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Mathematical Analysis of Some (3+1)-D Models via Rangaig Transform

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Abstract:
In this research, we suggest a hybrid method to solve (3+1)-D PDEs arising in various applications of sciences and engineering. The plausibility of the proposed method is demonstrated by considering the “Rangaig Transform” and the classical “Homotopy Analysis Technique”. Some experimental work has been performed to demonstrate the accuracy and simplicity of the proposed hybrid scheme.

Keywords: The “Rangaig Transform”, HAM, “(3+1)-D Telegraph Equation”, “(3+1)-D Diffusion Equation”, “(3+1)-D Schrodinger Equations”, Test Examples.

1. INTRODUCTION: -

The remaining higher-dimensional partial differential equations (PDEs) are in great demand in the areas of mathematical physics, engineering, and many other applied branches. In general, such equations are derived from some problems that include three spatial and one temporal coordinate, such as fluid dynamics, quantum, electromagnetic, or wave theory. In less trivial (significantly non-linear) situations, what these equations look like can be quite complicated indeed, and getting hold of exact or approximate solutions is relatively difficult. This is achieved using advanced mathematics, which writes these equations in a form that is much more accessible for some math to be performed.

Both the Rangaig Transform and Homotopy Analysis Method (HAM) are very potent in manipulating the solution of (3+1)-dimensional PDEs. The Rangaig Transform is an integral transform that goes beyond the conventional traditional employs of Fourier and Laplace type transforms and can be used to offer a new perspective at linear, nonlinear PDEs putting them in simpler algebraical forms. On the other hand, HAM [15-17] is an analytic method that converges rapidly to the approximate solutions for the nonlinear PDEs, where it can easily deform continuously from a simple problem to another complex problem and control the convergence of the solution.

From the papers cited, as well as from the applications seen in this section, it can be ascertained that both methods are efficacious for nonlinear (3+1)-D PDEs; they could be wave equations, heat equations, fluid dynamic schemes, and others. The use of the concept can be followed from abstract theoretical mathematical physics right through to engineering divisions, including machine learning, as well as industrial divisions, such as financial and computational sciences. Applying the “Rangaig Transform” with the help of HAM can offer exact & approximate sophisticated multi-dimensional global problems. It is possible, and such highly nontrivial, (3+1)-dimensional systems, sample methods can be formulated and employed here, encouraging their possible use in the numerical solution of some actual complex physical partial differential equation.

Integral transforms have significantly evolved over the years, providing powerful methods for solving differential equations. The Rangaig Transform, introduced by Rangaig et al. (2017), is one such innovative tool that simplifies the resolution of partial differential equations ([1], [22]). Similarly, Aboodh's work on the Aboodh Transform offers an effective alternative for integral equations, advancing the computational efficiency of these solutions [2]. The Homotopy Analysis Method (HAM) has also been utilized to solve nonlinear Schrödinger equations, with significant contributions from Alomari, Noorani, and Nazar [3]. Eltayeb and Kilicman further explored the utility of the Sumudu Transform in differential equations, expanding the scope of applicable mathematical problems [4]. On a parallel track, Elzaki's contributions introduced the Elzaki Transform, applied to both linear and nonlinear equations, underscoring the versatility of integral transforms in mathematical modelling ([5], [6], [24]). The Homotopy Analysis Method (HAM) continues to be an effective technique in solving nonlinear problems, as evidenced by the research of Ganjiani [7], Gupta and Kumar [8], and Jafari and Seifi [9]. Furthermore, new developments, such as the Shehu Transform proposed by Maitama and Zhao [18] and generalizations of the Rangaig Transform by Mansour and Kuffi ([19], [20]), have opened new possibilities for addressing complex mathematical challenges.

In this study, the potential of the "Rangaig Transform"-based "Homotopy Analysis Method" (RT-HAM) to solve some examples in (3+1) dimensions PDEs is investigated. In the end, we will scrutinize its possibility of use in various physical problems, comparing it with previous methods that solve versatility and convergence control over various linear and nonlinear complex scenario problems.

The remainder of this paper is organized as follows: Section 2 gives the basic definitions and concepts utilized in the "Rangaig Transform". Within Section 3, the rebuilt resilient homotopy examination system has been mentioned. The "Homotopy Analysis" and "Rangaig Transform Method" for solving (3+1)-D PDEs are described in Section 4. We provide Test problems of (3+1)-D "Telegraph Equation", "Diffusion Equation", and "Klein Gordan Equation" to be solved in section 5. The discussion of a conclusion is in section 6.

2. BASIC CONCEPT OF "RANGAIG TRANSFORM" [19-22]

The following part introduces the fundamental principles and features of another transform, the "Rangaig Transform", which will be introduced in this work. In this study, we introduce the "Rangaig Transform" as a novel transformation for exponentially ordered functions within the set H.

$$H = \{\eta(t)\ \exists N,\ A_1, A_2 > 0, |\eta(t)| > Ne^{A_i|t|}, t \in (-1)^{i-1} \times (-\infty, 0)\} \tag{1}$$

N, the arbitrary constant One can have an infinite or endlessly finite value for the arbitrary constant A_1, A_2. We now introduce a new transformation that may be included in (1) as follows:

$$\mathcal{R}[\eta(t)] = T(\varpi) = \frac{1}{\varpi}\int_{-\infty}^{0} c^{\varpi t}\, h(t)dt, \frac{1}{A_1} \le \varpi \le \frac{1}{A_2}, \tag{2}$$

There is a transformation process known as the "Rangaig Transform". The definition is really a statement that factor ϖ takes the place of the variable t in the function h. On the other hand, it is possible to claim that there is a transition to a description of the function $\eta(t)$ in the Pi-variant space $T(\varpi)$. The following demonstrates the application of the “Rangaig Transform” in obtaining results for a certain type of function [19-22].

The following “Rangaig Transform” is accomplished in some core forms. The general function is:

- $\eta(t) = \mathcal{R}\{\eta(t)\}$

i. $\mathcal{R}\{\eta(t)\} = \mathcal{R}\{1\} = \frac{1}{\varpi^2}$

ii. $\mathcal{R}\{1\} = -\frac{1}{\varpi^3}$

iii. $\mathcal{R}\{t\} = -\frac{1}{\varpi^3}$

iv. $\mathcal{R}\{t^n\, , n \geq 0\} = \frac{(-1)^2 n!}{\varpi^{n+2}}$

v. $\mathcal{R}\{sin(t)\} = -\frac{1}{\varpi(\varpi^2+1)}$

vi. $\mathcal{R}\{cos(t)\} = \frac{1}{(\varpi^2+1)}$

vii. $\mathcal{R}\{t^n\, , n \leq 0\} = \frac{(-1)^{n+1}\Gamma(-n)}{\varpi^n}$

viii. $\mathcal{R}\{e^{at}\} = \frac{1}{\varpi(\varpi+a)}$

ix. $\mathcal{R}\{M(t-a)\} = \frac{1}{\varpi^2} e^{at}$

Theorem 1: - [22] (Transformation of Rangaig Derivatives) If $\eta(t), \eta^1(t), \eta^n(t) \in H,$ then

$$\mathcal{R}[\eta^n(t)] = T(\varpi) = (-1)^n \varpi^n T(\varpi) + (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^{k} \varpi^{n-2-k} \eta^{(k)}(0) \qquad (3)$$

Theorem 2: - [22] (Transformation of Rangaig Integrals), If

$$m^n(t) = \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \cdots \int_0^{t_{n+1}} \eta(\tau)(d\tau)^n \ ,$$

so that $m(\mathrm{t}) \in \mathrm{H}$. Next, we define the Rangaig transform of $m^{\mathrm{n}}(\mathrm{t})$ as follows:

$$\mathcal{R}[m^n(t)] = T_n(\varpi) = \left(\frac{-1}{\varpi}\right)^n T(\varpi)$$

Theorem 3: - [22] The convolution identity's Rangaig transform is provided by:

$$\mathcal{R}[(\eta * \kappa)(t)] = -\varpi T_1(\varpi) T_2(\varpi)\ ,$$

Where $(\eta * \kappa)(t) = \int_0^t \eta\,(t-\tau)\kappa(\tau) d\tau,\ T_1(\varpi)\ and\ T_2(\varpi)$ is, respectively, the Rangaig transform of $\eta(t)$ and $\kappa(t)$.

Theorem 4: - [22] (*Duality relation of R-Transform and L-Transform)* The transformation relation between "Rangaig Transform" $T(\varpi)$ and "Laplace Transform" $F(\varpi)$ of $\eta(t)$ if $\eta(t)$ and $\eta(-t)$ exist over H.

$$T(\varpi) = \frac{1}{\varpi} F(-\varpi)$$

Proposition 1. If $\frac{\partial w(x,t)}{\partial t}$ exist, and we can apply integration by parts, we get the following:

$$\mathcal{R}\left[\frac{\partial w(x,t)}{\partial t}\right] = -\varpi T(x,\varpi) + \frac{1}{\varpi} w(x,0) \tag{4}$$

Proof. To illustrate it, we use the integration by parts and formula (2).

Proposition 2. If we assume that $T(x,\varpi)$ is the "Rangaig Transform" of $\mu(x,t)$, we obtain:

$$\mathcal{R}\left[\frac{\partial w(x,t)}{\partial t}\right] = (-1)^n \varpi^n T(x,\varpi) + (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \varpi^{n-2-k} \frac{\partial^k w(x,0)}{\partial t^k} \tag{5}$$

Proof: - We demonstrate mathematical induction to show that (5) is valid.

Using the formula (5) and assuming $n = 1$, we get:

$$\left[\frac{\partial w(x,t)}{\partial t}\right] = -\varpi T(x,\varpi) + \frac{1}{\varpi} w(x,0) \tag{6}$$

Thus, we observe that the formula holds when $n = 1$ based on (4).

Make the inductive assumption that the formula is valid for n so that

$$\mathcal{R}\left[\frac{\partial w(x,t)}{\partial t}\right] = (-1)^n \varpi^n T(x,\varpi) + (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \varpi^{n-2-k} \frac{\partial^k w(x,0)}{\partial t^k} \tag{7}$$

demonstrate that it remains valid at rank $n+1$. Assume $\frac{\partial^n w(x,t)}{\partial t^n} = v(x,t)$ and according to (4) and (7), we have:

$$= \mathcal{R}\left[\frac{\partial^{n+1} w(x,t)}{\partial t^{n+1}}\right] = \mathcal{R}\left[\frac{\partial v(x,t)}{\partial t}\right]$$

$$= -\varpi \mathcal{R}[v(x,t)] + \frac{1}{\varpi} v(x,0)$$

$$= -\varpi \left[(-1)^n \varpi^n T(x,\varpi) + (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \varpi^{n-2-k} \frac{\partial^k w(x,0)}{\partial t^k}\right] + \frac{1}{\varpi} \frac{\partial^n w(x,t)}{\partial t^n}$$

$$= (-1)^{n+1} \varpi^{n+1} T(x,\varpi) + (-1)^{n+2} \sum_{k=0}^{n-1} (-1)^k \varpi^{n-1-k} \frac{\partial^k w(x,0)}{\partial t^k} + \frac{1}{\varpi} \frac{\partial^n w(x,t)}{\partial t^n}$$

$$= (-1)^{n+1} \varpi^{n+1} T(x,\varpi) + (-1)^{n+2} \sum_{k=0}^{n} (-1)^k \varpi^{n-1-k} \frac{\partial^k w(x,0)}{\partial t^k}$$

Therefore, the formula (5) holds for every $n \geq 1$ according to the principle of mathematical induction.

3. HOMOTOPY ANALYSIS METHOD [13-17]

As a subsequent step, the next nonlinear differential equation was considered.

$$\mathcal{N}[w(\Upsilon, t)] = 0 \qquad (8)$$

When $\mathcal{N}$ is a nonlinear operator, $w(\Upsilon, t)$ is an unknown function, and Υ might be $\{x, y, z\}$. The observed independent variables for time and space are, in that order, the variables x, y, z and t with Liao's invention, the conventional homotopy technique.

$$(1-\wp)\mathcal{L}[\delta(\Upsilon, t\,;\wp) - w_0(\Upsilon, t)] = \wp h H(\Upsilon, t)\mathcal{N}[\delta(\Upsilon, t\,;\wp)] \qquad (9)$$

First, we use $w_0(\Upsilon, t)$ as the initial estimate for the equilibrium concentrations of $w(\Upsilon, t)$. Lastly, h is an added parameter which is not the zero while $\wp$ is the embedding parameter ranging from 0 to 1; $\mathcal{L}$ is an added linear operator; $H(\Upsilon, t) = 1$ is an auxiliary function; $h \neq 0$ is an auxiliary parameter; while the $\delta(\Upsilon, t; \wp)$ is an unknown function.

If $\wp = 0$ & $\wp = 1$, we obtain

$$\delta(\Upsilon, t\,; 0) = w_0(\Upsilon, t)\,,$$

and

$$\delta(\Upsilon, t\,; 1) = w(\Upsilon, t)\,,$$

As a consequence, solution $\delta(\Upsilon, t; \wp)$ shifts from starting guess $w_0(\Upsilon, t)$ to exact result $w(\Upsilon, t)$ as $\wp$ traverses range [0, 1]. When we expand $\delta(\Upsilon, t; \wp)$ in the Taylor series with respect to $\wp$, we obtain

$$\delta(\Upsilon, t; \wp) = w_0(\Upsilon, t) + \sum_{m=1}^{\infty} w_m\,(\Upsilon, t)\wp^m \qquad (10)$$

Where,

$$w_m(\Upsilon, t) = \frac{1}{m!}\,\frac{\partial^m \delta(\Upsilon, t; \wp)}{\partial \wp^m}\bigg|_{\wp=0}$$

If the auxiliary function, auxiliary linear operator, auxiliary parameter h, and initial estimate are all correctly chosen, then the series (10) converges at $\wp = 1$, and we obtain

$$w(\Upsilon, t) = w_0(\Upsilon, t) + \sum_{m=1}^{\infty} w_m\,(\Upsilon, t), \qquad (11)$$

This ought to be an acceptable solution for the initial nonlinear equation. By definition (11) the governing equation might be obtained from the 0-order deformation equation (9).

Explain the vector.

$$\overrightarrow{w_n} = w_0(\Upsilon, t), w_1(\Upsilon, t),\ w_2(\Upsilon, t) \ldots\ldots\ldots\ldots . w_n(\Upsilon, t)$$

Considering the embedding parameter $\wp$, differentiating the 0-order deformation equation (9) m-times. After splitting $m!$ by $\wp = 0$, the equation for the m^{th}-order deformation looks like this:

$$\mathcal{L}[w_m(\Upsilon, t) - \chi_m\, w_{m-1}\,(\Upsilon, t)] = h\mathcal{R}_m[w_{m-1}\,(\Upsilon, t)]$$

Whereas

$$\mathcal{R}_m\,(\overrightarrow{w_{m-1}}) = \frac{1}{m-1\,!}\,\frac{\partial^{m-1}\mathcal{N}\,[\delta(\Upsilon, t\,;\wp)}{\partial\wp^{m-1}}\quad\Bigg|_{\wp\,=\,0}$$

and

$$\chi_m = \begin{cases} 0, & m\ \le 1 \\ 1, & m\ > 1 \end{cases}.$$

4. RANGAIG TRANSFORM BASED HOMOTOPY ANALYSIS METHOD (RT-HAM)

Take the following partial differential equation of 3-D.

$$\mu_{tt}(\Upsilon) = \{\mathcal{L}(\mu) + \mathcal{N}(\mu) + f(\Upsilon)\}. \tag{12}$$

Where $f(\Upsilon)$ refers to the functions of $x, y, z,$ while $\mathcal{L}$ indicates the linear and $\mathcal{N}$ nonlinear parts. respectively. Considering both sides of the “Rangaig Transform”, we obtain

$$\mathcal{R}\{\mu_{tt}(\Upsilon)\} = \mathcal{R}\{\mathcal{L}(\mu) + \mathcal{N}(\mu) + f(\Upsilon)\}. \tag{13}$$

With the help of “Rangaig Transform” and an initial condition, we get

$$\mathcal{R}\{\mu(\Upsilon, t)\} = \frac{1}{w^2}\mu_0(\Upsilon) - \frac{1}{w^3}\frac{\partial\mu_0}{\partial t} + \frac{1}{w^2}\mathcal{R}\{\mathcal{L}(\mu) + \mathcal{N}(\mu) + f(\Upsilon)\}. \tag{14}$$

Taking the nonlinear part as:

$$\mathcal{N}[\delta(\Upsilon, t; \wp)] = \mathcal{R}\{\mu(\Upsilon, t)\} - \left[\frac{1}{w^2}\mu_0(\Upsilon) - \frac{1}{w^3}\frac{\partial\mu_0}{\partial t} + \frac{1}{w^2}\mathcal{R}\{\mathcal{L}(\mu) + \mathcal{N}(\mu) + f(\Upsilon)\}\right].$$

We begin by setting up the zero-order deformation equation; we have

$$(1-\wp)\mathcal{R}\{\delta(\Upsilon, t) - \mu_0(\Upsilon, t)\} = \wp h H(\Upsilon, t)\mathcal{N}[\delta(\Upsilon, t; \wp)]. \tag{15}$$

$H(\Upsilon, t) = 1$ is an auxiliary function, $h \neq 0$ is an auxiliary parameter, $\mathcal{R}$ is an auxiliary linear Rangaig operator.

When $\wp = 0$ & $\wp = 1$, we get,

$$\begin{cases} \delta(\Upsilon, t; 0) = \mu_0(\Upsilon, 0) \\ \delta(\Upsilon, t; 1) = \mu(\Upsilon, t). \end{cases}$$

As a result, we have an equation of deformation of order m.

$$\mathcal{R}\{\mu_m(\Upsilon, t) - \chi_m\mu_{m-1}\,(\Upsilon, t)\} = h\,\mathcal{R}_m\big(\overrightarrow{\mu_{m-1}}(\Upsilon, t)\big). \tag{16}$$

Applying the “Inverse Rangaig Transform” to both sides of Equation (16), we obtain

$$\mu_m(\Upsilon, t) - \chi_m\mu_{m-1}\,(\Upsilon, t) = \mathcal{R}^{-1}\big\{h\,\mathcal{R}_m\,\big(\overrightarrow{\mu_{m-1}}\,(\Upsilon, t)\big)\big\}. \tag{17}$$

From the above Equation we get

$$\mu_1(\Upsilon, t) = -\mathcal{R}^{-1}\big\{\mathcal{R}_1\big(\overrightarrow{\mu_0}(\Upsilon, t)\big)\big\},$$

$$\mu_2(\Upsilon,t)=\mu_1(\Upsilon,t)-\mathcal{R}^{-1}\left\{\mathcal{R}_2\left(\overrightarrow{\mu_1}(\Upsilon,t)\right)\right\},$$
$$\mu_3(\Upsilon,t)=\mu_2(\Upsilon,t)-\mathcal{R}^{-1}\left\{\mathcal{R}_3\left(\overrightarrow{\mu_2}(\Upsilon,t)\right)\right\},$$
$$\vdots$$

Therefore, the solution is:

$$\mu(\Upsilon,t)=\mu_0+\mu_1+\mu_2+\cdots \tag{18}$$

3. TEST EXAMPLES:

Within this segment, we will adopt both the “Rangaig Transform” and the traditional “Homotopy Analysis Technique (HAM)” to solve a couple of examples of semi-analytical solutions for (3+1)-D PDEs including the “(3+1)-D Telegraph Equation”, “(3+1)-D Diffusion Equation”, and “(3+1)-D Klein Gordan Equation”.

Example 1: Consider the (3+1)-D Telegraph Equation of the form:

$$\mu_{tt}-2\pi^2\mu\ =\frac{1}{3}(\mu_{xx}+\mu_{yy}+\mu_{zz}) \tag{19}$$

With initial condition $\mu(\Upsilon,0)=sin\,\pi x\ sin\,\pi y\ sin\,\pi z$, we use $\Upsilon=(x,y,z)$,

The exact solution is:

$$\mu(\Upsilon,t)=e^{\pi t}\ \sin(\pi x)\sin(\pi y)\sin(\pi z)$$

Rephrase the stated issue as follows:

$$\mu_{tt}\ =\frac{1}{3}\left(\mu_{xx}+\mu_{yy}+\mu_{zz}\right)+2\pi^2\mu \tag{20}$$

When the “Rangaig Transform” is carried out to both sides of Equation (20), we get,

$$\mathcal{R}\left[\mu_{tt}\right]=\mathcal{R}\left[\frac{1}{3}(\mu_{xx}+\mu_{yy}+\mu_{zz})+2\pi^2\mu\right]$$

This implies

$$(-1)^2\varpi^2\mathcal{R}[\mu]+(-1)^3\sum_{k=0}^{1}\frac{(-1)^k}{\varpi^k}\frac{\partial^k\mu}{\partial t^k}=\mathcal{R}\left[\frac{1}{3}(\mu_{xx}+\mu_{yy}+\mu_{zz})+2\pi^2\mu\right]$$

This expression can be expressed as:

$$\varpi^2\mathcal{R}[\mu]\ -\left[\frac{1}{\varpi^0}\mu(\Upsilon,0)+\frac{(-1)^1}{\varpi^1}\frac{\partial}{\partial t}\ \mu(\Upsilon,0)\right]=\mathcal{R}\left[\frac{1}{3}(\mu_{xx}+\mu_{yy}+\mu_{zz})+2\pi^2\mu\right]$$

When starting conditions are applied, we get

$$\varpi^2\mathcal{R}[\mu]\ -\left(1-\frac{\pi}{\varpi^1}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z)=\mathcal{R}\left[\frac{1}{3}(\mu_{xx}+\mu_{yy}+\mu_{zz})+2\pi^2\mu\right]$$

That suggests

$$\mathcal{R}[\mu]\ -\left(\frac{1}{\varpi^2}-\frac{\pi}{\varpi^3}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z)=\frac{1}{\varpi^2}\mathcal{R}\left[\frac{1}{3}(\mu_{xx}+\mu_{yy}+\mu_{zz})+2\pi^2\mu\right]$$

Or

$$\mathcal{R}[\mu]=\left(\frac{1}{\varpi^2}-\frac{\pi}{\varpi^3}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z)+\frac{1}{\varpi^2}\mathcal{R}\left[\frac{1}{3}(\mu_{xx}+\mu_{yy}+\mu_{zz})+2\pi^2\mu\right]$$

Here is how we define the nonlinear component:

$$\mathcal{N}[\delta(\Upsilon,t;\wp)]= \mathcal{R}[\mu]-\left(\frac{1}{\varpi^2}-\frac{\pi}{\varpi^3}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z) -\frac{1}{\varpi^2}\mathcal{R}\left[\frac{1}{3}\left(\mu_{xx}+\mu_{yy}+\mu_{zz}\right)+2\pi^2\mu\right] \tag{21}$$

We begin by setting up the 0-order deformation according to the assumption adopted herein, namely, $H(\Upsilon,t)=1$; we have

$$(1-\wp)\mathcal{R}\{\delta(\Upsilon,t)-\mu_0(\Upsilon,t)\}=\wp h\mathcal{N}[\delta(\Upsilon,t;\wp)]$$

When $\wp=0$ & $\wp=1$, we have

$$\begin{cases}\delta(\Upsilon,t;0)=\mu_0(\Upsilon,0)\\ \delta(\Upsilon,t;1)=\mu(\Upsilon,t)\end{cases}$$

So, the m^{th}-order deformation eqn.

$$\mathcal{R}\{\mu_m(\Upsilon,t)-\chi_m\mu_{m-1}(\Upsilon,t)\}=h\mathcal{R}_m(\overrightarrow{\mu_{m-1}}(\Upsilon,t)) \tag{22}$$

When the "Inverse Rangaig Transform" is carried out to both sides of Equation (22), we get,

$$\mu_m(\Upsilon,t)-\chi_m\mu_{m-1}(\Upsilon,t)=\mathcal{R}^{-1}\left\{h\mathcal{R}_m\left(\overrightarrow{\mu_{m-1}}(\Upsilon,t)\right)\right\} \tag{23}$$

With $h=-1$, we can get from Equation (23)

$$\mu_1(\Upsilon,t)=-\mathcal{R}^{-1}\left\{\mathcal{R}_1\left(\overrightarrow{\mu_0}(\Upsilon,t)\right)\right\},$$

$$\mu_2(\Upsilon,t)=\mu_1(\Upsilon,t)-\mathcal{R}^{-1}\left\{\mathcal{R}_2\left(\overrightarrow{\mu_1}(\Upsilon,t)\right)\right\},$$

$$\mu_3(\Upsilon,t)=\mu_2(\Upsilon,t)-\mathcal{R}^{-1}\left\{\mathcal{R}_3\left(\overrightarrow{\mu_2}(\Upsilon,t)\right)\right\},$$

$$\vdots$$

Where

$$\mathcal{R}_1\left(\overrightarrow{\mu_0}(\Upsilon,t)\right)=\left(\frac{\pi}{\varpi^3}-\frac{\pi^2}{\varpi^4}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z),$$

$$\mathcal{R}_2\left(\overrightarrow{\mu_1}(\Upsilon,t)\right)=\left(-\frac{\pi}{\varpi^3}+\frac{\pi^2}{\varpi^4}+\frac{\pi^3}{\varpi^5}-\frac{\pi^4}{\varpi^6}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z),$$

$$\mathcal{R}_3\left(\overrightarrow{\mu_2}(\Upsilon,t)\right)=\left(-\frac{\pi^3}{\varpi^5}+\frac{\pi^4}{\varpi^6}+\frac{\pi^5}{\varpi^7}-\frac{\pi^6}{\varpi^8}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z),$$

$$\vdots$$

Consequently,

$$\mu_1(\Upsilon,t) = \left(\pi t + \frac{(\pi t)^2}{2!}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z),$$

$$\mu_2(\Upsilon,t) = \left(\frac{(\pi t)^3}{3!} + \frac{(\pi t)^4}{4!}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z)$$

$$\mu_3(\Upsilon,t) = \left(\frac{(\pi t)^5}{5!} + \frac{(\pi t)^6}{6!}\right)\sin(\pi x)\sin(\pi y)\sin(\pi z)$$

$$\vdots$$

Consequently, the solution is:

$$\mu(\Upsilon,t) = \mu_0 + \mu_1 + \mu_2 + \cdots$$

Or

$$\mu(\Upsilon,t) = \left\{1 + \pi t + \frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!} + \frac{(\pi t)^4}{4!}\right\}\sin(\pi x)\sin(\pi y)\sin(\pi z)$$

Or

$$\mu(\Upsilon,t) = e^{\pi t}\sin(\pi x)\sin(\pi y)\sin(\pi z)$$

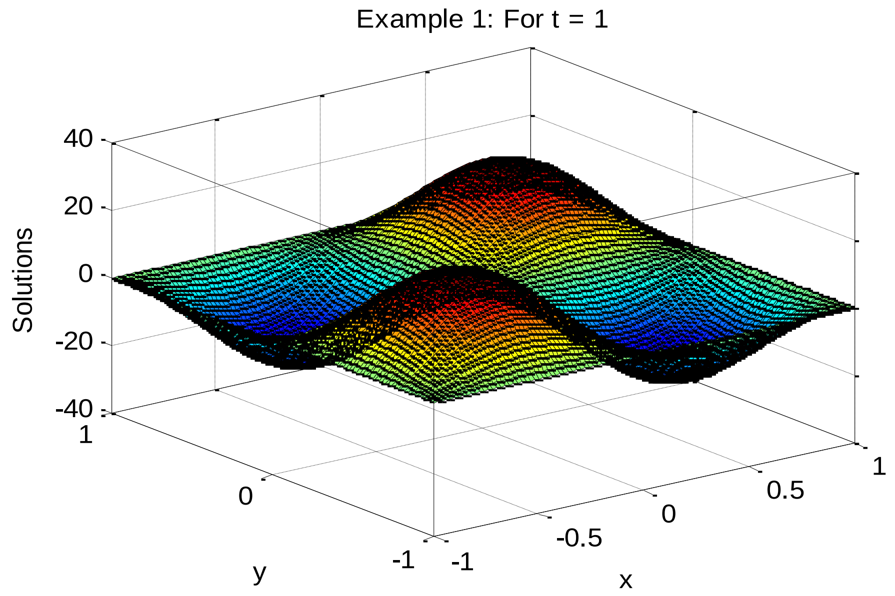

Figure 1: Example 1's solutions' physical behavior at $t = 1,\ z = \frac{1}{2}$

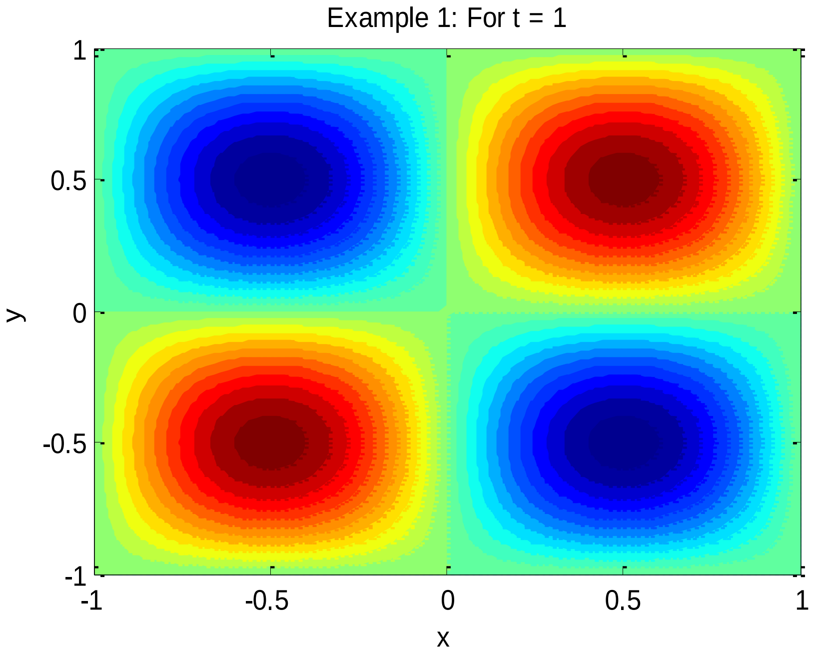

Figure 2: The contour diagram obtained from solving Example 1 at $\boldsymbol{t = 1,\ z = \frac{1}{2}}$

Figures 1 and 2 depict the physical and dynamic behaviour of the solutions found at $z = \frac{1}{2}$ and $t = 1$ using the "Homotopy Analysis Method" based on the "Rangaig Transform."

Example 2: Consider the (3+1)-D Diffusion Equation of the form:

$$\mu_t = \frac{1}{3\pi^2}\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right) \tag{24}$$

With initial condition $\mu(\Upsilon, 0) = e^{-t}\sin\pi x \sin\pi y \sin\pi z$, we use $\Upsilon = (x, y, z)$,

The exact solution is:

$$\mu(\Upsilon, t) = e^{-t}\sin\pi x\ \sin\pi y \sin\pi z$$

Applying the “Rangaig Transform” to both sides of Equation (24), we obtain,

$$\mathcal{R}[\mu_t] = \mathcal{R}\left[\frac{1}{3\pi^2}\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right)\right]$$

This implies

$$(-1)\varpi\,\mathcal{R}[\mu] + \frac{1}{\varpi}\mu\,(\Upsilon, 0) = \mathcal{R}\left[\frac{1}{3\pi^2}\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right)\right].$$

This implies

$$\mathcal{R}[\mu] - \frac{1}{\varpi^2}\mu\,(\Upsilon, 0) = -\frac{1}{\varpi}\mathcal{R}\left[\frac{1}{3\pi^2}\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right)\right]$$

When starting conditions are applied, we get

$$\mathcal{R}[\mu] = \frac{1}{\varpi^2}\sin\pi x\ \sin\pi y \sin\pi z\ - \frac{1}{\varpi}\mathcal{R}\left[\frac{1}{3\pi^2}\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right)\right]$$

Here is how we define the nonlinear component:

$$\mathcal{N}[\delta(\Upsilon, t\,;\wp)] = \mathcal{R}[\mu]\ - \frac{1}{\varpi^2}\sin\pi x\ \sin\pi y\sin\pi z + \frac{1}{\varpi}\mathcal{R}\left[\frac{1}{3\pi^2}\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right)\right] \qquad (25)$$

We begin by setting up the 0-order deformation according to the assumption adopted herein, namely, $H(\Upsilon, t) = 1$; we have

$$(1-\wp)\mathcal{R}\{\delta(\Upsilon, t) - \mu_0(\Upsilon, t)\} = \wp h \mathcal{N}[\delta(\Upsilon, t; \wp)]$$

When $\wp = 0$ & $\wp = 1$, we have

$$\begin{cases}\delta(\Upsilon, t; 0) = \mu_0(\Upsilon, 0)\\ \delta(\Upsilon, t; 1) = \mu(\Upsilon, t)\end{cases}$$

So, the m^{th}-order deformation eqn.

$$\mathcal{R}\{\mu_m(\Upsilon, t) - \chi_m\mu_{m-1}(\Upsilon, t)\} = h\mathcal{R}_m(\overrightarrow{\mu_{m-1}}(\Upsilon, t)) \qquad (26)$$

When the "Inverse Rangaig Transform" is carried out to both sides of Equation (26), we get,

$$\mu_m(\Upsilon, t) - \chi_m\mu_{m-1}(\Upsilon, t) = \mathcal{R}^{-1}\left\{h\mathcal{R}_m\left(\overrightarrow{\mu_{m-1}}(\Upsilon, t)\right)\right\} \qquad (27)$$

With $h = -1$, we can get from Equation (27)

$$\mu_1(\Upsilon, t) = -\mathcal{R}^{-1}\left\{\mathcal{R}_1\left(\overrightarrow{\mu_0}(\Upsilon, t)\right)\right\},$$

$$\mu_2(\Upsilon, t) = \mu_1(\Upsilon, t) - \mathcal{R}^{-1}\left\{\mathcal{R}_2\left(\overrightarrow{\mu_1}(\Upsilon, t)\right)\right\},$$

$$\mu_3(\Upsilon, t) = \mu_2(\Upsilon, t) - \mathcal{R}^{-1}\left\{\mathcal{R}_3\left(\overrightarrow{\mu_2}(\Upsilon, t)\right)\right\},$$

$$\vdots$$

Where

$$\mathcal{R}_1\left(\overrightarrow{\mu_0}(\Upsilon, t)\right) = -\frac{1}{w^3}\sin\pi x\ \sin\pi y\sin\pi z,$$

$$\mathcal{R}_2\left(\overrightarrow{\mu_1}(\Upsilon, t)\right) = \left(\frac{1}{\varpi^3} - \frac{1}{\varpi^4}\right)\sin\pi x\ \sin\pi y\sin\pi z,$$

$$\mathcal{R}_3\left(\overrightarrow{\mu_2}(\Upsilon, t)\right) = \left(\frac{1}{\varpi^4} - \frac{1}{\varpi^5}\right)\sin\pi x\ \sin\pi y\sin\pi z,$$

$$\vdots$$

Therefore,

$$\mu_1(\Upsilon, t) = -\ \sin\pi x\ \sin\pi y\sin\pi z,$$

$$\mu_2(\Upsilon, t) = \frac{t^2}{2!}\ \sin\pi x\ \sin\pi y\sin\pi z,$$

$$\mu_3(\Upsilon, t) = -\frac{t^3}{3!}\ \sin\pi x\ \sin\pi y\sin\pi z,$$

$$\vdots$$

The solution is:

$$\mu(\Upsilon,t)=\mu_0+\mu_1+\mu_2+\cdots$$

Or

$$\mu(\Upsilon,t)=\left\{1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right\}\sin\pi x\ \sin\pi y\sin\pi z=e^{-t}\sin\pi x\ \sin\pi y\sin\pi z$$

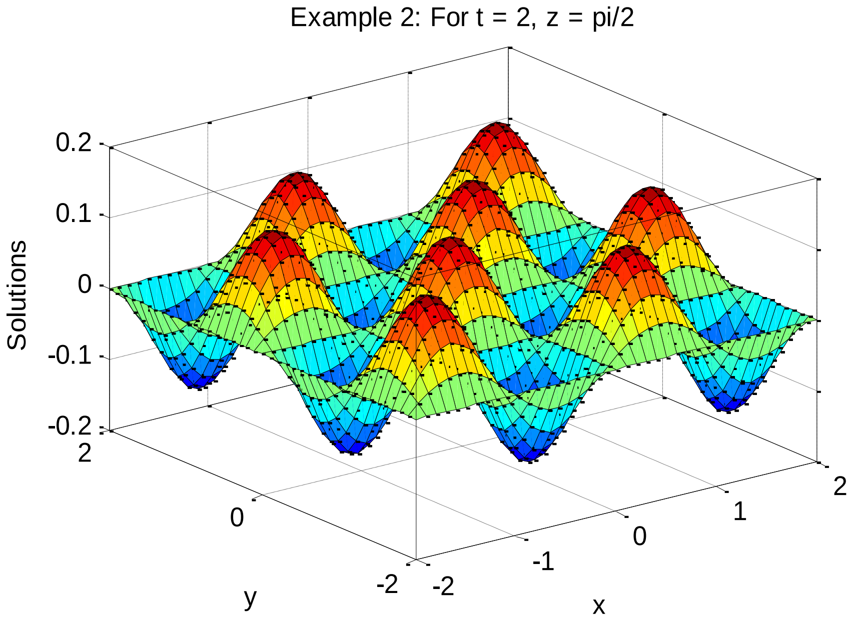

Figure 3: Example 2's solutions' physical behavior at $t=2,\ z=\frac{\pi}{2}$

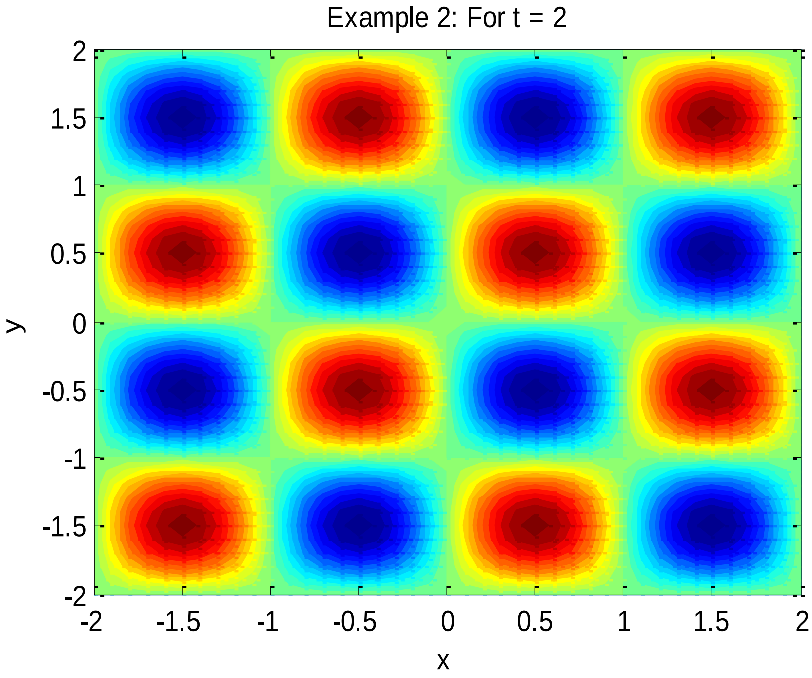

Figure 4: The contour diagram obtained from solving Example 2 at $t=2,\ z=\frac{\pi}{2}$

Figures 3 and 4 depict the physical and dynamic behaviour of the solutions found at $z = 1$ and $t = \frac{\pi}{2}$ using the "Homotopy Analysis Method" based on the "Rangaig transform."

Example 3: Consider the (3+1)-D “Klein Gordan Equation” of the form:

$$\mu_{tt} - \left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right) + \mu = 2(\sin x + \sin y + \sin z) \tag{28}$$

With initial condition ($\mu(\Upsilon, 0) = \sin x + \sin y + \sin z$, we use $\Upsilon = (x, y, z)$,

The exact solution is:

$$\mu(\Upsilon, t) = \sin x + \sin y + \sin z + \sin t$$

Rewrite the given problem as:

$$\mu_{tt} = \left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right) - \mu + 2(\sin x + \sin y + \sin z) \tag{29}$$

Applying the “Rangaig Transform” to both sides of Equation (29), we obtain,

$$\mathcal{R}[\mu_{tt}] = \mathcal{R}\left[\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right) - \mu + 2(\sin x + \sin y + \sin z)\right]$$

This implies

$$(-1)^2 \varpi^2 \mathcal{R}[\mu] + (-1)^3 \sum_{k=0}^{1} \frac{(-1)^k}{\varpi^k} \frac{\partial^k \mu}{\partial t^k} = \mathcal{R}\left[\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right) - \mu + 2\,(\sin x + \sin y + \sin z)\right].$$

This implies

$$\varpi^2 \mathcal{R}[\mu] - \left[\frac{1}{\varpi^0} \mu(\Upsilon, 0) + \frac{(-1)^1}{\varpi^1} \frac{\partial}{\partial t}\, \mu(\Upsilon, 0)\right]$$
$$= \frac{1}{\varpi^2} \mathcal{R}\left[\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right) - \mu + 2\,(\sin x + \sin y + \sin z)\right]$$

Or

$$\mathcal{R}[\mu] - \left[\frac{1}{\varpi^2} \mu(\Upsilon, 0) + \frac{(-1)^1}{\varpi^3} \frac{\partial}{\partial t}\, \mu(\Upsilon, 0)\right]$$
$$= \frac{1}{\varpi^2} \mathcal{R}\left[\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right) - \mu + 2\,(\sin x + \sin y + \sin z)\right]$$

When the initial conditions are applied, we get

$$\mathcal{R}[\mu] = \left(\frac{1}{\varpi^2}\right)(\sin x + \sin y + \sin z) - \frac{1}{\varpi^3}$$
$$+ \frac{1}{\varpi^2} \mathcal{R}\left[\left(\mu_{xx} + \mu_{yy} + \mu_{zz}\right) - \mu + 2(\sin x + \sin y + \sin z)\right]$$

We define the nonlinear component as follows:

$$\mathcal{N}[\delta(\Upsilon, t; p)] = \mathcal{R}[\mu] - \left(\frac{1}{\varpi^2}\right)(\sin x + \sin y + \sin z) + \frac{1}{\varpi^3} \tag{30}$$

$$-\frac{1}{\varpi^2}\mathcal{R}\left[\left(\mu_{xx}+\mu_{yy}+\mu_{zz}\right)-\mu+2(\sin x+\sin y+\sin z)\right]$$

We begin by setting up the 0-order deformation according to the assumption adopted herein, namely, $H(\Upsilon, t) = 1$; we have

$$(1-p)\mathcal{R}\{\delta(\Upsilon,t)-\mu_0(\Upsilon,t)\} = ph\mathcal{N}[\delta(\Upsilon,t;p)]$$

When $p = 0$ & $p = 1$, we have

$$\begin{cases}\delta(\Upsilon,t;0)=\mu_0(\Upsilon,0)\\ \delta(\Upsilon,t;1)=\mu(\Upsilon,t)\end{cases}$$

Thus, the equation for m^{th}-order deformation.

$$\mathcal{R}\{\mu_m(\Upsilon,t)-\chi_m\mu_{m-1}(\Upsilon,t)\} = h\mathcal{R}_m(\overrightarrow{\mu_{m-1}}(\Upsilon,t)) \tag{31}$$

When the “Inverse Rangaig Transform” is carried out to both sides of Equation (31), we get,

$$\mu_m(\Upsilon,t)-\chi_m\mu_{m-1}(\Upsilon,t) = \mathcal{R}^{-1}\left\{h\mathcal{R}_m\left(\overrightarrow{\mu_{m-1}}(\Upsilon,t)\right)\right\} \tag{32}$$

With $h = -1$, we can get from Equation (32)

$$\mu_1(\Upsilon,t) = -\mathcal{R}^{-1}\left\{\mathcal{R}_1\left(\overrightarrow{\mu_0}(\Upsilon,t)\right)\right\},$$

$$\mu_2(\Upsilon,t) = \mu_1(\Upsilon,t)-\mathcal{R}^{-1}\left\{\mathcal{R}_2\left(\overrightarrow{\mu_1}(\Upsilon,t)\right)\right\},$$

$$\mu_3(\Upsilon,t) = \mu_2(\Upsilon,t)-\mathcal{R}^{-1}\left\{\mathcal{R}_3\left(\overrightarrow{\mu_2}(\Upsilon,t)\right)\right\},$$

$$\vdots$$

Where

$$\mathcal{R}_1\left(\overrightarrow{\mu_0}(\Upsilon,t)\right) = \frac{1}{w^3},$$

$$\mathcal{R}_2\left(\overrightarrow{\mu_1}(\Upsilon,t)\right) = \left(-\frac{1}{\varpi^3}-\frac{1}{\varpi^5}\right),$$

$$\mathcal{R}_3\left(\overrightarrow{\mu_2}(\Upsilon,t)\right) = \left(\frac{1}{\varpi^5}+\frac{1}{\varpi^7}\right),$$

$$\vdots$$

Therefore,

$$\mu_1(\Upsilon,t) = t,$$

$$\mu_2(\Upsilon,t) = -\frac{t^3}{3!},$$

$$\mu_3(\Upsilon,t) = \frac{t^5}{5!},$$

$$\vdots$$

The solution is:

$$\mu(\Upsilon, t) = \mu_0 + \mu_1 + \mu_2 + \cdots$$

Or

$$\mu(\Upsilon, t) = \sin x + \sin y + \sin z + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

Or

$$\mu(\Upsilon, t) = \sin x + \sin y + \sin z + \sin t$$

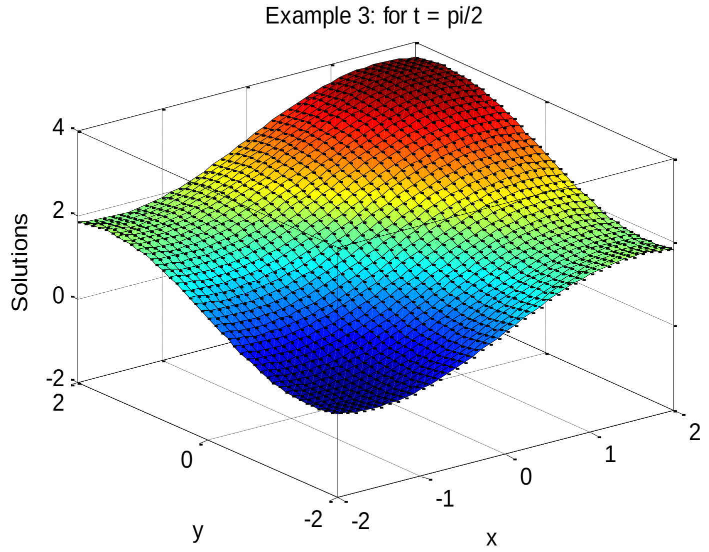

Figure 5: Example 3's solutions' physical behavior at $z = 1,\ t = \frac{\pi}{2}$

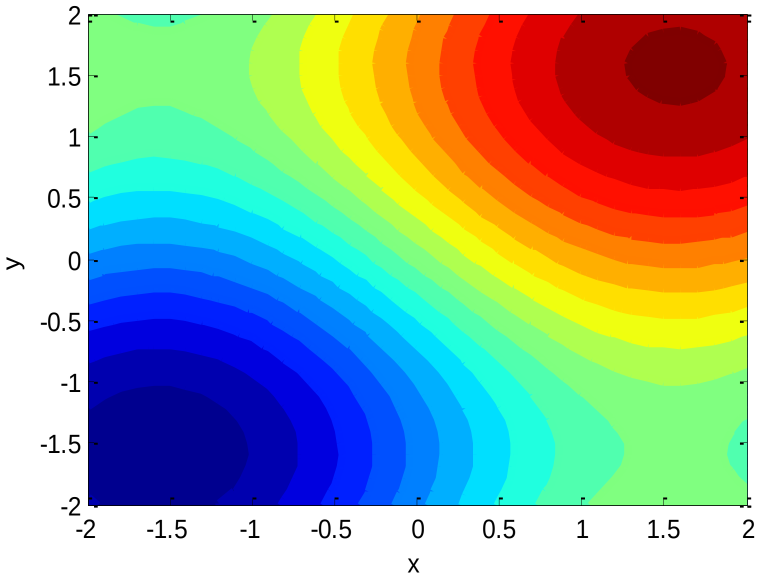

Figure 6: The contour diagram obtained from solving Example 3 at $z = 1, t = \frac{\pi}{2}$

Figures 5 and 6 depict the physical and dynamic behaviour of the solutions found at $z = 1$ and $t = \frac{\pi}{2}$ using the "Homotopy Analysis Method" based on the "Rangaig Transform."

6. CONCLUSION

The computational results established from the above data indicate that the “Rangaig Transforms” with the “Homotopy Analysis Method” is a powerful and simple way to find exact solutions of a few “(3+1)-D Telegraph Equations”, “(3+1)-D Diffusion Equations”, and “(3+1)- D Klein Gordan Equations”. These equations are used in various disciplines of science and engineering. This approach will be valid in the future for fractional PDEs and other problems.

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