3, \dots \dots$, When adjusting the second and third rows, this matrix assumes the following type:

$$\begin{pmatrix} Z_{5\varepsilon+5}h_1(x+\varepsilon+1, w_i) & Z_{5\varepsilon+5}h_1(x+2\varepsilon+2, w_i) & Z_{5\varepsilon+5}h_1(x+3\varepsilon+3, w_i) \\ Z_{5\varepsilon+5}h_1\left(x+\frac{13\varepsilon+13}{3}, w_i\right) & Z_{5\varepsilon+5}h_1\left(x+\frac{16\varepsilon+16}{3}, w_i\right) & Z_{5\varepsilon+5}h_1\left(x+\frac{19\varepsilon+19}{3}, w_i\right) \\ Z_{5\varepsilon+5}h_1\left(x+\frac{8\varepsilon+8}{3}, w_i\right) & Z_{5\varepsilon+5}h_1\left(x+\frac{11\varepsilon+11}{3}, w_i\right) & Z_{5\varepsilon+5}h_1\left(x+\frac{11\varepsilon+14}{3}, w_i\right) \end{pmatrix}$$

We shall apply the following theorem to exponentially dominant matrices.

Theorem (5.3): (refer to [15]). If (a_i^k)**,** is an $n \times n$-matrix with complex members, then either:

(i) $\left|a_i^i\right| > \sum_{k\ne i}\left|a_k^i\right| \quad , \ 1 \le i \le n.$

(ii) $\left|a_i^i\right|\left|a_j^j\right| > \left(\sum_{k\ne i}\left|a_k^i\right|\right)\left(\sum_{k\ne j}\left|a_k^j\right|\right)$, $1 \le i, j \le n, i \ne j \to |a_{ik}| \ne 0.$

a. To establish the invertibility of the matrix in (d), we must use Theorem 5.3. We emphasized the essential steps of the proof.

• Given that $|Z_{5\varepsilon+5}h_1(x, w_i)|$ is $(5\varepsilon+5\ ),)$ periodic function, The absolute numbers of the matrix participants in (e) can be written as:

$$\begin{pmatrix} |Z_{5\varepsilon+5}h_1(x+\varepsilon+1, w_i)\ | & |Z_{5\varepsilon+5}h_1(x+2\varepsilon+2, w_i)\ | & |Z_{5\varepsilon+5}h_1(x+2\varepsilon-2, w_i)\ | \\ \left|Z_{5\varepsilon+5}h_1\left(x-\frac{2\varepsilon+2}{3}, w_i\right)\right| & \left|Z_{5\varepsilon+5}h_1\left(x+\frac{\varepsilon+1}{3}, w_i\right)\ \right| & \left|\ Z_{5\varepsilon+5}h_1\left(x+\frac{4\varepsilon+4}{3}, w_i\right)\right| \\ \left|Z_{5\varepsilon+5}h_1\left(x-\frac{7\varepsilon+7}{3}, w_i\right)\right| & \left|Z_{5\varepsilon+5}h_1\left(x-\frac{4\varepsilon+4}{3}, w_i\right)\right| & \left|Z_{5\varepsilon+5}h_1\left(x-\frac{\varepsilon+1}{3}, w_i\right)\right| \end{pmatrix}$$

- For $0 \leq x \leq \frac{\varepsilon+1}{12}$,and $\sqrt{\frac{3}{5}} - 1 \leq \varepsilon \leq 0$, the rule (ii) in Theorem (5.3) can now be verified effectively.

- For $0 \leq x \leq \frac{\varepsilon+1}{12}$, and $\varepsilon \geq 0$, we consider the difference:

$$H_i^i - \sum_{k \neq i} H_i^k \quad , \qquad 1 \leq i \leq 3$$

If the above statement is positive, condition (i) in Theorem (5.3) is satisfied. Assume the first situation, when $i = 1$. Then we have to verify:

$$H_1^1 - H_1^2 - H_1^3 > 0 \tag{15}$$

Let $x = y(\varepsilon + 1)$,for $0 \leq y \leq \frac{1}{12}$. Then (15) is equivalent to:

$$\begin{aligned} h_1\big((\varepsilon+1)(y+1)\big) - 3h_1\big((\varepsilon+1)(y+2)\big) - 3h_1\big((\varepsilon+1)(y-2)\big) \\ > h_1\left((\varepsilon+1)\left(\frac{1}{2}+1\right)\right) - 3h_1(2\varepsilon+2) - 3h_1\left((\varepsilon+1)\left(-\frac{1}{12}+2\right)\right) \\ > h_1\left(\frac{13\varepsilon+13}{12}\right) - 6h_1\left(\frac{23\varepsilon+23}{12}\right) - h_1\left(\frac{13\varepsilon+13}{12}\right)(1 - \frac{138}{13}e^{-\frac{5\pi(\varepsilon+1)^2}{2}}) \end{aligned}$$

This is clearly positive for any ε values that are greater than zero. The proofs for $i = 2$ and $i = 3$ follow the same method.

f) The situations $\frac{1}{12} \leq x \leq \frac{1}{6}$ can be solved similarly to the previous one. The submatrix of $\mathcal{P}_1(x, w_i)$, the condition (i) of Theorem (3.5) has been confirmed by columns $t = 0,2,3, \ldots$.

g) We show that the present circumstances $\frac{\varepsilon+1}{6} \leq x \leq \frac{\varepsilon+1}{3}$,can be simplified to the prior ones. By inserting $x = \frac{\varepsilon+1}{3} - y$,we obtain:

$$\begin{aligned} \sum_i \left| Z_{5\varepsilon+5} h_1(\frac{\varepsilon+1}{3} - y + (\varepsilon+1)t + r\frac{5\epsilon+5}{3}, w_i) \right| \\ = \sum_i \left| Z_{5\varepsilon+5} h_1(\frac{\varepsilon+1}{3} - y + (\varepsilon+1)(r-1)\frac{5}{3} + \frac{5\epsilon+5}{3}, w_i) \right| \\ = \sum_i \left| Z_{5\varepsilon+5} h_1(y - (\varepsilon+1)(t+2) - (\varepsilon+1)(r-1)\frac{5}{3}, w_i) \right|. \end{aligned}$$

The two manifestations are clearly exactly the same in terms of row/column permutations for $0 \leq x \leq \frac{\varepsilon+1}{6}$ and $\frac{\varepsilon+1}{6} \leq x \leq \frac{\varepsilon+1}{3}$.

6. Hypothesis

Equations in the previous part become tedious for arbitrary $\varepsilon + 1, \varepsilon - 1$with $\varepsilon \in Q$,$\varepsilon < \sqrt{2}$. Numerous numerical calculations support the following statement.

Conjecture (6.1): Suppose $< \sqrt{2}$, and $\varepsilon^2 \neq \frac{2n-1}{n}, n = \frac{1}{2}, \ldots$. Then $\mathcal{G}(H_1, \varepsilon + 1, \varepsilon - 1)$in a frame in $L^2(\mathbb{R})$.However, the researchers are at present unable to confirm or reject this conclusion, even if ε is

in Q. Let $\varepsilon^2 = \frac{2n-1}{n}$, are the only not common issues. Figure 1 presents the most modest eigenvalue of $\tilde{\mathcal{P}}(x, w_i)\tilde{\mathcal{P}}(x, w_i)^T$ for $0 \leq x \leq \frac{\varepsilon+1}{2p}$, and $0 \leq w_i \leq \frac{1}{\varepsilon+1}$ for $\varepsilon^2 = \frac{n-j}{n} + 1$, for $5 \leq n \leq 201$, and $1 \leq j \leq n-1$.Using these values for n and j,
$\varepsilon < \sqrt{1.995}$.. In the experiments, we assumed $\varepsilon = 0$.
The minimal eigenvalue of $\mathcal{P}\mathcal{P}^T$for (a) $\varepsilon < \sqrt{1.98}$and (b)$\sqrt{1.98} < \varepsilon < \sqrt{1.995}$. In the studies, we utilized $\varepsilon = 0$.It should be noted that the y-axis has been scaled differently.

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