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Algebraic Identities on Generalized Derivations in Prime Rings with Involution

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Abstract:

This research paper aims to investigate the commutativity properties of prime rings in relation to generalized derivations and a left multiplier that fulfil specific algebraic conditions involving involution. Additionally, we present various examples studies to illustrate that the constraints imposed in our theorems are indeed necessary and cannot be omitted without compromising the validity of our results.

Introduction: Consider a ring $\mathfrak{S}$ that satisfies the associative property, and let $Z(\mathfrak{S})$ represent its center. An involution denoted by $*$, which is an additive function mapping $\mathfrak{S}$ to itself. This involution has specific properties: for any α and β in $\mathfrak{S}$, applying the involution twice returns the original element ($(\alpha^*)^* = \alpha$), it distributes over addition ($(\alpha + \beta)^* = \alpha^* + \beta^*$), and it reverses the order of multiplication ($(\alpha\beta)^* = \beta^*\alpha^*$). We categorize elements as hermitian when they remain unchanged under the involution ($\alpha^* = \alpha$), and as skew-hermitian when they change sign ($\alpha^* = -\alpha$). We use $\mathscr{H}(\mathfrak{S})$ to represent the collection of all hermitian elements in $\mathfrak{S}$, and $S(\mathfrak{S})$ for all skew-hermitian elements. The involution is classified as first kind if $\mathscr{H}(\mathfrak{S})$ is a subset of the center of $\mathfrak{S}$, denoted as $Z(\mathfrak{S})$. If this is not the case, it's considered second kind, and in this scenario, the intersection of $\mathscr{H}(\mathfrak{S})$ and $Z(\mathfrak{S})$ contains more than just the zero element. We also define several types of mappings on ring $\mathfrak{S}$. A left multiplier, $\triangle$, is an additive map where $\triangle(\upsilon\omega) = \triangle(\upsilon)\omega$, $\forall \upsilon, \omega \in \mathfrak{S}$. A derivation, ψ, is an additive mapping that satisfies $\psi(\upsilon\omega) = \psi(\upsilon)\omega + \upsilon\psi(\omega)$, $\forall \upsilon, \omega \in \mathfrak{S}$. Extending this concept, we define a generalized derivation, Γ, which is linked to a derivation ψ. This function satisfies $\Gamma(\upsilon\omega) = \Gamma(\upsilon)\omega + \upsilon\psi(\omega)$, $\forall \upsilon, \omega \in \mathfrak{S}$. It's worth noting that any derivation can be considered a generalized derivation associated with itself.

Objectives: In this study, we intend to examine the commutativity of a prime ring $\mathfrak{S}$ by utilizing generalized derivations Γ_1, Γ_2, and a left multiplier $\triangle$, while adhering to specific algebraic identities that involve involution. Specifically, we will delve into the commutativity of rings $\mathfrak{S}$ that fulfill the following algebraic conditions:

- $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) + \triangle(\upsilon \circ \upsilon^*) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] + \triangle(\upsilon \circ \upsilon^*) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $\Gamma(\upsilon\upsilon^*) \pm \Gamma(\upsilon)\Gamma(\upsilon^*) + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;

- $\Gamma(\upsilon\upsilon^*) \pm \Gamma(\upsilon^*)\Gamma(\upsilon) + \triangle(\upsilon \circ \upsilon^*) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$.

Lastly, we offer examples to illustrate that the constraints applied to our hypotheses are necessary and not redundant.

Results: Building on the work of Nejjar ([8], Theorems 3.5, 3.8), who demonstrated that a 2-torsion-free prime ring with involution and a derivation ψ satisfying certain conditions must be commutative, we explore broader generalizations of these conditions. Specifically, Nejjar showed that if the derivation ψ meets either of the following criteria: $[\psi(\upsilon), \psi(\upsilon)] \pm \upsilon \circ \upsilon \in Z(\mathfrak{S}), \quad \forall \upsilon \in \mathfrak{S}$ or $\psi(\upsilon) \circ \psi(\upsilon) \pm \upsilon \circ \upsilon \in Z(\mathfrak{S}), \quad \forall \upsilon \in \mathfrak{S}$, then $\mathfrak{S}$ is necessarily commutative. Our work extends these findings by introducing new identities for pairs of generalized derivations that are connected to a left multiplier $\triangle$. Finally, we offer examples to illustrate that the constraints applied to our hypotheses are necessary and not redundant.

Conclusions: In this research, we investigate the commutativity of prime rings $\mathfrak{S}$ admitting an involution and generalized derivations satisfying some algebraic identities. We can conclude our paper with an open question.

Open question: are these results correct if we replace the generalized derivation by the generalized (α, β) −derivation, where α and β are automorphisms of ring $\mathfrak{S}$?

Keywords: Generalized derivation, prime ring with involution, integral domain.

1. Introduction

Consider a ring $\mathfrak{S}$ that satisfies the associative property, and let $Z(\mathfrak{S})$ represent its center. We'll use the notation $[\alpha, \beta]$ to represent the commutator $\alpha\beta - \beta\alpha, \forall \alpha, \beta \in \mathfrak{S}$. Similarly, we'll denote the anti-commutator $\alpha \circ \beta$, defined as $\alpha\beta + \beta\alpha$. A ring $\mathfrak{S}$ is considered prime if, for any two elements $\alpha, \beta \in \mathfrak{S}$, the condition $\alpha\mathfrak{S}\beta = 0$ necessitates that either $\alpha = 0$ or $\beta = 0$.

In the context of $\mathfrak{S}$, an involution denoted by $*$, which is an additive function mapping $\mathfrak{S}$ to itself. This involution has specific properties: for any α and β in $\mathfrak{S}$, applying the involution twice returns the original element ($(\alpha^*)^* = \alpha$), it distributes over addition ($(\alpha + \beta)^* = \alpha^* + \beta^*$), and it reverses the order of multiplication ($(\alpha\beta)^* = \beta^*\alpha^*$). We categorize elements as hermitian when they remain unchanged under the involution ($\alpha^* = \alpha$), and as skew-hermitian when they change sign ($\alpha^* = -\alpha$).

We use $\triangle(\mathfrak{S})$ to represent the collection of all hermitian elements in $\mathfrak{S}$, and $S(\mathfrak{S})$ for all skew-hermitian elements. The involution is classified as first kind if $\triangle(\mathfrak{S})$ is a subset of the center of $\mathfrak{S}$, denoted as $Z(\mathfrak{S})$. If this is not the case, it's considered second kind, and in this scenario, the intersection of $\triangle(\mathfrak{S})$ and $Z(\mathfrak{S})$ contains more than just the zero element.

We also define several types of mappings on ring $\mathfrak{S}$. A left multiplier, $\triangle$, is an additive map where $\triangle(\upsilon\omega) = \triangle(\upsilon)\omega, \ \forall \upsilon, \omega \in \mathfrak{S}$. A derivation, ψ, is an additive mapping that satisfies $\psi(\upsilon\omega) = \psi(\upsilon)\omega + \upsilon\psi(\omega), \ \forall \upsilon, \omega \in \mathfrak{S}$. Extending this concept, we define a generalized derivation, Γ, which is linked to a derivation ψ. This function satisfies $\Gamma(\upsilon\omega) = \Gamma(\upsilon)\omega + \upsilon\psi(\omega), \quad \forall \upsilon, \omega \in \mathfrak{S}$. It's worth noting that any derivation can be considered a generalized derivation associated with itself.

In recent decades, numerous mathematicians have explored the relationship between the commutativity of a ring $\mathfrak{S}$ and certain types of additive mappings, such as automorphisms and generalized derivations acting on subsets of rings. The seminal work of Posner established the most significant theorem on commuting and related mappings. Posner demonstrated that a prime ring $\mathfrak{S}$ is commutative if it possesses a nonzero derivation ψ such that $[\psi(\upsilon), \upsilon] \in Z(\mathfrak{S}), \quad \forall \upsilon \in \mathfrak{S}$. This foundational result has been further refined and extended by various researchers, as documented in the comprehensive bibliography provided in [2], [3], [13], and [7]. More recently, some authors have investigated these concepts in the context of rings with involution. For instance in [1], Ali and Dar obtained a $*$-version of Posner's second theorem. Specifically, they showed that a ring $\mathfrak{S}$ is commutative if it possesses a derivation ψ with an involution $*$, where $char(\mathfrak{S}) \neq 2$, $\psi(S(\mathfrak{S}) \cap Z(\mathfrak{S})) \neq 0$, and $[\psi(\upsilon), \upsilon^*] \in Z(\mathfrak{S})$, $\forall \upsilon \in \mathfrak{S}$.

In this study, we intend to examine the commutativity of a prime ring $\mathfrak{S}$ by utilizing generalized derivations Γ_1, Γ_2, and a left multiplier $\triangle$, while adhering to specific algebraic identities that involve involution. Specifically, we will delve into the commutativity of rings $\mathfrak{S}$ that fulfill the following algebraic conditions:

- $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) + \triangle(\upsilon \circ \upsilon^*) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] + \triangle(\upsilon \circ \upsilon^*) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $\Gamma(\upsilon\upsilon^*) \pm \Gamma(\upsilon)\Gamma(\upsilon^*) + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$;
- $\Gamma(\upsilon\upsilon^*) \pm \Gamma(\upsilon^*)\Gamma(\upsilon) + \triangle(\upsilon \circ \upsilon^*) \in Z(\mathfrak{S})$, for all $\upsilon \in \mathfrak{S}$.

Lastly, we offer examples to illustrate that the constraints applied to our hypotheses are necessary and not redundant.

2. Preliminaries

In this section, we commence our analysis by introducing several well-established results that will be extensively leveraged in the proof of our theorems. Specifically, for all $\upsilon, \omega, z \in \mathfrak{S}$, the following identities hold:

1. $[\upsilon,, \omega z] = \omega[\upsilon, z] + [\upsilon, \omega]z$; $[[\upsilon\omega, z] = [\upsilon, z]\omega + \upsilon[\omega, z]$.
2. $\upsilon\omega \circ z = (\upsilon \circ z)\omega + \upsilon[\omega, z] = \upsilon(\omega \circ z) - [\upsilon, z]\omega$; $\upsilon \circ, \omega z = \omega(\upsilon \circ z) + [\upsilon, \omega]z = (\upsilon \circ \omega)z + \omega[z, \upsilon]$.

Lemma 1 [[5] Lemmas 2.1 and 2.2] "Let $\mathfrak{S}$ be a prime ring with involution * of the second kind such that $char(\mathfrak{S}) \neq 2$, the following assertions are equivalents:"

1. $[\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \quad \forall \upsilon \in \mathfrak{S}$.
2. $\upsilon \circ \upsilon^* \in Z(\mathfrak{S}) \quad \forall \upsilon \in \mathfrak{S}$.
3. $\mathfrak{S}$ is an integral domain.

Lemma 2 [[11] Lemma 2.5] "Let $\mathfrak{S}$ be a prime ring with involution of the second kind such that $char(\mathfrak{S}) \neq 2$. Let ψ be a derivation of $\mathfrak{S}$ such that $\psi(h) = 0$ for all $h \in \triangle(\mathfrak{S}) \cap Z(\mathfrak{S})$. Then $\psi(\upsilon) = 0 \quad \forall \upsilon \in \mathfrak{S}$".

Lemma 3 [[4] Lemma 2] "Let $\mathfrak{S}$ be a prime ring. If $ab \in Z(\mathfrak{S})$ for some $0 \neq a \in Z(\mathfrak{S})$, then $b \in Z(\mathfrak{S})$. In particular, if $ab = 0$, then $b = 0$".

Lemma 4 [[12] Lemma 2.2] "Let $\mathfrak{S}$ be a ring and ψ be a multiplicative derivation of $\mathfrak{S}$. Then $\psi(Z(\mathfrak{S})) \subseteq Z(\mathfrak{S})$."

3. Main results

Building on the work of Nejjar ([8], Theorems 3.5, 3.8), who demonstrated that a 2-torsion-free prime ring with involution and a derivation ψ satisfying certain conditions must be commutative, we explore broader generalizations of these conditions. Specifically, Nejjar showed that if the derivation ψ meets either of the following criteria: $[\psi(\upsilon),\psi(\upsilon)]\pm\upsilon\circ\upsilon\in Z(\mathfrak{S}),\quad\forall\upsilon\in\mathfrak{S}$ or $\psi(\upsilon)\circ\psi(\upsilon)\pm\upsilon\circ\upsilon\in Z(\mathfrak{S}),\ \ \forall\upsilon\in\mathfrak{S}$, then $\mathfrak{S}$ is necessarily commutative. Our work extends these findings by introducing new identities for pairs of generalized derivations that are connected to a left multiplier $\triangle$.

Throughout the following results, let $\mathfrak{S}$ be a prime ring with involution, $\triangle$ be a left multiplier, and (Γ_1,Γ_2) be two generalized derivations, which are linked to nonzero derivations (ψ_1,ψ_2), respectively.

Theorem 1 *The following statements are equivalent:*

(i) $[\Gamma_1(\upsilon),\Gamma_2(\upsilon^*)]+\triangle([\upsilon,\upsilon^*])\in Z(\mathfrak{S}),\quad\forall\upsilon\in\mathfrak{S}.$
(ii) $\Gamma_1(\upsilon)\circ\Gamma_2(\upsilon^*)+\triangle(\upsilon\circ\upsilon^*)\in Z(\mathfrak{S})\quad\forall\upsilon\in\mathfrak{S}.$
(iii) $[\Gamma_1(\upsilon),\Gamma_2(\upsilon^*)]+\triangle([\upsilon\circ\upsilon^*])\in Z(\mathfrak{S})\quad\forall\upsilon\in\mathfrak{S}.$
(iv) $\Gamma_1(\upsilon)\circ\Gamma_2(\upsilon^*)+\triangle([\upsilon,\upsilon^*])\in Z(\mathfrak{S})\quad\forall\upsilon\in\mathfrak{S}.$
(v) $\mathfrak{S}$ *is an integral domain.*

Proof. It is necessary to demonstrate that $(i),(ii),(iii)$ and $(iv)\Longrightarrow(v)$.

$(i)\Longrightarrow(v)$ Suppose that

$$[\Gamma_1(\upsilon),\Gamma_2(\upsilon^*)]+\triangle([\upsilon,\upsilon^*])\in Z(\mathfrak{S})\quad\forall\upsilon\in\mathfrak{S}.\tag{1}$$

Replacing υ by $\upsilon+\omega$ in (1) and using it, we obtain

$$[\Gamma_1(\upsilon),\Gamma_2(\omega^*)]+[\Gamma_1(\omega),\Gamma_2(\upsilon^*)]+\triangle([\upsilon,\omega^*])+\triangle([\omega,\upsilon^*])\in Z(\mathfrak{S})\quad\forall\upsilon,\omega\in\mathfrak{S}.\tag{2}$$

Replacing ω by ωh in (2) where $h\in Z(\mathfrak{S})\cap\triangle(\mathfrak{S})$ and using it again, we get

$$[\Gamma_1(\upsilon),\omega^*]\psi_2(h)+[\omega,\Gamma_2(\upsilon^*)]\psi_1(h)\in Z(\mathfrak{S})\quad\forall\upsilon,\omega\in\mathfrak{S}.\tag{3}$$

Replacing υ by υh in (3) where $h\in Z(\mathfrak{S})\cap\triangle(\mathfrak{S})$ and using it again, we obtain

$$[\upsilon,\omega^*]\psi_1(h)\psi_2(h)+[\omega,\upsilon^*]\psi_1(h)\psi_2(h)\in Z(\mathfrak{S})\quad\forall\upsilon,\omega\in\mathfrak{S}.\tag{4}$$

Taking $\omega=\upsilon$ in (4), we arrive at

$$2[\upsilon,\upsilon^*]\psi_1(h)\psi_2(h)\in Z(\mathfrak{S})\quad\forall\upsilon\in\mathfrak{S}.\tag{5}$$

Since $char(\mathfrak{S})\neq 2$, and by using Lemme 1 ,we get $[\upsilon,\upsilon^*]\in Z(\mathfrak{S})$ or $\psi_1(h)\psi_2(h)=0$, by primness that leads to $\mathfrak{S}$ is an integral domain or $\psi_1(h)=0$ or $\psi_2(h)=0$ for all $h\in Z(\mathfrak{S})$.

If $\psi_1(h)=0$ for all $h\in Z(\mathfrak{S})\cap\triangle(\mathfrak{S})$, Lemma 2 leads us to $\psi_1(\upsilon)=0,\ \ \forall\upsilon\in\mathfrak{S}$, a contradiction. Using the same technique, we obtain that $\psi_2(Z(\mathfrak{S})\cap\triangle(\mathfrak{S}))\neq 0$.

$(ii)\Longrightarrow(v)$. Suppose that

$$\Gamma_1(\upsilon)\circ\Gamma_2(\upsilon^*)+\triangle(\upsilon\circ\upsilon^*)\in Z(\mathfrak{S})\quad\forall\upsilon\in\mathfrak{S}.\tag{6}$$

Replacing υ by $\upsilon+\omega$ in (6) and using it, we obtain

$$\Gamma_1(\upsilon)\circ\Gamma_2(\omega^*)+\Gamma_1(\omega)\circ\Gamma_2(\upsilon^*)+\triangle(\upsilon\circ\omega^*)+\triangle(\omega\circ\upsilon^*)\in Z(\mathfrak{S})\quad\forall\upsilon,\omega\in\mathfrak{S}.\tag{7}$$

Replacing ω by $\omega\tau$ in (7) where $\tau\in Z(\mathfrak{S})\cap\triangle(\mathfrak{S})$ and we using it again, we get

$$(\Gamma_1(\upsilon)\circ\omega^*)\psi_2(\tau)+(\omega,\Gamma_2(\upsilon^*))\psi_1(\tau)\in Z(\mathfrak{S})\quad\forall\upsilon,\omega\in\mathfrak{S}.\tag{8}$$

Replacing υ by $\upsilon\tau$ in (8) where $\tau\in Z(\mathfrak{S})\cap\triangle(\mathfrak{S})$ and we using it again, we have

$$(\upsilon\circ\omega^*)\psi_1(\tau)\psi_2(\tau)+(\omega\circ\upsilon^*)\psi_1(\tau)\psi_2(\tau)\in Z(\mathfrak{S})\quad\forall\upsilon,\omega\in\mathfrak{S}.\tag{9}$$

Taking $\omega=\upsilon$ in (9), we arrive at

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$$2(\upsilon \circ \upsilon^*)\psi_1(\tau)\psi_2(\tau) \in Z(\mathfrak{S}) \quad \forall \upsilon \in \mathfrak{S}. \tag{10}$$

Since $char(\mathfrak{S}) \neq 2$, and the primeness of $\mathfrak{S}$, then $\upsilon \circ \upsilon^* \in Z(\mathfrak{S})$ or $\psi_1(\tau)\psi_2(\tau) = 0$, by Lemme 1 in case $\upsilon \circ \upsilon^* \in Z(\mathfrak{S})$, then $\mathfrak{S}$ is an integral domain or $\psi_1(\tau) = 0$ or $\psi_2(\tau) = 0 \;\; \forall \tau \in Z(\mathfrak{S})$. Continue using the same technique, using the first case to arrive at $\mathfrak{S}$ is an integral domain.

Using the same technique, we discover that the other identities lead to $\mathfrak{S}$ being an integral domain.

It's obvious that $\pm I_{\mathfrak{S}}$ and $0_{\mathfrak{S}}$ are multipliers of $\mathfrak{S}$. Therefore, the theorem 1 leads to the following corollary:.

Corollary 1 *The following statements are equivalent:*

(i) $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \;\; \forall \; \upsilon \in \mathfrak{S}$. (*In particular,* $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] \pm [\upsilon, \upsilon^*] = 0 \;\; \forall \; \upsilon \in \mathfrak{S}$.)
(ii) $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) \pm \upsilon \circ \upsilon^* \in Z(\mathfrak{S}) \;\; \forall \; \upsilon \in \mathfrak{S}$. (In particular, $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) \pm \upsilon \circ \upsilon^* = 0 \;\; \forall \; \upsilon \in \mathfrak{S}$.)
(iii) $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] \pm [\upsilon \circ \upsilon^*] \in Z(\mathfrak{S}) \;\; \forall \; \upsilon \in \mathfrak{S}$. (In particular, $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] \pm [\upsilon \circ \upsilon^*] = 0 \;\; \forall \; \upsilon \in \mathfrak{S}$.)
(iv) $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \;\; \forall \; \upsilon \in \mathfrak{S}$. (In particular, $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) \pm [\upsilon, \upsilon^*] = 0 \;\; \forall \; \upsilon \in \mathfrak{S}$.)
(v) $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] \in Z(\mathfrak{S}) \;\; \forall \; \upsilon \in \mathfrak{S}$. (In particular, $[\Gamma_1(\upsilon), \Gamma_2(\upsilon^*)] = 0 \;\; \forall \; \upsilon \in \mathfrak{S}$.)
(vi) $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) \in Z(\mathfrak{S}) \;\; \forall \; \upsilon \in \mathfrak{S}$. (In particular, $\Gamma_1(\upsilon) \circ \Gamma_2(\upsilon^*) = 0 \;\; \forall \; \upsilon \in \mathfrak{S}$.)
(vii) $\mathfrak{S}$ is an integral domain.

A derivation ψ is classified as a generalized derivation. By setting $\Gamma = \psi$ and $\triangle = \pm I_{\mathfrak{S}}$ in the preceding theorem, we arrive at the following corollary, which holds significant importance in the work of [[8]].

Corollary 2 [[8] *Theorems* 3.5 *and* 3.8] *Let $\mathfrak{S}$ be a prime ring with involution. If $\mathfrak{S}$ admits a derivation ψ, then the following assertions are equivalents:*

(i) $[\psi(\upsilon), \psi(\upsilon^*)] \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \;\; \forall \;\; \upsilon \in \mathfrak{S}$.
(ii) $\psi(\upsilon) \circ \psi(\upsilon^*) \pm \upsilon \circ \upsilon^* \in Z(\mathfrak{S}) \;\; \forall \;\; \upsilon \in \mathfrak{S}$.
(iii) $[\psi(\upsilon), \psi(\upsilon^*)] \pm [\upsilon \circ \upsilon^*] \in Z(\mathfrak{S}) \;\; \forall \;\; \upsilon \in \mathfrak{S}$.
(iv) $\psi(\upsilon) \circ \psi(\upsilon^*) \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \;\; \forall \;\; \upsilon \in \mathfrak{S}$.
(v) $\mathfrak{S}$ *is an integral domain.*

Theorem 2 *The following statements are equivalent:*

(i) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon)\Gamma_2(\upsilon^*) + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S}) \;\; \forall \;\; \upsilon \in \mathfrak{S}$.
(ii) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon)\Gamma_2(\upsilon^*) + \triangle(\upsilon \circ \upsilon^*) \in Z(\mathfrak{S}) \;\; \forall \;\; \upsilon \in \mathfrak{S}$.
(iii) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon^*)\Gamma_2(\upsilon) + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S}) \;\; \forall \;\; \upsilon \in \mathfrak{S}$.
(iv) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon^*)\Gamma_2(\upsilon) + \triangle(\upsilon \circ \upsilon^*) \in Z(\mathfrak{S}) \;\; \forall \;\; \upsilon \in \mathfrak{S}$.
(v) $\mathfrak{S}$ *is an integral domain.*

Proof. It is sufficient for us to demonstrate that $(i), (ii), (iii)$, and $(iv) \Longrightarrow (v)$.

We begin with $(i) \Longrightarrow (v)$ and suppose that

$$\Gamma_1(\upsilon\upsilon^*) + \Gamma_2(\upsilon)\Gamma_2(\upsilon^*) + \triangle([\upsilon, \upsilon^*]) \in Z(\mathfrak{S}) \;\; \forall \upsilon \in \mathfrak{S}. \tag{11}$$

Replacing υ by $\upsilon + \omega$ in (11) and using it, we obtain

$$\Gamma_1(\upsilon\omega^*) + \Gamma_1(\omega\upsilon^*) + \Gamma_2(\upsilon)\Gamma_2(\omega^*) + \Gamma_2(\omega)\Gamma_2(\upsilon^*) + \triangle([\upsilon, \omega^*]) + \triangle([\omega, \upsilon^*]) \in Z(\mathfrak{S}) \quad \forall \upsilon, \omega \in \mathfrak{S}. \tag{12}$$

Replacing ω by $\omega\tau$ in (12) where $\tau \in Z(\mathfrak{S}) \cap \triangle(\mathfrak{S})$ and using it, we get

$$(\upsilon\omega^* + \omega\upsilon^*)\psi_1(\tau) + \Gamma_2(\upsilon)\omega^*\psi_2(\tau) + \omega\Gamma_2(\upsilon^*)\psi_2(\tau) \in Z(\mathfrak{S}) \quad \forall \upsilon, \omega \in \mathfrak{S}. \tag{13}$$

Replacing υ by $\upsilon\iota$ in (13) where $\iota \in Z(\mathfrak{S}) \cap \triangle(\mathfrak{S})$ and using it, we obtain

$$(\upsilon\omega^* + \omega\upsilon^*)\psi_2(\iota)\psi_2(\tau) \in Z(\mathfrak{S}) \quad \forall \upsilon, \omega \in \mathfrak{S}. \tag{14}$$

Taking $\omega = \upsilon$ in (14), we get

$$2(\upsilon \circ \upsilon^*)\psi_2(\iota)\psi_2(\tau) \in Z(\mathfrak{S}) \quad \forall \upsilon \in \mathfrak{S}. \tag{15}$$

Since $char(\mathfrak{S}) \neq 2$, and by Lemme 1, we get $\upsilon \circ \upsilon^* \in Z(\mathfrak{S})$ or $\psi_2(\iota)\psi_2(\tau) = 0$, by primness that leads to $\mathfrak{S}$ is an integral domain or $\psi_2(\tau) = 0\ \forall \tau \in Z(\mathfrak{S})$, then $\psi_2 = 0$, it's a contradiction. Using the same technique, we discover that the other identities lead to $\mathfrak{S}$ being an integral domain.

Replacing $\triangle$ by $\pm I_{\mathfrak{S}}$ and $0_{\mathfrak{S}}$ in above theorem, we get the following corollary.

Corollary 3 *The following assertions are equivalents:*

(i) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon)\Gamma_2(\upsilon^*) \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(ii) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon)\Gamma_2(\upsilon^*) \pm \upsilon \circ \upsilon^* \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(iii) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon^*)\Gamma_2(\upsilon) \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(iv) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon^*)\Gamma_2(\upsilon) \pm \upsilon \circ \upsilon^* \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(v) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon)\Gamma_2(\upsilon^*) \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(vi) $\Gamma_1(\upsilon\upsilon^*) \pm \Gamma_2(\upsilon^*)\Gamma_2(\upsilon) \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(vii) $\mathfrak{S}$ *is an integral domain.*

Corollary 4 [[6], $Theorems$ 1, 2, 3] *The following assertions are equivalents:*

(i) $\Gamma(\upsilon\upsilon^*) \pm \Gamma(\upsilon)\Gamma(\upsilon^*)] \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(ii) $\Gamma(\upsilon\upsilon^*) \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(iii) $\Gamma(\upsilon\upsilon^*) \pm \upsilon\upsilon^* \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(iv) $\Gamma(\upsilon\upsilon^*) \pm \upsilon^*\upsilon \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(v) $\Gamma(\upsilon)\Gamma(\upsilon^*) \pm [\upsilon, \upsilon^*] \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(vi) $\Gamma(\upsilon)\Gamma(\upsilon^*) \pm \upsilon\upsilon^* \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(vii) $\Gamma(\upsilon)\Gamma(\upsilon^*) \pm \upsilon^*\upsilon \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(viii) $\Gamma(\upsilon\upsilon^*) \pm \Gamma(\upsilon^*)\Gamma(\upsilon) \pm \upsilon \circ \upsilon^* \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(ix) $\Gamma(\upsilon^*)\Gamma(\upsilon) \pm \upsilon \circ \upsilon^* \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(x) $\Gamma(\upsilon^*)\Gamma(\upsilon) \pm \upsilon\upsilon^* \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(xi) $\Gamma(\upsilon^*)\Gamma(\upsilon) \pm \upsilon^*\upsilon \in Z(\mathfrak{S}) \quad \forall \quad \upsilon \in \mathfrak{S}.$
(xii) $\mathfrak{S}$ *is an integral domain*.

The following example shows that the assumption of * is of the second kind in Theorem 1 and the primness of $\mathfrak{S}$ in Theorem 2, are not superfluous.

Example 1 Let $\mathbb{Z}$ be the set of integers.

1. Let us define $\mathfrak{S}$ and $\Gamma, \psi, \triangle, ^* : \mathfrak{S} \to \mathfrak{S}$ as follows:

$$\mathfrak{S} = \left\{ \begin{pmatrix} \upsilon & \omega \\ 0 & z \end{pmatrix} \mid \upsilon, \omega, z \in \mathbb{Z} \right\}, \begin{pmatrix} \upsilon & \omega \\ 0 & z \end{pmatrix}^* = \begin{pmatrix} \upsilon & -\omega \\ 0 & z \end{pmatrix}.$$

Taking

$$\Gamma_1 = \Gamma_2 = \Gamma \quad such \quad \Gamma\begin{pmatrix} \upsilon & \omega \\ 0 & z \end{pmatrix} = \begin{pmatrix} \upsilon & -\omega \\ 0 & z \end{pmatrix}, \psi_1 = \psi_2 = \psi \quad such \quad \psi\begin{pmatrix} \upsilon & \omega \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & -2\omega \\ 0 & 0 \end{pmatrix}$$

$$and \quad \triangle\begin{pmatrix} \upsilon & \omega \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}.$$

It can be confirmed that Γ serves as a generalized derivation linked to a non-zero derivation ψ, where $\triangle$ acts as the left multiplier. Furthermore, for any two elements M and N in $\mathfrak{S}$, the following conditions are consistently satisfied

(i) $[\Gamma(M),\Gamma(M^*)]+\triangle([M,M^*])\in Z(\mathfrak{S}),\ \forall M\in\mathfrak{S}.$
(ii) $\Gamma(M)\circ\Gamma(M^*)+\triangle(M\circ M^*)\in Z(\mathfrak{S}),\ \forall M\in\mathfrak{S}.$
(iii) $[\Gamma(M),\Gamma(M^*)]+\triangle(M\circ M^*)\in Z(\mathfrak{S}),\forall M\in\mathfrak{S}.$
(iv) $\Gamma(M)\circ\Gamma(M^*)+\triangle([M,M^*])\in Z(\mathfrak{S}),\ \forall M\in\mathfrak{S}.$

However, $\mathfrak{S}$ is not an integral domain due to * is an involution of the first kind, not the second kind.

2. Suppose that $T=\mathfrak{S}\times\mathbb{C}$ is a non-prime ring, where $\mathfrak{S}$ is the same as in Part 1 and $\mathbb{C}$ is the ring of complex numbers with conjugate involution $\star$. We define in T the involution τ of the second kind such that $\tau(\upsilon,\omega)=(\upsilon^*,\lambda^\star)$ such that * is the same involution in part 1. We can easily demonstrate that the map G, defined in T by $G(M,N)=(\Gamma(M),0)$, is a generalized derivation links to the derivation $g(M,N)=(\psi(M),0)$, where ψ is the same in part 1. It can be easily verified that:

(i) $G(MM^*)-G(M)G(M^*)+\triangle([M,M^*])\in Z(\mathfrak{S})\quad\forall\quad M\in\mathfrak{S}.$
(ii) $G(MM^*)-G(M)G(M^*)+\triangle(M\circ M^*)\in Z(\mathfrak{S})\quad\forall\quad M\in\mathfrak{S}.$
(iii) $G(MM^*)-G(M^*)G(M)+\triangle([M,M^*])\in Z(\mathfrak{S})\quad\forall\quad M\in\mathfrak{S}.$
(iv) $G(MM^*)-G(M^*)G(M)+\triangle(M\circ M^*)\in Z(\mathfrak{S})\quad\forall\quad M\in\mathfrak{S}.$

However, T is not an integral domain due to it's not prime ring.

Conflicts of Interest: "The author declares that has has no conflicts of interest."

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