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Stability of a General Quadratic-Cubic Functional Equation in Non-Archimedean 2-Normed Spaces

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Abstract

This research aims to investigate the Hyers-Ulam stability of the mixed-type quadratic-cubic functional equation in non-Archimedean 2-normed spaces using the fixed-point method. Additionally, some counter-examples are illustrated for instability. The exciting possibilities of this cutting-edge research and unlocking new frontiers in mathematical analysis are explored.

Keywords: Fixed point method, Hyers-Ulam stability, Non-Archimedean 2-normed spaces, p-adic field, Quadratic-cubic functional equation.

1. Introduction

Functional equations play an essential and fascinating role in mathematics, employing simple algebraic procedures that lead to intriguing solutions. The theory of functional equations is also applied in developing other domains such as analysis, algebra, geometry, and more. New approaches and techniques are utilized in problem-solving across fields like IT, finance, geometry, wireless sensor networks, and beyond. Ulam stability is a crucial concept in studying functional equations and their solutions. This theory examines whether a function that approximately satisfies a certain functional equation is close to a function that exactly satisfies the equation. Numerous researchers across various fields have explored different types of functional equation stability, such as Hyers-Ulam (H-U) stability, Hyers-Ulam-Rassias (H-U-R) stability, and generalized Hyers-Ulam stability, particularly in the context of different functional equations and mixed types over recent decades. Additionally, many authors have investigated the stability of various functional equations, yielding fascinating results in the classical (Archimedean) case. In recent years, the stability problems of these functional equations in non-Archimedean (NA) spaces have also been examined. In the last few decades, researchers have become much more interested in the study of Hyers-Ulam stability of functional equations (FEs). Numerous high-quality research papers have been published on this topic, as evidenced by references [1-3] and the associated citations in references [4-6]. As the field continues to grow, various approaches, including direct methods, fixed-point techniques, and others, have been developed and applied to solve different types of FEs [7-9].

Typically, when employing the direct method to establish stability outcomes for FEs, one must satisfy either of the two conditions:

$$\left\|\mathfrak{f}(\omega_1)-\frac{1}{\alpha}\mathfrak{f}(\alpha\omega_1)\right\|\leq\mathfrak{u}(\omega_1)\ \text{ or }\ \left\|\mathfrak{f}(\omega_1)-\alpha\mathfrak{f}\left(\frac{\omega_1}{\alpha}\right)\right\|\leq\mathfrak{u}\left(\frac{\omega_1}{\alpha}\right).$$

The choice between these conditions depends on specific assumptions, necessitating distinctions to apply an appropriate approach to solving distinct problems [10-12]. It was Ulam [13] who first raised the stability problem in 1940, and it was Hyers [14] who responded affirmatively in 1941 for a Banach space. According to Rassias, the bound for the norm of the Cauchy difference is weakened after the FE

$$\| \mathfrak{g}(\omega_1 + \omega_2) - \mathfrak{g}(\omega_1) - \mathfrak{g}(\omega_2) \| \leq \varepsilon(\| \omega_1 \|^p + \| \omega_2 \|^p)$$

which leads to the generalized H-U stability theorem for additive mapping. Concerning Hyers' theorem of additive mappings, papers were published by Aoki [15] and Rassias [16]. Gavruta presented a generalization of the Rassias theorem in 1994 [17]. The FE

$$\mathfrak{g}(\omega_1 + \omega_2) = \mathfrak{g}(\omega_1) + \mathfrak{g}(\omega_2) \quad 1$$

is referred to as an additive FE. Specifically, every solution of the additive FE is termed an additive mapping. In the work by Gahler [18], the theory of 2-norms and n-norms within a linear space was introduced. Subsequently, Gahler and White proposed the idea of 2-Banach spaces [19]. Kim and Park [20] investigated the generalized H-U stability of additive FEs in NA 2-normed space in 2014. In 2020, Wand, Park, and Shin [21] examined the H-U stability of additive ρ-functional equations in NA 2-normed space. Recently, Ghali and Kabbaj [22] investigated the hyperstability of the Cauchy-Jensen FE in NA 2-Banach spaces and some of its applications. In 2020, Cho, Gordji, and Zolfaghari [23] investigated the solution and stability of a generalized mixed-type quadratic-cubic$(\acute{Q}_2 - \acute{C}_3)$ functional equation in random normed spaces. In 2022, Mohiuddine, Tamilvannan, Mursaleen, and Alotaibi [24] examined the stability of a quartic functional equation in modular spaces using Hyers and fixed-point methods.

For example, Hensel [25] discovered the p-adic numbers in 1897 as a number theory equivalent to power series in complex analysis. He created a field with a valuation standard that lacks the Archimedean property. Numbers with p-adic numbers are the best examples of NA spaces. Those who work with p-adic numbers understand that they do not adhere to Archimedean properties, which state that for any positive number n, there exists an integer x such that $nx > y$ for another positive integer y. In the past 30 years, physicists have shown increased interest in the theory of NA spaces, especially with problems in quantum physics, p-adic physics, and superstring theory. The NA version of several results in the usual normed spaces theory differs significantly in their proofs, requiring a new level of understanding. Notably, in every valuation field where $|n| \leq 1$, every triangle is isosceles, and there may not be a unit vector in a non-Archimedean space. These details highlight the remarkable structure of NA spaces.

In this present article, we will use the alternative fixed point method to investigate the H-U stability of the general quadratic-cubic FE over NA 2-normed spaces.

2. Preliminaries

Some basic definitions and theorems are presented in this section as a reminder.

Definition 1: [21] Let $\mathcal{W}$ be a vector space over a scalar field $\mathbb{K}$ with an NA non-trivial valuation $|.|$. A function $\|.\|$ from $\mathcal{W}$ to $\mathbb{R}$ is called an NA norm (valuation) if it satisfies the following conditions:

(1) $\| \omega_1 \| = 0$ if and only if $\omega_1 = 0$;

(2) $\| r\omega_1 \| = |r| \, \| \omega_1 \|$ for all $r \in \mathbb{K}$, $\omega_1 \in \mathcal{W}$;

(3) $\| \omega_1, \omega_2 \| \leq \max\{\| \omega_1 \|, \| \omega_2 \|\}$ (satisfying the strong triangle inequality or ultra-metric property)

for all $\omega_1, \omega_2 \in \mathcal{W}$. Then $(\mathcal{W}, \|.\|)$ is called an NA-normed space.

Example 1: Let ρ be a fixed prime number. For any non-zero rational number ω_1, there is a unique integer $n_{\omega_1} \in \mathbb{Z}$ such that

$$\omega_1 = \frac{\alpha}{\beta}\rho^{n_{\omega_1}},$$

where α and β are integers not divisible by ρ. Then, the function $|.|_\rho: \mathbb{Q}_\rho \to [0, +\infty)$ defined by

$$|\omega_1| = \begin{cases} 0, & \omega_1 = 0, \\ \rho^{-n_{\omega_1}}, & \omega_1 \neq 0 \end{cases}$$

is an NA valuation on $\mathbb{Q}_\rho$.

Example 2: Let $\omega_1 = \frac{70}{13}$. In this case, let us find its 5-adic absolute value $(\rho = 5)$ as given below

$$\omega_1 = \frac{70}{13} = 5^1.\frac{14}{13}$$

which means $|\omega_1|_5 = \frac{1}{5}$ or 5^{-1}.

It will be simple to $|\omega_1|_{13} = 13$, because

$$\omega_1 = 13^{-1}.75$$

$$|\omega_1|_{13} = \frac{1}{13^{-1}} = 13.$$

which means $|\omega_1|_{13} = 13$.

Definition 2: [18] Let $\mathcal{W}$ be a vector space over a scalar field $\mathbb{K}$ with an NA non-trivial valuation $|.|$ with dim $\mathcal{W} > 1$. A function $\|.,.\|$ from $\mathcal{W}$ to $\mathbb{R}$ is called an NA 2-norm (valuation) if it satisfies the following conditions:

(1) $\| \omega_1, \omega_2 \| = 0$ if and only if ω_1, ω_2 are linearly dependent;

(2) $\| \omega_1, \omega_2 \| = \| \omega_2, \omega_1 \|$;

(3) $\| r\ \omega_1, \omega_2 \| = |r| \ \| \omega_1, \omega_2 \|$ for all $r \in \mathbb{K}, \omega_1, \omega_2 \in \mathcal{W}$;

(4) $\| \omega_1, \omega_2 + \upsilon \| \leq \max\{\| \omega_1, \omega_2 \|, \| \omega_1, \upsilon \|\}$ for all $\omega_1, \omega_2, \upsilon \in \mathcal{W}$

Then $(\mathcal{W}, \|.,.\|)$ is called an NA 2-normed space.

The following lemma follows from Definition 2.

Lemma 1: [18] Let $(\mathcal{W}, \|.,.\|)$ be an NA 2-normed space. If $\omega_1 \in \mathcal{W}$ and $\| \omega_1, \omega_2 \| = 0$ for all $\omega_2 \in \mathcal{W}$, then $\omega_1 = 0$.

Definition 3: [18] A sequence $\{\omega_{1_\pi}\}$ in an NA 2-normed space $(\mathcal{W}, \|.,.\|)$ is called a Cauchy sequence if there are two linearly independent points $\omega_1, \omega_2 \in \mathcal{W}$ such that

$$\lim_{m,\pi} \| \omega_{1_\pi} - \omega_{1_m}, \omega_2 \| = 0 \text{ and } \lim_{m,\pi} \| \omega_{1_\pi} - \omega_{1_m}, \omega_2 \| = 0.$$

Definition 4: [18] A sequence $\{\omega_{1_\pi}\}$ in an NA 2-normed space $(\mathcal{W}, \|.,.\|)$ is called a convergent sequence if there exists an $\omega_1 \in \mathcal{W}$ such that

$$\lim_{\pi} \| \omega_{1_\pi} - \omega_1, \omega_2 \| = 0 \quad \text{for all } \omega_1, \omega_2 \in \mathcal{W}.$$

In this case, recall that $\{\omega_{1_\pi}\}$ converges to ω_1 or that ω_1 is the limit of $\{\omega_{1_\pi}\}$, write $\{\omega_{1_\pi}\} \to \omega_1$ as $\pi \to \infty$ or $\lim_{\pi\to\infty} \omega_{1_\pi} = \omega_1$.

By Definition 2 (4), we have

$$\| \omega_{1_n} - \omega_{1_m}, \omega_2 \| \leq \max\{\| \omega_{1_{j+1}} - \omega_{1_j}, \omega_2 \| : m \leq j \leq n-1\}, \quad (n > m),$$

for all $\omega_1, \omega_2 \in \mathcal{W}$. Hence, a sequence $\{\omega_{1_n}\}$ in Cauchy in $(\mathcal{W}, \|.,.\|)$ if and only if $\{\omega_{1_{n+1}} - \omega_{1_n}\}$ converges to 0 in an NA 2-normed space $(\mathcal{W}, \|.,.\|)$.

Remark 1: [18] Let $(\mathcal{W}, \|.,.\|)$ be an NA 2-normed space. One can show that conditions (2) and (4) in Definition 2 imply that

$$\| \omega_1 + \omega_2, \upsilon \| \leq \| \omega_1, \upsilon \| + \| \omega_2, \upsilon \| \quad \text{and} \quad | \, \| \omega_1 - \upsilon \| - \| \omega_2, \upsilon \| \, | \leq \| \omega_1 - \omega_2, \upsilon \|$$

for all $\omega_1, \omega_2, \upsilon \in \mathcal{W}$. It is effortless to get the following lemma by using Remark 1.

Lemma 2: [18] For a convergent sequence $\{\omega_{1_n}\}$ in an NA 2-normed space $(\mathcal{W}, \|.,.\|)$,

$$\lim_{n\to\infty} \| \omega_{1_n}, \omega_2 \| = \| \lim_{n\to\infty} \omega_{1_n}, \omega_2 \| \quad \text{for all } \omega_2 \in \mathcal{W}.$$

Definition 5: [18] If every Cauchy sequence in $\mathcal{W}$ converges, then the NA 2-normed space $\mathcal{W}$ is called an NA 2-Banach space or an ultrametric 2-Banach space.

Definition 6: [26] Let $\mathcal{W}$ be a set. A function $\eth: \mathcal{W} \times \mathcal{W} \to [0, \infty]$ is called a generalized metric(GM) on $\mathcal{W}$ if $\eth$ satisfies the following conditions:

(1) $\eth(\omega_1, \omega_2) = 0$ if and only if $\omega_1 = \omega_2$;

(2) $\eth(\omega_1, \omega_2) = \eth(\omega_2, \omega_1)$ for all $\omega_1, \omega_2 \in \mathcal{W}$;

(3) $\eth(\omega_1, \upsilon) \leq \eth(\omega_1, \omega_2) + \eth(\omega_2, \upsilon)$ for all $\omega_1, \omega_2, \upsilon \in \mathcal{W}$.

Theorem 1: [26] Let $(\mathcal{W}, \eth)$ be complete and $\Lambda: \mathcal{W} \to \mathcal{W}$ be strictly contractive mapping with $0 < \pounds < 1$. Then for each given element $\omega_1 \in \mathcal{W}$, either

$$\eth(\Lambda^n \omega_1, \Lambda^{n+1} \omega_1) = \infty \text{ for all } n \geq 0$$

or there exists a natural number n_0 such that

(1) $\eth(\Lambda^n \omega_1, \Lambda^{n+1} \omega_1) < \infty$, for all $n \geq n_0$;

(2) the sequence $\{\Lambda^n \omega_1\}$ converges to a fixed point ω_2^* of Λ;

(3) ω_2^* is the unique fixed point of Λ in the set

$$\Psi = \{\omega_2 \in \mathcal{W} \; : \; \eth(\Lambda^{n_0} \omega_1, \omega_2) < \infty\};$$

(4) $\eth(\omega_2, \omega_2^*) \leq \frac{1}{1-\pounds} \eth(\omega_2, \Lambda \omega_2)$, for all $\omega_2 \in \Psi$."

In this article, let $\mathcal{W}$ be an NA 2-normed space with $\dim \mathcal{W} > 1$ and $\mathcal{Z}$ be an NA 2-Banach space with $\dim \mathcal{Z} > 1$. Suppose for a mapping $g: \mathcal{W} \to \mathcal{Z}$,

$$Dg(\omega_1, \omega_2) := g(\omega_1 + m\omega_2) + g(\omega_1 - m\omega_2) - m^2[g(\omega_1 + \omega_2) + g(\omega_1 - \omega_2)] - \frac{2(m^2-1)}{m^2(m-2)} g(m\omega_1) + \frac{m^3 - m^2 - m + 1}{2(m-2)} g(2\omega_1) - (g(2\omega_2) - g(-2\omega_2)) + 8(g(\omega_2) - g(-\omega_2))2$$

for all $\omega_1, \omega_2 \in \mathcal{W}$.

3. Results and Discussion

Stability of the FE – Even case

One can easily demonstrate this by applying the conditons for even and odd functions in this section. The condition for an even function is $\mathfrak{g}(-a) = \mathfrak{g}(a)$, and for and odd function, it is $\mathfrak{g}(-a) = -\mathfrak{g}(a)$. An even mapping $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ with $\mathfrak{g}(0) = 0$ satisfies Eq 2, if and only if the even mapping $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ is a quadratic ($\acute{Q}_2$) mapping, that is,

$$\mathfrak{g}(\omega_1 + \omega_2) + \mathfrak{g}(\omega_1 + \omega_2) = 2\mathfrak{g}(\omega_1) + 2\mathfrak{g}(\omega_2)$$

and an odd mapping $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ satisfies Eq 2 if and only if the odd mapping $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ is a cubic ($\acute{C}_3$) mapping, that is,

$$\mathfrak{g}(2\omega_1 + \omega_2) + \mathfrak{g}(2\omega_1 + \omega_2) = 2\mathfrak{g}(\omega_1 + \omega_2) + 2\mathfrak{g}(\omega_1 + \omega_2) + 12\mathfrak{g}(\omega_1)$$

It was proved in [23], that $\mathfrak{g}(\omega_1) = f(2\omega_1) - 4f(\omega_1)$ and $\mathfrak{g}(\omega_1) = f(2\omega_1) - 8f(\omega_1)$ are $\acute{Q}_2$ and $\acute{C}_3$ mappings respectively.

In this section, to prove the generalized H-U stability of the FE $D\mathfrak{g}(\omega_1, \omega_2) = 0$ in NA 2-normed space is discussed for even case.

Theorem 2: Let $\Phi: \mathcal{W} \times \mathcal{W} \to [0, \infty)$ be an even function such that there exists a constant $0 < £ < 1$ with

$$\Phi(\mathfrak{m}\omega_1, \mathfrak{m}\omega_2) \leq |\mathfrak{m}|^2 £\, \Phi(\omega_1, \omega_2) \qquad 3$$

for all $\omega_1, \omega_2 \in \mathcal{W}$. Let $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ be an even mapping satisfying

$$\|D\mathfrak{g}(\omega_1, \omega_2), \upsilon\| \leq \Phi(\omega_1, \omega_2) \qquad 4$$

for all $\omega_1, \omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Then there is a unique quadratic mapping $\acute{Q}_2: \mathcal{W} \to \mathcal{Z}$ such that

$$\left\|\mathfrak{g}(\omega_1) - \acute{Q}_2(\omega_1), \upsilon\right\| \leq \frac{1}{|2|\ |\mathfrak{m}|^2\ (1-£)} \Phi(0, \omega_1) \qquad 5$$

for all $\omega_1 \in \mathcal{W}$, $\upsilon \in \mathcal{Z}$.

Proof: Putting $\omega_1 = 0$ in Eq 2, implies

$$\left\|2\mathfrak{g}(\mathfrak{m}\omega_2) - 2\mathfrak{m}^2\mathfrak{g}(\omega_2), \upsilon\right\| \leq \Phi(0, \omega_2) \qquad 6$$

for all $\omega_2 \in \mathcal{W}$, $\upsilon \in \mathcal{Z}$. Interchanging ω_2 by ω_1 in Eq 6 and dividing on both sides of Eq 6 by 2, gives

$$\left\|\mathfrak{g}(\mathfrak{m}\omega_1) - \mathfrak{m}^2\mathfrak{g}(\omega_1), \upsilon\right\| \leq \frac{1}{|2|} \Phi(0, \omega_1)$$

$$\left\|\frac{\mathfrak{g}(\mathfrak{m}\omega_1)}{\mathfrak{m}^2} - \mathfrak{g}(\omega_1), \upsilon\right\| \leq \frac{1}{|2|} \frac{1}{|\mathfrak{m}|^2} \Phi(0, \omega_1) \qquad 7$$

for all $\omega_1 \in \mathcal{W}$, $\upsilon \in \mathcal{Z}$. Consider the set

$$\Gamma = \{\mathfrak{f}: \mathcal{W} \to \mathcal{Z}\} \qquad 8$$

and define the generalized metric ð in Γ by

$$ð(\mathfrak{f}, \mathfrak{h}) = \inf\{\sigma \in (0, \infty): \|\mathfrak{f}(\omega_1) - \mathfrak{h}(\omega_1), \upsilon\| \leq \sigma\ \Phi(0, \omega_1),\ \forall\ \omega_1 \in \mathcal{W}\}. \qquad 9$$

It is simple to prove that $(\Gamma, ð)$ is complete [26].

Now, define the function $\Lambda : \Gamma \to \Gamma$ such that

$$\Lambda \mathfrak{f}(\omega_1) = \frac{1}{\mathfrak{m}^2}\ \mathfrak{f}(\mathfrak{m}\omega_1) \tag{10}$$

for all $\omega_1 \in \mathcal{W}$. Let $\mathfrak{f}, \mathfrak{h} \in \Gamma$ be given such that $\eth(\mathfrak{f}, \mathfrak{h}) = \epsilon$. Then

$$\|\mathfrak{f}(\omega_1) - \mathfrak{h}(\omega_1), \upsilon\| \leq \epsilon\, \Phi(0, \omega_1) \tag{11}$$

for all $\omega_1 \in \mathcal{W}$, $\upsilon \in \mathcal{Z}$. Hence

$$\|\Lambda\mathfrak{f}(\mathfrak{m}\omega_1) - \Lambda\mathfrak{h}(\mathfrak{m}\omega_1), \upsilon\| = \left\|\frac{1}{\mathfrak{m}^2}\mathfrak{f}(\mathfrak{m}\omega_1) - \frac{1}{\mathfrak{m}^2}\mathfrak{h}(\mathfrak{m}\omega_1), \upsilon\right\|$$
$$\leq \frac{1}{|\mathfrak{m}|^2}.\epsilon.\Phi(0, \mathfrak{m}\omega_1)$$
$$\leq \frac{1}{|\mathfrak{m}|^2}.\epsilon.|\mathfrak{m}|^2.\pounds.\Phi(0, \omega_1)$$
$$\leq \epsilon.\pounds.\Phi(0, \omega_1)$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$, that is $\eth(\Lambda\mathfrak{f}, \Lambda\mathrm{h}) \leq \pounds\varepsilon$. Therefore

$$\eth(\Lambda\mathfrak{f}, \Lambda\mathrm{h}) \leq \pounds\ \eth(\mathfrak{f}, \mathrm{h})$$

for all $\mathfrak{f}, \mathfrak{h} \in \Gamma$. According to Eq 7

$$\eth(\mathfrak{g}, \Lambda\mathfrak{g}) \leq \frac{1}{|2|}.\frac{1}{|\mathfrak{m}|^2} < +\infty. \tag{12}$$

By Theorem 1, there is a mapping $\acute{Q}_2 : \mathcal{W} \to \mathcal{Z}$ satisfying the following conditions:

(1) $\acute{Q}_2$ is a fixed point of Λ, that is,

$$\acute{Q}_2(\mathfrak{m}\omega_1) = \mathfrak{m}^2\ \acute{Q}_2(\omega_1) \tag{13}$$

$\acute{Q}_2$ is a unique fixed point of the set denoted by Λ.

$$\mathrm{S} = \{\mathfrak{h} \in \Gamma : \eth(\mathfrak{f}, \mathfrak{h}) < \infty\}.$$

This indicates that $\acute{Q}_2$ is a unique mapping satisfying Eq 13 such that there is a $\sigma \in (0, \infty)$ satisfying

$$\left\|\mathfrak{g}(\omega_1) - \acute{Q}_2(\omega_1), \upsilon\right\| \leq \sigma\ .\ \Phi(0, \omega_1), \quad \forall\ \omega_1 \in \mathcal{W},\ \upsilon \in \mathcal{Z}.$$

(2) $\eth(\Lambda^{\mathrm{n}}\mathfrak{g}, \acute{Q}_2) \to 0$ as $\mathrm{n} \to \infty$. This gives that,

$$\lim_{\mathrm{n}\to\infty}(\Lambda^{\mathrm{n}}\mathfrak{g})(\omega_1) = \lim_{\mathrm{n}\to\infty}\frac{\mathfrak{g}(\mathfrak{m}^{\mathrm{n}}\omega_1)}{\mathfrak{m}^{2\mathrm{n}}} = \acute{Q}_2(\omega_1), \quad \forall\ \omega_1 \in \mathcal{W}.$$

(3) $\eth(\mathfrak{g}, \acute{Q}_2) \leq \frac{1}{1-\pounds}\eth(\mathfrak{g}, \Lambda^{\mathrm{n}}\mathfrak{g})$, which gives the inequality

$$\eth\left(\mathfrak{g}, \acute{Q}_2\right) \leq \frac{1}{1-\pounds}\eth(\mathfrak{g}, \Lambda\mathfrak{g}) \leq \frac{1}{|2|}\frac{1}{|\mathfrak{m}|^2}\frac{1}{(1-\pounds)}.$$

This indicates that the inequality Eq 5 remains valid.

According to Eq 3 and Eq 4

$$\|\ \mathrm{D}\acute{Q}_2(\omega_1, \omega_2), \upsilon\ \| = \lim_{\mathrm{n}\to\infty}\|\ \mathfrak{m}^{-2\mathrm{n}}\mathrm{D}\mathfrak{g}(\mathfrak{m}^{\mathrm{n}}\omega_1, \mathfrak{m}^{\mathrm{n}}\omega_2), \upsilon\ \|$$
$$\leq \lim_{\mathrm{n}\to\infty}\frac{1}{|\mathfrak{m}|^{2\mathrm{n}}}\Phi(\mathfrak{m}^{\mathrm{n}}\omega_1, \mathfrak{m}^{\mathrm{n}}\omega_2)$$

$$\leq \lim_{n\to\infty} \frac{1}{|\mathfrak{m}|^{2n}} £^n |\mathfrak{m}|^{2n} \Phi(\omega_1, \omega_2)$$
$$\leq \lim_{n\to\infty} £^n \Phi(\omega_1, \omega_2) = 0$$

for all $\omega_1, \omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$, and $n \in \mathbb{N}$. So, $\| D\acute{Q}_2(\omega_1, \omega_2), \upsilon \| = 0$.

Thus the mapping $\acute{Q}_2: \mathcal{W} \to \mathcal{Z}$ is $\acute{Q}_2$ as desired.

Corollary 1: Let $\theta \geq 0$ and $\tau = s + t$ be a positive real number with $\tau < 2$. Let $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ be an even mapping with $\mathfrak{g}(0) = 0$ satisfying

$$\| D_{\mathfrak{g}}(\omega_1, \omega_2), \upsilon \| \leq \theta(\| \omega_1 \|^\tau + \| \omega_2 \|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t)$$

for all $\omega_1, \omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$, and $\acute{Q}_2: \mathcal{W} \to \mathcal{Z}$ is a quadratic mapping such that

$$\| \mathfrak{g}(\omega_1) - \acute{Q}_2(\omega_1) \| \leq \frac{|\mathfrak{m}|^\tau}{|\mathfrak{m}|^2(|\mathfrak{m}|^\tau - |\mathfrak{m}|^2)} \frac{\theta \| \omega_1 \|^\tau}{|2|}$$

for all $\omega_1 \in \mathcal{W}$.

Proof: Assuming

$$\Phi(\omega_1, \omega_2) := \theta \ (\|\omega_1\|^\tau + \|\omega_2\|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t)$$

for all $\omega_1, \omega_2 \in \mathcal{W}$, and by choosing $£ = |\mathfrak{m}|^{2-\tau}$, the expected result can be obtained by Theorem 2.

Theorem 3: Let $\Phi: \mathcal{W} \times \mathcal{W} \to [0, \infty)$ be an even function such that there is a constant $0 < £ < 1$ with

$$\Phi\left(\frac{\omega_1}{\mathfrak{m}}, \frac{\omega_2}{\mathfrak{m}}\right) \leq \frac{£}{|\mathfrak{m}|^2} \Phi(\omega_1, \omega_2)$$

for all $\omega_1, \omega_2 \in \mathcal{W}$. Let $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ be an even mapping satisfying Eq 2. Then there is a unique quadratic mapping $\acute{Q}_2: \mathcal{W} \to \mathcal{Z}$ such that

$$\left\| \mathfrak{g}(\omega_1) - \acute{Q}_2(\omega_1), \upsilon \right\| \leq \frac{1}{|2|} \frac{1}{|\mathfrak{m}|^2} \frac{£}{(1-£)} \ \Phi(0, \omega_1) \qquad 14$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$.

Proof: Putting $\omega_1 = 0$ in Eq 2 implies

$$\left\| 2\mathfrak{g}(\mathfrak{m}\omega_2) - 2\mathfrak{m}^2 \mathfrak{g}(\omega_2), \upsilon \right\| \leq \Phi(0, \omega_2), \qquad 15$$

for all $\omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Replacing ω_2 by $\left(\frac{\omega_1}{\mathfrak{m}}\right)$ in Eq 15 and dividing on both sides of Eq 6 by 2, it gives

$$\left\| \mathfrak{g}(\omega_1) - \mathfrak{m}^2 \mathfrak{g}\left(\frac{\omega_1}{\mathfrak{m}}\right), \upsilon \right\| \leq \frac{1}{|2|} \Phi\left(0, \frac{\omega_1}{\mathfrak{m}}\right), \qquad 16$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Let $(\Gamma, \eth)$ be the GMS as defined by Theorem 2.

Now, define the function $\Lambda: \Gamma \to \Gamma$ such that

$$\Lambda\mathfrak{f}(\omega_1) = \mathfrak{m}^2 \mathfrak{f}\left(\frac{\omega_1}{\mathfrak{m}}\right) \qquad 17$$

for all $\omega_1 \in \mathcal{W}$. So $\eth(\mathfrak{g}, \Lambda\mathfrak{g}) \leq \frac{£}{|2||\mathfrak{m}|^2}$. According to Eq 16

$$\eth(\mathfrak{g}, \acute{Q}_2) \leq \frac{1}{|2|} . \frac{£}{|\mathfrak{m}|^2(1 - £)}$$

for all $\omega_1 \in \mathcal{W}$. This proof follows the same pattern as Theorem 2.

Corollary 2: Let $\theta \geq 0$ and $\tau = s + t$ be a positive real number with $\tau > 2$. Let $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ be an even mapping with $\mathfrak{g}(0) = 0$ satisfying

$$\| D\mathfrak{g}(\omega_1, \omega_2), \upsilon \| \leq \theta(\| \omega_1 \|^\tau + \| \omega_2 \|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t),$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Then there is a unique $\acute{Q}_2$ mapping $\acute{Q}_2: \mathcal{W} \to \mathcal{Z}$ such that

$$\| \mathfrak{g}(\omega_1) - \acute{Q}_2(\omega_1), \upsilon \| \leq \frac{|\mathfrak{m}|^\tau}{|\mathfrak{m}|^2(|\mathfrak{m}|^2 - |\mathfrak{m}|^\tau)} \frac{\theta \| \omega_1 \|^\tau}{|2|}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$.

Proof: Assuming

$$\Phi(\omega_1, \omega_2) := \theta \ (\|\omega_1\|^\tau + \|\omega_2\|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t)$$

for all $\omega_1, \omega_2 \in \mathcal{W}$, and by choosing $£ = |\mathfrak{m}|^{\tau-2}$, the expected result can be obtained by Theorem 3.

Example 3: Let $\rho > 2$ be a prime number and $\mathcal{W} = \mathcal{Z} = \mathbb{Q}_p$. Define $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ by $\mathfrak{g}(\omega\_1) = \omega_1^2 + 1$ for all $\omega_1 \in \mathcal{W}$. Since $|2^n|_\rho = 1$.

$$|D\mathfrak{g}(\omega_1, \omega_2)| = \left|\frac{88}{9}\right| \leq \theta(\|\omega_1\|^\tau + \|\omega_2\|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t) \quad (\forall \ \omega_1, \omega_2 \in \mathcal{W})$$

and

$$\left\|\frac{h(2^n\omega_1)}{2^{2n}} - \frac{h(2^{n-1}\omega_1)}{2^{2(n-1)}}\right\| = |9| \neq 0.$$

Hence $\{2^{-2n}h(2^n\omega_1)\}$ is not a Cauchy sequence. Where $h(\omega_1) = \mathfrak{g}(2\omega_1) - 4\mathfrak{g}(\omega_1)$.

Stability of the FE Eq 2: Odd case

Theorem 4: Let $\Phi: \mathcal{W} \times \mathcal{W} \to [0, \infty)$ be an odd function such that there is a constant $0 < £ < 1$ with

$$\Phi(\mathfrak{m}\omega_1, \mathfrak{m}\omega_2) \leq |\mathfrak{m}|^3 £ \Phi(\omega_1, \omega_2) \tag{18}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$. Let $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ be an odd mapping satisfying

$$\|D\mathfrak{g}(\omega_1, \omega_2), \upsilon\| \leq \Phi(\omega_1, \omega_2) \tag{19}$$

for all $\omega_1, \omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Then there is a unique $\acute{C}_3$ mapping $\acute{C}_3: \mathcal{W} \to \mathcal{Z}$ such that

$$\|\mathfrak{g}(\omega_1) - \acute{C}_3(\omega_1), \upsilon\| \leq \frac{1}{|\mathfrak{m}|^3(1-£)} \ \overline{\Phi(\omega_1)}, \tag{20}$$

where

$$\overline{\Phi(\omega_1)} = \max\left\{\frac{|\mathfrak{m}^2(\mathfrak{m}-1)|}{|2|.|4|}\Phi(0, \omega_1), \frac{|\mathfrak{m}^2(\mathfrak{m}-2)|}{|2(1-\mathfrak{m}^2)|}\Phi(\omega_1, 0)\right\}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$.

Proof: Replacing $\omega_1 = 0$ in Eq 2, obtains

$$\|2\mathfrak{g}(2\omega_2) - 16\mathfrak{g}(\omega_2), \upsilon\| \leq \Phi(0, \omega_2) \tag{21}$$

for all $\omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Interchanging ω_2 by ω_1 in Eq 21 and dividing on both sides of Eq 21 by 2, which gives

$$\|\mathfrak{g}(2\omega_1) - 8\mathfrak{g}(\omega_1), \upsilon\| \leq \frac{1}{|2|} \ \Phi(0, \omega_1) \qquad 22$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Letting $\omega_2 = 0$ in Eq 2, it gives

$$\left\|2(1-\mathfrak{m}^2)\mathfrak{g}(\omega_1) + \frac{2(1-\mathfrak{m}^2)}{\mathfrak{m}^2(\mathfrak{m}-2)}\mathfrak{g}(\mathfrak{m}\omega_1) + \frac{\mathfrak{m}^3-\mathfrak{m}^2-\mathfrak{m}+1}{2(\mathfrak{m}-2)}\mathfrak{g}(2\omega_1), \upsilon\right\| \leq \Phi(\omega_1, 0) \qquad 23$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Therefore

$$\left\|\mathfrak{m}^2(\mathfrak{m}-2)\mathfrak{g}(\omega_1) + \mathfrak{g}(\mathfrak{m}\omega_1) - \frac{\mathfrak{m}^2(\mathfrak{m}-1)}{4}\mathfrak{g}(2\omega_1), \upsilon\right\| \leq \frac{|\mathfrak{m}^2(\mathfrak{m}-2)|}{|2(1-\mathfrak{m}^2)|}\Phi(\omega_1, 0) \qquad 24$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. According to Eq 22 and Eq 24

$$\| \mathfrak{g}(\mathfrak{m}\omega_1) - \mathfrak{m}^3\mathfrak{g}(\omega_1), \upsilon \| \leq \max\left\{\frac{|\mathfrak{m}^2(\mathfrak{m}-1)|}{|2|.|4|}\Phi(0, \omega_1), \frac{|\mathfrak{m}^2(\mathfrak{m}-2)|}{|2(1-\mathfrak{m}^2)|}\Phi(\omega_1, 0)\right\} \qquad 25$$

where

$$\overline{\Phi(\omega_1)} = \max\left\{\frac{|\mathfrak{m}^2(\mathfrak{m}-1)|}{|2|.|4|}\Phi(0, \omega_1), \frac{|\mathfrak{m}^2(\mathfrak{m}-2)|}{|2(1-\mathfrak{m}^2)|}\Phi(\omega_1, 0)\right\}.$$

Then,

$$\| \mathfrak{g}(\mathfrak{m}\omega_1) - \mathfrak{m}^3\mathfrak{g}(\omega_1), \upsilon \| \leq \overline{\Phi(\omega_1)} \qquad 26$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. According to Eq 26, obtains

$$\left\|\frac{\mathfrak{g}(\mathfrak{m}\omega_1)}{\mathfrak{m}^3} - \mathfrak{g}(\omega_1), \upsilon\right\| \leq \frac{1}{|\mathfrak{m}|^3}\overline{\Phi(\omega_1)} \qquad 27$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Consider the set

$$\Gamma = \{\mathfrak{f}: \mathcal{W} \to \mathcal{Z}\} \qquad 28$$

and define the generalized metric $\eth$ in Γ by

$$\eth(\mathfrak{f}, \mathfrak{h}) = \inf\{\sigma \in (0, \infty): \|\mathfrak{f}(\omega_1) - \mathfrak{h}(\omega_1), \upsilon\| \leq \sigma \ \overline{\Phi(\omega_1)}, \forall \ \omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}\}. \qquad 29$$

It is simple to prove that $(\Gamma, \eth)$ is complete [26].

Now, define the function $\Lambda: \Gamma \to \Gamma$ such that

$$\Lambda\mathfrak{f}(\omega_1) = \frac{1}{\mathfrak{m}^3} \ \mathfrak{f}(\mathfrak{m}\omega_1) \qquad 30$$

for all $\omega_1 \in \mathcal{W}$. Let $\mathfrak{f}, \mathfrak{h} \in \Gamma$ be given such that $\eth(\mathfrak{f}, \mathfrak{h}) = \epsilon$. Then

$$\|\mathfrak{f}(\omega_1) - \mathfrak{h}(\omega_1), \upsilon\| \leq \epsilon \ \overline{\Phi(\omega_1)} \qquad 31$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Hence

$$\begin{aligned}\|\Lambda\mathfrak{f}(\mathfrak{m}\omega_1) - \Lambda\mathfrak{h}(\mathfrak{m}\omega_1), \upsilon\| &= \left\|\frac{1}{\mathfrak{m}^3}\mathfrak{f}(\mathfrak{m}\omega_1) - \frac{1}{\mathfrak{m}^3}\mathfrak{h}(\mathfrak{m}\omega_1), \upsilon\right\| \\ &\leq \frac{1}{|\mathfrak{m}|^3} \ \epsilon \ \overline{\Phi(\mathfrak{m}\omega_1)} \\ &\leq \frac{1}{|\mathfrak{m}|^3} \ \epsilon \ |\mathfrak{m}|^3 £ \ \overline{\Phi(\omega_1)} \\ &\leq \epsilon \ £ \ \overline{\Phi(\omega_1)}\end{aligned}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$, that is $\eth(\Lambda\mathfrak{f}, \Lambda\mathfrak{h}) \leq £\varepsilon$. Therefore

$$\eth(\Lambda\mathfrak{f}, \Lambda\mathfrak{h}) \leq £\ \eth(\mathfrak{f}, \mathfrak{h})$$

for all $\mathfrak{f}, \mathfrak{h} \in \Gamma$. According to Eq 27

$$\eth(\mathfrak{g}, \Lambda\mathfrak{g}) \leq \frac{1}{|\mathfrak{m}|^3} < +\infty. \qquad 32$$

By Theorem 1, there is a function $\acute{C}_3: \mathcal{W} \to \mathcal{Z}$ satisfying the following conditions:

(1) $\acute{C}_3$ is a fixed point of Λ, that is,

$$\acute{C}_3(\mathfrak{m}\omega_1) = \mathfrak{m}^3\ \acute{C}_3(\omega_1) \qquad 33$$

for all $\omega_1 \in \mathcal{W}$. $\acute{C}_3$ is a unique fixed point of the set denoted by Λ

$$S = \{\mathfrak{h} \in \Gamma : \eth(\mathfrak{f}, \mathfrak{h}) < \infty\}.$$

This indicates that $\acute{C}_3$ is a unique mapping satisfying Eq 33 such that there is a $\sigma \in (0, \infty)$ satisfying

$$\left\|\mathfrak{g}(\omega_1) - \acute{C}_3(\omega_1), \upsilon\right\| \leq \sigma\, \overline{\Phi(\omega_1)} \qquad \forall\ \omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}.$$

(2) $\eth(\Lambda^n\mathfrak{g}, \acute{C}_3) \to 0$ as $n \to \infty$. This indicates the equality,

$$\lim_{n\to\infty}(\Lambda^n\mathfrak{g})(\omega_1) = \lim_{n\to\infty}\frac{\mathfrak{g}(\mathfrak{m}^n\omega_1)}{\mathfrak{m}^{3n}} = \acute{C}_3(\omega_1), \qquad \forall\ \omega_1 \in \mathcal{W}.$$

(3) $\eth(\mathfrak{g}, \acute{C}_3) \leq \frac{1}{1-£}\eth(\mathfrak{g}, \Lambda^n\mathfrak{g})$, which implies

$$\eth\left(\mathfrak{g}, \acute{C}_3\right) \leq \frac{1}{1-£}\eth(\mathfrak{g}, \Lambda\mathfrak{g}) \leq \frac{1}{|\mathfrak{m}|^3}\frac{1}{(1-£)}.$$

This indicates that the inequality Eq 20 remains valid.

According to Eq 18 and Eq 19

$$\begin{aligned}\| D\acute{C}_3(\omega_1, \omega_2), \upsilon \| &= \lim_{n\to\infty} \| \mathfrak{m}^{-3n} D\mathfrak{g}(\mathfrak{m}^n\omega_1, \mathfrak{m}^n\omega_2), \upsilon \| \\ &\leq \lim_{n\to\infty}\frac{1}{|\mathfrak{m}|^{3n}}\Phi(\mathfrak{m}^n\omega_1, \mathfrak{m}^n\omega_2) \\ &\leq \lim_{n\to\infty}\frac{1}{|\mathfrak{m}|^{3n}}£^n|\mathfrak{m}|^{3n}\Phi(\omega_1, \omega_2) \\ &\leq \lim_{n\to\infty}£^n\Phi(\omega_1, \omega_2) = 0.\end{aligned}$$

for all $\omega_1, \omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$ and $n \in \mathbb{N}$. So $\| D\acute{C}_3(\omega_1, \omega_2), \upsilon \| = 0$.

Thus the mapping $\acute{C}_3: \mathcal{W} \to \mathcal{Z}$ is $\acute{C}_3$ as desired.

Corollary 3: Let $\theta \geq 0$ and $\tau = s + t$ be a positive real number with $\tau < 3$. Let $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ be an odd mapping with $\mathfrak{g}(0) = 0$ satisfying

$$\| D\mathfrak{g}(\omega_1, \omega_2), \upsilon \| \leq \theta(\| \omega_1 \|^\tau + \| \omega_2 \|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t)$$

for all $\omega_1, \omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Then there is a unique $\acute{C}_3$ mapping $\acute{C}_3: \mathcal{W} \to \mathcal{Z}$ such that

$$\| \mathfrak{g}(\omega_1) - \acute{C}_3(\omega_1), \upsilon \| \leq \frac{|\mathfrak{m}|^\tau}{|\mathfrak{m}|^3(|\mathfrak{m}|^\tau - |\mathfrak{m}|^3)}\max\left\{\frac{|\mathfrak{m}^2(\mathfrak{m}-1)|}{|2|.|4|}\theta \| \omega_1 \|^\tau, \frac{|\mathfrak{m}^2(\mathfrak{m}-2)|}{|2(1-\mathfrak{m}^2)|}\theta \| \omega_1 \|^\tau\right\}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$.

Proof: Assuming

$$\Phi(\omega_1,\omega_2) := \theta\ (\|\omega_1\|^{\tau} + \|\omega_2\|^{\tau} + \|\omega_1\|^{s}.\|\omega_2\|^{t})$$

for all $\omega_1,\omega_2 \in \mathcal{W}$, and by choosing $£ = |\mathfrak{m}|^{3-\tau}$, the expected result can be obtained by Theorem 4.

Theorem 5: Let $\Phi: \mathcal{W}\times\mathcal{W} \to [0,\infty)$ be an odd function such that there is a constant $0 < £ < 1$ with

$$\Phi\left(\frac{\omega_1}{\mathfrak{m}},\frac{\omega_2}{\mathfrak{m}}\right) \le \frac{£}{|\mathfrak{m}|^3}\Phi(\omega_1,\omega_2) \tag{34}$$

for all $\omega_1,\omega_2 \in \mathcal{W}$. Let $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ be an odd mapping satisfying Eq 19. Then there is a unique $\acute{C}_3$ mapping $\acute{C}_3: \mathcal{W} \to \mathcal{Z}$ such that

$$\left\|\mathfrak{g}(\omega_1) - \acute{C}_3(\omega_1),\upsilon\right\| \le \frac{£}{|\mathfrak{m}|^3(1-£)}\ \overline{\Phi(\omega_1)} \tag{35}$$

where

$$\overline{\Phi(\omega_1)} = \max\left\{\frac{|\mathfrak{m}^2(\mathfrak{m}-1)|}{|2|.|4|}\Phi(0,\omega_1), \frac{|\mathfrak{m}^2(\mathfrak{m}-2)|}{|2(1-\mathfrak{m}^2)|}\Phi(\omega_1,0)\right\}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$.

Proof: According to Eq 26,

$$\left\|\mathfrak{g}(\omega_1) - \mathfrak{m}^3\mathfrak{g}\left(\frac{\omega_1}{\mathfrak{m}}\right),\upsilon\right\| \le \overline{\Phi\left(\frac{\omega_1}{\mathfrak{m}}\right)} \tag{36}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$.

Let $(\Gamma, \eth)$ be the GMS as defined by Theorem 2.

Now, define the function $\Lambda: \Gamma \to \Gamma$ such that

$$\Lambda\mathfrak{f}(\omega_1) = \mathfrak{m}^3\mathfrak{f}\left(\frac{\omega_1}{\mathfrak{m}}\right) \tag{37}$$

for all $\omega_1 \in \mathcal{W}$. Let $\mathfrak{f},\mathfrak{h} \in \Gamma$, be given such that $\eth(\mathfrak{f},\mathfrak{h}) = \epsilon$. Then

$$\|\mathfrak{f}(\omega_1) - \mathfrak{h}(\omega_1),\upsilon\| \le \epsilon\,\overline{\Phi\left(\frac{\omega_1}{\mathfrak{m}}\right)} \tag{38}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Hence

$$\begin{aligned}\|\Lambda\mathfrak{f}(\omega_1) - \Lambda\mathfrak{h}(\omega_1),\upsilon\| &= \left\|\mathfrak{m}^3\mathfrak{f}\left(\frac{\omega_1}{\mathfrak{m}}\right) - \mathfrak{m}^3\mathfrak{h}\left(\frac{\omega_1}{\mathfrak{m}}\right),\upsilon\right\| \\ &\le |\mathfrak{m}|^3\epsilon\,\overline{\Phi\left(\frac{\omega_1}{\mathfrak{m}}\right)} \\ &\le |\mathfrak{m}|^3\epsilon\,\frac{£}{|\mathfrak{m}|^3}.\overline{\Phi(\omega_1)}\end{aligned}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$, that is $\eth(\Lambda\mathfrak{f},\Lambda\mathfrak{h}) \le £\epsilon$. Therefore

$$\eth(\Lambda\mathfrak{f},\Lambda\mathfrak{h}) \le £\eth(\mathfrak{f},\mathfrak{h}) \tag{39}$$

for all $\mathfrak{f},\mathfrak{h} \in \Gamma$. According to Eq 36,

$$\eth(\mathfrak{g},\Lambda\mathfrak{g}) \le \frac{£}{|\mathfrak{m}|^3} < \infty. \tag{40}$$

So

$$ð(\mathfrak{g}, \acute{C}_3) \leq \frac{£}{|\mathfrak{m}|^3(1-£)}.$$

This gives us the inequality Eq 35. This proof follows the same pattern as Theorem 4.

Corollary 4: Let $\theta \geq 0$ and $\tau = s + t$ be a positive real number with $\tau > 3$. Let $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ be an odd mapping with $\mathfrak{g}(0) = 0$ satisfying

$$\| D\mathfrak{g}(\omega_1, \omega_2), \upsilon \| \leq \theta(\| \omega_1 \|^\tau + \| \omega_2 \|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t)$$

for all $\omega_1, \omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Then there is a unique $\acute{C}_3$ mapping $\acute{C}_3: \mathcal{W} \to \mathcal{Z}$ such that

$$\| \mathfrak{g}(\omega_1) - \acute{C}_3(\omega_1), \upsilon \| \leq \frac{|\mathfrak{m}|^\tau}{|\mathfrak{m}|^3(|\mathfrak{m}|^3 - |\mathfrak{m}|^\tau)} \max\left\{\frac{|\mathfrak{m}^2(\mathfrak{m}-1)|}{|2|.|4|} \theta \| \omega_1 \|^\tau, \frac{|\mathfrak{m}^2(\mathfrak{m}-2)|}{|2(1-\mathfrak{m}^2)|} \theta \| \omega_1 \|^\tau\right\}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$.

Proof: Assuming

$$\Phi(\omega_1, \omega_2) := \theta \ (\|\omega_1\|^\tau + \|\omega_2\|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t)$$

for all $\omega_1, \omega_2 \in \mathcal{W}$, and by choosing $£ = |\mathfrak{m}|^{\tau-3}$, the expected result can be obtained by Theorem 5.

Example 4: Let $\rho > 2$ be a prime number and $\mathcal{W} = \mathcal{Z} = \mathbb{Q}_\mathfrak{p}$. Define $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ by $\mathfrak{g}(\omega_1) = \omega_1^3 + 1$ for all $\omega_1 \in \mathcal{W}$. Since $|2^n|_\rho = 1$.

$$|D\mathfrak{g}(\omega_1, \omega_2)| = \left|\frac{88}{9}\right| \leq \theta(\|\omega_1\|^\tau + \|\omega_2\|^\tau + \| \omega_1 \|^s . \| \omega_2 \|^t) \qquad (\forall \ \omega_1, \omega_2 \in \mathcal{W}),$$

and

$$\left\|\frac{h(2^n\omega_1)}{2^{3n}} - \frac{h(2^{n-1}\omega_1)}{2^{3(n-1)}}\right\| = |49| \neq 0.$$

Hence $\{2^{-3n}h(2^n\omega_1)\}$ is not a Cauchy sequence. Where $h(\omega_1)=\mathfrak{g}(2\omega_1) - 8\mathfrak{g}(\omega_1)$.

Stability of the FE Eq 2: Mixed case

Our goal in this section will be to establish the generalized H-U stability of the $\acute{Q}_2 - \acute{C}_3$ FE Eq 2, in NA 2-normed spaces. For a given mapping $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$, let

$$\mathfrak{g}_o(\omega_1) = \frac{\mathfrak{g}(\omega_1) - \mathfrak{g}(-\omega_1)}{2} \quad \text{and} \quad \mathfrak{g}_e(\omega_1) = \frac{\mathfrak{g}(\omega_1) + \mathfrak{g}(-\omega_1)}{2}.$$

Then $\mathfrak{g}_o$ is odd and $\mathfrak{g}_e$ is even.

Theorem 6: Let $\Phi: \mathcal{W} \times \mathcal{W} \to [0, \infty)$ be a function such that there is a constant $0 < £ < 1$ with

$$\Phi(\mathfrak{m}\omega_1, \mathfrak{m}\omega_2) \leq |\mathfrak{m}|^3 £ \ \Phi(\omega_1, \omega_2) \tag{41}$$

for all $\omega_1, \omega_2 \in \mathcal{W}$. Suppose $\mathfrak{g}: \mathcal{W} \to \mathcal{Z}$ is a mapping satisfying the inequality

$$\|D\mathfrak{g}(\omega_1, \omega_2), \upsilon\| \leq \Phi(\omega_1, \omega_2) \tag{42}$$

for all $\omega_1, \omega_2 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Then there are a unique $\acute{Q}_2$ mapping $\acute{Q}_2: \mathcal{W} \to \mathcal{W}$ and a unique $\acute{C}_3$ mapping $\acute{C}_3: \mathcal{W} \to \mathcal{Z}$ such that

$$\| \mathfrak{g}(\omega_1) - \acute{Q}_2(\omega_1) - \acute{C}_3(\omega_1), \upsilon \|$$

$$\leq \max\left\{\frac{1}{|2|} \max\left\{\frac{1}{|2|.|\mathfrak{m}|^2.(1-£)} \Phi(0, \omega_1) \ \Phi(0, -\omega_1)\right\}, \frac{1}{|2|} \max\left\{\frac{1}{|\mathfrak{m}|^3(1-£)} \ \overline{\Phi(\omega_1)}, \frac{1}{|\mathfrak{m}|^3(1-£)} \ \overline{\Phi(-\omega_1)}\right\}\right\} \tag{43}$$

where $\overline{\Phi(\omega_1)} = \max\left\{\frac{|\mathfrak{m}^2(\mathfrak{m}-1)|}{|2|.|4|}\Phi(0,\omega_1), \frac{|\mathfrak{m}^2(\mathfrak{m}-2)|}{|2(1-\mathfrak{m}^2)|}\Phi(\omega_1,0)\right\}$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$.

Proof: Assume that $\mathfrak{g}(\omega_1) = \mathfrak{g}_e(\omega_1) + \mathfrak{g}_o(\omega_1)$. Let

$$\Phi(\omega_1,\omega_2) = \frac{1}{|2|}\max\{\Phi(\omega_1,\omega_2), \phi(-\omega_1,-\omega_2)\}$$

then by Eq 41, and Eq 42, which gives

$$\Phi(\mathfrak{m}\omega_1,\mathfrak{m}\omega_2) \le |\mathfrak{m}|^3 \ \pounds \ \Phi(\omega_1,\omega_2) \le |\mathfrak{m}|^2 \ \pounds \ \Phi(\omega_1,\omega_2)$$

$$\| D\mathfrak{g}_o(\omega_1,\omega_2 \ ),\upsilon \| \le \Phi(\omega_1,\omega_2), \quad \| D\mathfrak{g}_e(\omega_1,\omega_2),\upsilon \| \le \Phi(\omega_1,\omega_2),$$

Hence by Theorem 2 and Theorem 4, there are a unique $\acute{Q}_2$ mapping $\acute{Q}_2: \mathcal{W} \to \mathcal{Z}$ and a unique $\acute{C}_3$ mapping $\acute{C}_3: \mathcal{W} \to \mathcal{Z}$ such that

$$\| \mathfrak{g}_o(\omega_1) - \acute{Q}_2(\omega_1),\upsilon \| \le \frac{1}{|2|.|\mathfrak{m}|^2.(1-\pounds)}\Phi(0,\omega_1)$$

and

$$\| \mathfrak{g}_e(\omega_1) - \acute{C}_3(\omega_1),\upsilon \| \le \frac{1}{|\mathfrak{m}|^3(1-\pounds)} \ \overline{\Phi(\omega_1)}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. Therefore

$$\| \mathfrak{g}(\omega_1) - \acute{C}_3(\omega_1) - \acute{Q}_2(\omega_1),\upsilon \| = \left\|\mathfrak{g}_o(\omega_1) + \mathfrak{g}_e(\omega_1) - \acute{C}_3(\omega_1) - \acute{Q}_2(\omega_1),\upsilon\right\|$$

$$\le \max\left\{\| \mathfrak{g}_o(\omega\_1) - \acute{C}_3(\omega\_1),\upsilon \|, \| \mathfrak{g}_e(\omega\_1) - \acute{Q}_2(\omega\_1),\upsilon \|\right\}$$

$$\le$$

$$\max\left\{\frac{1}{|2|}\max\left\{\frac{1}{|2|.|\mathfrak{m}|^2.(1-\pounds)}\Phi(0,\omega_1), \frac{1}{|2|.|\mathfrak{m}|^2.(1-\pounds)}\Phi(0,-\omega_1)\right\}, \frac{1}{|2|}\max\left\{\frac{1}{|\mathfrak{m}|^3(1-\pounds)} \ \overline{\Phi(\omega_1)}, \frac{1}{|\mathfrak{m}|^3(1-\pounds)} \ \overline{\Phi(-\omega_1)}\right\}\right\}$$

for all $\omega_1 \in \mathcal{W}, \upsilon \in \mathcal{Z}$. This completes the proof.

4. Conclusion

Lorem In this article, we investigated the H-U stability of the quadratic-cubic functional equation (Eq 2) in NA 2-normed spaces using fixed-point methods and provided a suitable counterexample. In light of this work, it has become possible to gain a more comprehensive understanding of the stability problems of functional equations within the context of NA 2-normed spaces.

Conflict of interest

The authors declare that they have no competing interests.

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