

Complex cubic intuitionistic fuzzy set applied to subbisemirings of bisemirings using homomorphism

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Abstract

We develop and analyze the concept of complex cubic intuitionistic fuzzy subbisemiring (ComCIFSBS). We investigate ComCIFSBS its characteristics and homomorphic properties. We suggest the ComCIFSBS level sets of bisemiring. A cubic complex intuitionistic fuzzy set subset Ξ of bisemiring \mathcal{B} , if and only if each non-empty level set $R^{(t,s)}$, where $R = (\widehat{\mu}_Z \cdot e^{i2\pi\beta_Z}, \widehat{\nu}_Z \cdot e^{i2\pi\gamma_Z}, \mu_Z \cdot e^{i2\pi\beta_Z}, \nu_Z \cdot e^{i2\pi\gamma_Z})$ is a ComCIFSBS of \mathcal{B} . Let v be a ComCIFSBS bisemiring \mathcal{B} . If Υ is a ComCIFSBS of $\mathcal{B} \times \mathcal{B}$, then Z is a ComCIFSBS of bisemiring \mathcal{B} . Let Z denote the strongest complex intuitionistic fuzzy relation bisemiring \mathcal{B} . It is proved that all ComCIFSBSs have homomorphic images as well as homomorphic pre-images. Examples are presented to demonstrate how our findings are applied.

Keywords: CCNSBS, CCNNSBS, SBS, homomorphism.

1 Introduction

Fuzzy set (FS) theory was developed by Zadeh¹ and is most effective in managing ambiguity and uncertainty. If an element in an FS has a single value inside the interval, it is regarded as a member. The degree of non-membership may not always equal one minus the degree of membership, though, as resistance can occur in real-life situations. An increasing number of hybrid fuzzy models are being developed as FS theory develops significantly. The uncertainties have led to the development of a number of uncertain theories, such as FS,¹ intuitionistic FS (IFS),² Pythagorean FS (PFS),³ and spherical FS (SFS).⁴ MG sets, or sets with grades between 0 and 1, make up an FS. Despite an assertion made by Atanassov² that non-membership grades (NMG) can only have a value of 1, IFS is categorized as MG. The

total of MGs and NMGs may occasionally exceed 1 throughout a decision-making process. Yager³ used PFS logic to develop the generalized MG and NMG logic, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs. As the neutral state is neither positive nor negative, these theories are unable to describe it. Cuong⁵ et al. communicated to the picture FS used three grading points: positive, neutral, and negative. The sum of these grades could not be greater than 1. It also outperforms PFS and IFS in several situations. It addresses the truth, indeterminacy, and falsity of FS and IFS and is an autonomous generalization of three models.

To handle conflicting and unclear data, Smarandache⁶ developed the neutrosophic set (NS). The degree to which a proposition is true, ambiguous, or false is established using this logic. Ramot et al. introduce the idea of the complex fuzzy set (CFS).⁷ The membership functions of CFS's deals can have a very broad range of values. While the unit circle of a fuzzy membership function remains fixed, the unit circle of the complex plane is expanded to $[0, 1]$.

Rather than extending exclusively to $[0, 1]$, the membership function $x_X(x)$ of the CFS X extends to the unit circle in the complex plane. Hence, $x_X(x)$ is a complex-valued function that, for any element x in the discourse universe, provides a grade of membership of the type $\eta_X(x) \cdot e^{i2\pi x(x)}$, where $i = \sqrt{-1}$. The two real-valued variables, $\eta_X(x)$ and $2\pi x(x)$, where $\eta_X(x), 2\pi x(x) \in [0, 1]$, define the value of $x_X(x)$. Golan⁸ established the concept of semiring logic and its applications. Hussian et al.⁹ talked about the concept and use of bisemirings. Lee¹⁰ addresses bipolar-valued FSs and associated techniques. Fuzzy semirings were studied by Ahsan et al.¹¹ Sen et al.¹² introduced the concept of bisemirings. A fuzzy normal subbisemiring of bisemiring that is intuitionistic was presented by Palanikumar et al.¹³ Bisemiring was first proposed by Palanikumar et al.¹⁴ utilizing bipolar-valued neutrosophic normal sets. Novel concepts including the fuzzy extension set, neutrosophic set, and specific fuzzy set have been the subject of numerous recent writings^{15, 21}. Recently al-husban et al. discussed the new structures such neutrosophic and its applications^{22, 32}.

We will look at a few aspects of SBS and ComCIFSBS concepts and make some inferences. The article is divided into the following five sections. In Section 1, semirings and SBS are introduced. Section 2 contains information on semiring and SBS preparation. ComCIFSBS properties are enumerated in Section 3. It is recommended to use numerical examples to assess ComCIFSBS.

2 Preliminaries

Definition 2.1. An algebraic structure $(\mathcal{B}, \uplus, \ominus, \odot)$ is a bisemiring, if $(\mathcal{B}, \uplus, \ominus)$ and $(\mathcal{B}, \ominus, \odot)$ are semirings, ie., (\mathcal{B}, \uplus) , (\mathcal{B}, \ominus) and (\mathcal{B}, \odot) are semigroups and

1. $\varkappa_v \ominus (\varkappa_\zeta \uplus \varkappa_\eta) = (\varkappa_v \ominus \varkappa_\zeta) \uplus (\varkappa_v \ominus \varkappa_\eta)$,
2. $(\varkappa_\zeta \uplus \varkappa_\eta) \ominus \varkappa_v = (\varkappa_\zeta \ominus \varkappa_v) \uplus (\varkappa_\eta \ominus \varkappa_v)$,
3. $\varkappa_v \odot (\varkappa_\zeta \ominus \varkappa_\eta) = (\varkappa_v \odot \varkappa_\zeta) \ominus (\varkappa_v \odot \varkappa_\eta)$,
4. $(\varkappa_\zeta \ominus \varkappa_\eta) \odot \varkappa_v = (\varkappa_\zeta \odot \varkappa_v) \ominus (\varkappa_\eta \odot \varkappa_v)$, $\forall \varkappa_v, \varkappa_\zeta, \varkappa_\eta \in \mathcal{B}$.

Definition 2.2. A non-empty subset Ξ of a bisemiring $(\mathcal{B}, \uplus, \ominus, \odot)$ is a subbisemiring if $z_v \uplus z_\zeta \in \Xi$, $z_v \ominus z_\zeta \in \Xi$ and $z_v \odot z_\zeta \in \Xi$ for all $z_v, z_\zeta \in \Xi$.

Definition 2.3. Let $(\mathcal{B}, +, \cdot)$ be semiring. A fuzzy subset Ξ of \mathcal{B} is said to be fuzzy sub-bisemiring if

(i) $\pi_{\Xi}(u_1 + u_2) \geq \min\{\pi_{\Xi}(u_1), \pi_{\Xi}(u_2)\}$, (ii) $\pi_{\Xi}(u_1 \cdot u_2) \geq \min\{\pi_{\Xi}(u_1), \pi_{\Xi}(u_2)\}$ for all $u_1, u_2 \in \mathcal{B}$.

Definition 2.4. An intuitionistic fuzzy set Ξ in non-empty set U is defined, the form $\Xi = \{ \langle u, \pi_{\Xi}(u), \omega_{\Xi}(u) | u \in U \rangle \}$ where $\pi_{\Xi} : U \rightarrow [0, 1]$ (define the degree of membership) and $\omega_{\Xi} : U \rightarrow [0, 1]$ (degree of non-membership) for every $u \in U$ satisfying $0 \leq \pi_{\Xi}(u) + \omega_{\Xi}(u) \leq 1$.

Definition 2.5. A NS v in the universe U is $v = \{x, u_v(x), u_v^I(x), u_v^F(x) | x \in U\}$, where $u_v(x), u_v^I(x), u_v^F(x)$ represents the TD, ID and FD of v respectively. Consider the mapping $u_v : U \rightarrow [0, 1], u_v^I : U \rightarrow [0, 1], u_v^F : U \rightarrow [0, 1]$ and $0 \preceq u_v(x) + u_v^I(x) + u_v^F(x) \preceq 3$.

Definition 2.6. Let $\partial_1 = \langle u_{\partial_1}^T, u_{\partial_1}^I, u_{\partial_1}^F \rangle, \partial_2 = \langle u_{\partial_2}^T, u_{\partial_2}^I, u_{\partial_2}^F \rangle$ and $\partial_3 = \langle u_{\partial_3}^T, u_{\partial_3}^I, u_{\partial_3}^F \rangle$ be the three neutrosophic numbers over U . Then

1. $\partial_2 \sqcup \partial_3 = \langle \max(u_{\partial_2}^T, u_{\partial_3}^T), \min(u_{\partial_2}^I, u_{\partial_3}^I), \min(u_{\partial_2}^F, u_{\partial_3}^F) \rangle$,
2. $\partial_2 \sqcap \partial_3 = \langle \min(u_{\partial_2}^T, u_{\partial_3}^T), \max(u_{\partial_2}^I, u_{\partial_3}^I), \max(u_{\partial_2}^F, u_{\partial_3}^F) \rangle$,
3. $\partial_2 \succeq \partial_3$ iff $u_{\partial_2}^T \succeq u_{\partial_3}^T$ and $u_{\partial_2}^I \preceq u_{\partial_3}^I$ and $u_{\partial_2}^F \preceq u_{\partial_3}^F$,
4. $\partial_2 = \partial_3$ iff $u_{\partial_2}^T = u_{\partial_3}^T$ and $u_{\partial_2}^I = u_{\partial_3}^I$ and $u_{\partial_2}^F = u_{\partial_3}^F$.

Definition 2.7. For any NS $\partial = \{x, \chi_v^T(x), \chi_v^I(x), \chi_v^F(x)\}$ of U . Then (t, s) -cut is defined as $\{x \in U | \chi_v^T(x) \succeq t, \chi_v^I(x) \succeq s, \chi_v^F(x) \preceq s\}$.

Definition 2.8. Let V and Y be two NSs of \mathcal{B} . Then Cartesian product of V and Y is defined as $V \times Y = \{\chi_{V \times Y}^T(\chi, \partial), \chi_{V \times Y}^I(\chi, \partial), \chi_{V \times Y}^F(\chi, \partial) | \text{for all } \chi, \partial \in \mathcal{B}\}$, where $\chi_{V \times Y}^T(\chi, \partial) = \min\{\chi_V^T(x), \chi_Y^T(\partial)\}, \chi_{V \times Y}^I(\chi, \partial) = \frac{\chi_V^I(x) + \chi_Y^I(\partial)}{2}, \chi_{V \times Y}^F(\chi, \partial) = \max\{\chi_V^F(x), \chi_Y^F(\partial)\}$.

Definition 2.9. A fuzzy subset v of a bisemiring $(\mathcal{B}, \ddagger_1, \ddagger_2, \ddagger_3)$ is represents a fuzzy sub-bisemiring of \mathcal{B} if $\chi_v(\chi \ddagger_1 \varepsilon) \succeq \min\{\chi_v(x), \chi_v(\varepsilon)\}, \chi_v(\chi \ddagger_2 \varepsilon) \succeq \min\{\chi_v(x), \chi_v(\varepsilon)\}, \chi_v(\chi \ddagger_3 \varepsilon) \succeq \min\{\chi_v(x), \chi_v(\varepsilon)\}$, for all $\chi, \varepsilon \in \mathcal{B}$.

3 Complex cubic intuitionistic fuzzy subbisemiring

Here \mathcal{B} denotes bisemiring unless other stated.

Definition 3.1. The complex cubic IFS (ComCIFS) Z in universal set \mathcal{D} , $Z = \{\hbar, \widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)} : \hbar \in \mathcal{D}\}$, where $\widehat{\mu}_Z(\hbar) = [\mu_Z^L, \mu_Z^U], \widehat{\nu}_Z(\hbar) = [\nu_Z^L, \nu_Z^U]$ and $\mu, \nu : \mathcal{D} \rightarrow D[0, 1]$, also $\mu, \nu : \mathcal{D} \rightarrow [0, 1]$ represents the truth degree and false degree respectively. For ComCIF number $Z = \{\hbar, \widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)} : \hbar \in \mathcal{D}\}$.

Definition 3.2. Let $Z = \left\{ \widehat{h}, \widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \nu_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})} \right\}$

and $\mathbb{k} = \left\{ \widehat{h}, \widehat{\mu}_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\beta_{\mathbb{k}}(\widehat{h})}, \widehat{\nu}_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\widehat{h})}, \mu_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\beta_{\mathbb{k}}(\widehat{h})}, \nu_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\widehat{h})} \right\}$ be two ComCIFNs of \mathcal{D} . Then we define the intersection and union operation is defined as

(i) $Z \sqcap \mathbb{k} = \left\{ \left(\widehat{h}, \min\{\widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \widehat{\mu}_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\beta_{\mathbb{k}}(\widehat{h})}\}, \max\{\widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \widehat{\nu}_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\widehat{h})}\}, \min\{\mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \mu_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\beta_{\mathbb{k}}(\widehat{h})}\}, \max\{\nu_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \nu_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\widehat{h})}\} \right\} \mid \widehat{h} \in \mathcal{D}$.

(ii) $Z \sqcup \mathbb{k} = \left\{ \left(\widehat{h}, \max\{\widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \widehat{\mu}_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\beta_{\mathbb{k}}(\widehat{h})}\}, \min\{\widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \widehat{\nu}_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\widehat{h})}\}, \max\{\mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \mu_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\beta_{\mathbb{k}}(\widehat{h})}\}, \min\{\nu_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \nu_{\mathbb{k}}(\widehat{h}) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\widehat{h})}\} \right\} \mid \widehat{h} \in \mathcal{D}$.

Definition 3.3. For any ComCIF $Z = \left\{ \widehat{h}, \widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \nu_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})} \right\}$ of a universal set \mathcal{D} . Then (t, s) -cut is defined as $\left\{ \widehat{h} \in \mathcal{D} \mid \widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})} \succeq t, \widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})} \preceq s, \mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})} \succeq t, \nu_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})} \preceq s \right\}$.

Definition 3.4. The Cartesian product of Z and \mathbb{k} is defined as

$Z \times \mathbb{k} = \left\{ \widehat{\mu}_{Z \times \mathbb{k}}((\widehat{h}, \widehat{\delta})) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta})}, \widehat{\nu}_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta}) \cdot e^{i2\pi\gamma_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta})}, \mu_{Z \times \mathbb{k}}((\widehat{h}, \widehat{\delta})) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta})}, \nu_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta}) \cdot e^{i2\pi\gamma_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta})} \mid \text{for all } \widehat{h}, \widehat{\delta} \in S \right\}$, where Z and \mathbb{k} be the ComCIF of \mathcal{D} , where

$$\left\{ \begin{array}{l} \widehat{\mu}_{Z \times \mathbb{k}}((\widehat{h}, \widehat{\delta})) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta})} = \min \left\{ \widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \widehat{\mu}_{\mathbb{k}}(\widehat{\delta}) \cdot e^{i2\pi\beta_{\mathbb{k}}(\widehat{\delta})} \right\} \\ \widehat{\nu}_{Z \times \mathbb{k}}((\widehat{h}, \widehat{\delta})) \cdot e^{i2\pi\gamma_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta})} = \max \left\{ \widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \widehat{\nu}_{\mathbb{k}}(\widehat{\delta}) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\widehat{\delta})} \right\} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \mu_{Z \times \mathbb{k}}((\widehat{h}, \widehat{\delta})) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta})} = \min \left\{ \mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \mu_{\mathbb{k}}(\widehat{\delta}) \cdot e^{i2\pi\beta_{\mathbb{k}}(\widehat{\delta})} \right\} \\ \nu_{Z \times \mathbb{k}}((\widehat{h}, \widehat{\delta})) \cdot e^{i2\pi\gamma_{Z \times \mathbb{k}}(\widehat{h}, \widehat{\delta})} = \max \left\{ \nu_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \nu_{\mathbb{k}}(\widehat{\delta}) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\widehat{\delta})} \right\} \end{array} \right\}$$

Definition 3.5. For any ComCIF Z of \mathcal{B} is said to be a ComCIFSBS of \mathcal{B} if

$$\left\{ \begin{array}{l} \widehat{\mu}_Z((\widehat{h} \circ_1 \widehat{\delta})) \cdot e^{i2\pi\beta_Z((\widehat{h} \circ_1 \widehat{\delta}))} \succeq \min\{\widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \widehat{\mu}_Z(\widehat{\delta}) \cdot e^{i2\pi\beta_Z(\widehat{\delta})}\} \\ \widehat{\mu}_Z((\widehat{h} \circ_2 \widehat{\delta})) \cdot e^{i2\pi\beta_Z((\widehat{h} \circ_2 \widehat{\delta}))} \succeq \min\{\widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \widehat{\mu}_Z(\widehat{\delta}) \cdot e^{i2\pi\beta_Z(\widehat{\delta})}\} \\ \widehat{\mu}_Z((\widehat{h} \circ_3 \widehat{\delta})) \cdot e^{i2\pi\beta_Z((\widehat{h} \circ_3 \widehat{\delta}))} \succeq \min\{\widehat{\mu}_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \widehat{\mu}_Z(\widehat{\delta}) \cdot e^{i2\pi\beta_Z(\widehat{\delta})}\} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \widehat{\nu}_Z((\widehat{h} \circ_1 \widehat{\delta})) \cdot e^{i2\pi\gamma_Z((\widehat{h} \circ_1 \widehat{\delta}))} \preceq \max\{\widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \widehat{\nu}_Z(\widehat{\delta}) \cdot e^{i2\pi\gamma_Z(\widehat{\delta})}\} \\ \widehat{\nu}_Z((\widehat{h} \circ_2 \widehat{\delta})) \cdot e^{i2\pi\gamma_Z((\widehat{h} \circ_2 \widehat{\delta}))} \preceq \max\{\widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \widehat{\nu}_Z(\widehat{\delta}) \cdot e^{i2\pi\gamma_Z(\widehat{\delta})}\} \\ \widehat{\nu}_Z((\widehat{h} \circ_3 \widehat{\delta})) \cdot e^{i2\pi\gamma_Z((\widehat{h} \circ_3 \widehat{\delta}))} \preceq \max\{\widehat{\nu}_Z(\widehat{h}) \cdot e^{i2\pi\gamma_Z(\widehat{h})}, \widehat{\nu}_Z(\widehat{\delta}) \cdot e^{i2\pi\gamma_Z(\widehat{\delta})}\} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \mu_Z((\widehat{h} \circ_1 \widehat{\delta})) \cdot e^{i2\pi\beta_Z((\widehat{h} \circ_1 \widehat{\delta}))} \succeq \min\{\mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \mu_Z(\widehat{\delta}) \cdot e^{i2\pi\beta_Z(\widehat{\delta})}\} \\ \mu_Z((\widehat{h} \circ_2 \widehat{\delta})) \cdot e^{i2\pi\beta_Z((\widehat{h} \circ_2 \widehat{\delta}))} \succeq \min\{\mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \mu_Z(\widehat{\delta}) \cdot e^{i2\pi\beta_Z(\widehat{\delta})}\} \\ \mu_Z((\widehat{h} \circ_3 \widehat{\delta})) \cdot e^{i2\pi\beta_Z((\widehat{h} \circ_3 \widehat{\delta}))} \succeq \min\{\mu_Z(\widehat{h}) \cdot e^{i2\pi\beta_Z(\widehat{h})}, \mu_Z(\widehat{\delta}) \cdot e^{i2\pi\beta_Z(\widehat{\delta})}\} \end{array} \right\}$$

$$\left\{ \begin{aligned} \nu_Z((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\gamma_Z((\hbar \circledast_1 \delta))} &\preceq \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\delta) \cdot e^{i2\pi\gamma_Z(\delta)}\} \\ \nu_Z((\hbar \circledast_2 \delta)) \cdot e^{i2\pi\gamma_Z((\hbar \circledast_2 \delta))} &\preceq \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\delta) \cdot e^{i2\pi\gamma_Z(\delta)}\} \\ \nu_Z((\hbar \circledast_3 \delta)) \cdot e^{i2\pi\gamma_Z((\hbar \circledast_3 \delta))} &\preceq \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\delta) \cdot e^{i2\pi\gamma_Z(\delta)}\} \end{aligned} \right\}$$

for all $\hbar, \delta \in \mathcal{B}$.

Example 3.6. Consider the bisemiring $\mathcal{B} = \{x_1, x_2, x_3, x_4\}$ with the Cayley table:

\circledast_1	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_1	x_2
x_3	x_1	x_1	x_3	x_3
x_4	x_1	x_2	x_3	x_4

\circledast_2	x_1	x_2	x_3	x_4
x_1	x_1	x_2	x_3	x_4
x_2	x_2	x_2	x_4	x_4
x_3	x_3	x_4	x_3	x_4
x_4	x_4	x_4	x_4	x_4

\circledast_3	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_3	x_4
x_3	x_4	x_4	x_4	x_4
x_4	x_4	x_4	x_4	x_4

	$(w) = x_1$	$(w) = x_2$
$(\widehat{\mu}_Z, \widehat{\beta}_Z)(w)$	$[0.45e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.55)}]$	$[0.35e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.45)}]$
$(\widehat{\nu}_Z, \widehat{\gamma}_Z)(w)$	$[0.4e^{i2\pi(0.3)}, 0.45e^{i2\pi(0.35)}]$	$[0.45e^{i2\pi(0.45)}, 0.5e^{i2\pi(0.55)}]$

	$(w) = x_3$	$(w) = x_4$
$(\widehat{\mu}_Z, \widehat{\beta}_Z)(w)$	$[0.25e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.35)}]$	$[0.3e^{i2\pi(0.35)}, 0.35e^{i2\pi(0.4)}]$
$(\widehat{\nu}_Z, \widehat{\gamma}_Z)(w)$	$[0.6e^{i2\pi(0.55)}, 0.65e^{i2\pi(0.65)}]$	$[0.55e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.6)}]$

	$(w) = x_1$	$(w) = x_2$
$(\mu_Z, \beta_Z)(w)$	$0.5e^{i2\pi(0.55)}$	$0.45e^{i2\pi(0.5)}$
$(\nu_Z, \gamma_Z)(w)$	$0.45e^{i2\pi(0.4)}$	$0.5e^{i2\pi(0.45)}$

	$(w) = x_3$	$(w) = x_4$
$(\mu_Z, \beta_Z)(w)$	$0.3e^{i2\pi(0.45)}$	$0.35e^{i2\pi(0.4)}$
$(\nu_Z, \gamma_Z)(w)$	$0.6e^{i2\pi(0.55)}$	$0.55e^{i2\pi(0.5)}$

Hence, Z is a ComCIFSBS of \mathcal{B} .

Theorem 3.7. The intersection of a every ComCIFSBSs is again a ComCIFSBS of \mathcal{B} .

Proof. Let $\{\widehat{\Upsilon}_i : i \in I\}$ be the family of ComCIFSBSs of \mathcal{B} and $Z = \bigcap_{i \in I} \widehat{\Upsilon}_i$. Let $\hbar, \delta \in \mathcal{B}$.

Now,

$$\begin{aligned} \widehat{\mu}_Z((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar \circledast_1 \delta))} &= \inf_{i \in I} \widehat{\mu}_{\Upsilon_i}((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar \circledast_1 \delta))} \\ &\succeq \inf_{i \in I} \min\{\widehat{\mu}_{\Upsilon_i}(\hbar) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon_i}(\hbar)}, \widehat{\mu}_{\Upsilon_i}(\delta) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon_i}(\delta)}\} \\ &= \min\left\{ \inf_{i \in I} \widehat{\mu}_{\Upsilon_i}(\hbar) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon_i}(\hbar)}, \inf_{i \in I} \widehat{\mu}_{\Upsilon_i}(\delta) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon_i}(\delta)} \right\} \\ &= \min\{\widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar)}, \widehat{\mu}_Z(\delta) \cdot e^{i2\pi\widehat{\beta}_Z(\delta)}\} \end{aligned}$$

Similarly,

$$\begin{aligned} \widehat{\mu}_Z((\hbar \circledast_2 \delta)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar \circledast_2 \delta))} &\succeq \min\{\widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar)}, \widehat{\mu}_Z(\delta) \cdot e^{i2\pi\widehat{\beta}_Z(\delta)}\}, \\ \widehat{\mu}_Z((\hbar \circledast_3 \delta)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar \circledast_3 \delta))} &\succeq \min\{\widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar)}, \widehat{\mu}_Z(\delta) \cdot e^{i2\pi\widehat{\beta}_Z(\delta)}\}. \end{aligned}$$

Now,

$$\begin{aligned} \widehat{\nu}_Z((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\widehat{\gamma}_Z((\hbar \circledast_1 \delta))} &= \sup_{i \in I} \widehat{\nu}_{\Upsilon_i}((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\widehat{\gamma}_Z((\hbar \circledast_1 \delta))} \\ &\preceq \sup_{i \in I} \max\{\widehat{\nu}_{\Upsilon_i}(\hbar) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon_i}(\hbar)}, \widehat{\nu}_{\Upsilon_i}(\delta) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon_i}(\delta)}\} \\ &= \max\left\{ \sup_{i \in I} \widehat{\nu}_{\Upsilon_i}(\hbar) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon_i}(\hbar)}, \sup_{i \in I} \widehat{\nu}_{\Upsilon_i}(\delta) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon_i}(\delta)} \right\} \\ &= \max\{\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)}, \widehat{\nu}_Z(\delta) \cdot e^{i2\pi\widehat{\gamma}_Z(\delta)}\} \end{aligned}$$

Similarly,

$$\begin{aligned} \widehat{\nu}_Z((\hbar \circledast_2 \delta)) \cdot e^{i2\pi\widehat{\gamma}_Z((\hbar \circledast_2 \delta))} &\preceq \max\{\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)}, \widehat{\nu}_Z(\delta) \cdot e^{i2\pi\widehat{\gamma}_Z(\delta)}\} \text{ and} \\ \widehat{\nu}_Z((\hbar \circledast_3 \delta)) \cdot e^{i2\pi\widehat{\gamma}_Z((\hbar \circledast_3 \delta))} &\preceq \max\{\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)}, \widehat{\nu}_Z(\delta) \cdot e^{i2\pi\widehat{\gamma}_Z(\delta)}\}. \end{aligned}$$

Let $\{\Upsilon_i : i \in I\}$ be the family of ComCIFSBSs of \mathcal{B} and $Z = \bigcap_{i \in I} \Upsilon_i$. Let $\hbar, \delta \in \mathcal{B}$.

Now,

$$\begin{aligned} \mu_Z((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\beta_Z((\hbar \circledast_1 \delta))} &= \inf_{i \in I} \mu_{\Upsilon_i}((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\beta_Z((\hbar \circledast_1 \delta))} \\ &\succeq \inf_{i \in I} \min\{\mu_{\Upsilon_i}(\hbar) \cdot e^{i2\pi\beta_{\Upsilon_i}(\hbar)}, \mu_{\Upsilon_i}(\delta) \cdot e^{i2\pi\beta_{\Upsilon_i}(\delta)}\} \\ &= \min\left\{ \inf_{i \in I} \mu_{\Upsilon_i}(\hbar) \cdot e^{i2\pi\beta_{\Upsilon_i}(\hbar)}, \inf_{i \in I} \mu_{\Upsilon_i}(\delta) \cdot e^{i2\pi\beta_{\Upsilon_i}(\delta)} \right\} \\ &= \min\{\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \mu_Z(\delta) \cdot e^{i2\pi\beta_Z(\delta)}\} \end{aligned}$$

Similarly,

$$\begin{aligned} \mu_Z((\hbar \circledast_2 \delta)) \cdot e^{i2\pi\beta_Z((\hbar \circledast_2 \delta))} &\succeq \min\{\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \mu_Z(\delta) \cdot e^{i2\pi\beta_Z(\delta)}\}, \\ \mu_Z((\hbar \circledast_3 \delta)) \cdot e^{i2\pi\beta_Z((\hbar \circledast_3 \delta))} &\succeq \min\{\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \mu_Z(\delta) \cdot e^{i2\pi\beta_Z(\delta)}\}. \end{aligned}$$

Now,

$$\begin{aligned} \nu_Z((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\gamma_Z((\hbar \circledast_1 \delta))} &= \sup_{i \in I} \nu_{\Upsilon_i}((\hbar \circledast_1 \delta)) \cdot e^{i2\pi\gamma_Z((\hbar \circledast_1 \delta))} \\ &\preceq \sup_{i \in I} \max\{\nu_{\Upsilon_i}(\hbar) \cdot e^{i2\pi\gamma_{\Upsilon_i}(\hbar)}, \nu_{\Upsilon_i}(\delta) \cdot e^{i2\pi\gamma_{\Upsilon_i}(\delta)}\} \\ &= \max\left\{ \sup_{i \in I} \nu_{\Upsilon_i}(\hbar) \cdot e^{i2\pi\gamma_{\Upsilon_i}(\hbar)}, \sup_{i \in I} \nu_{\Upsilon_i}(\delta) \cdot e^{i2\pi\gamma_{\Upsilon_i}(\delta)} \right\} \\ &= \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\delta) \cdot e^{i2\pi\gamma_Z(\delta)}\} \end{aligned}$$

Similarly,

$$\begin{aligned} \nu_Z((\hbar \circledast_2 \delta)) \cdot e^{i2\pi\gamma_Z((\hbar \circledast_2 \delta))} &\preceq \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\delta) \cdot e^{i2\pi\gamma_Z(\delta)}\} \text{ and} \\ \nu_Z((\hbar \circledast_3 \delta)) \cdot e^{i2\pi\gamma_Z((\hbar \circledast_3 \delta))} &\preceq \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\delta) \cdot e^{i2\pi\gamma_Z(\delta)}\}. \end{aligned}$$

Thus, Z is a ComCIFSBS of \mathcal{B} .

Theorem 3.8. If Z and \mathbb{k} be the ComCIFSBSs of \mathcal{B}_1 and \mathcal{B}_2 respectively, then $\widehat{Z \times \mathbb{k}}$ is a ComCIFSBS of $\mathcal{B}_1 \times \mathcal{B}_2$.

Proof. Let $\hbar_1, \hbar_2 \in \mathcal{B}_1$ and $\eth_1, \eth_2 \in \mathcal{B}_2$. Then (\hbar_1, \eth_1) and (\hbar_2, \eth_2) are in $\mathcal{B}_1 \times \mathcal{B}_2$. Now

$$\begin{aligned} & \widehat{\mu_{Z \times k}}[((\hbar_1, \eth_1) \circ_1 (\hbar_2, \eth_2))] \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1) \circ_1 (\hbar_2, \eth_2))} \\ = & \widehat{\mu_{Z \times k}}((\hbar_1 \circ_1 \hbar_2, \eth_1 \circ_1 \eth_2)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1 \circ_1 \hbar_2, \eth_1 \circ_1 \eth_2))} \\ = & \min\{\widehat{\mu_Z}((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\widehat{\beta_Z}((\hbar_1 \circ_1 \hbar_2))}, \widehat{\mu_k}((\eth_1 \circ_1 \eth_2)) \cdot e^{i2\pi\widehat{\beta_k}((\eth_1 \circ_1 \eth_2))}\} \\ \succeq & \min\{\min\{\widehat{\mu_Z}(\hbar_1) \cdot e^{i2\pi\widehat{\beta_Z}(\hbar_1)}, \widehat{\mu_Z}(\hbar_2) \cdot e^{i2\pi\widehat{\beta_Z}(\hbar_2)}\}, \min\{\widehat{\mu_k}(\eth_1) \cdot e^{i2\pi\widehat{\beta_Z}(\eth_1)}, \widehat{\mu_k}(\eth_2) \cdot e^{i2\pi\widehat{\beta_Z}(\eth_2)}\}\} \\ = & \min\{\min\{\widehat{\mu_Z}(\hbar_1) \cdot e^{i2\pi\widehat{\beta_Z}(\hbar_1)}, \widehat{\mu_k}(\eth_1) \cdot e^{i2\pi\widehat{\beta_Z}(\eth_1)}\}, \min\{\widehat{\mu_Z}(\hbar_2) \cdot e^{i2\pi\widehat{\beta_Z}(\hbar_2)}, \widehat{\mu_k}(\eth_2) \cdot e^{i2\pi\widehat{\beta_Z}(\eth_2)}\}\} \\ = & \min\{\widehat{\mu_{Z \times k}}((\hbar_1, \eth_1)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1))}, \widehat{\mu_{Z \times k}}((\hbar_2, \eth_2)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_2, \eth_2))}\} \end{aligned}$$

$$\begin{aligned} \text{Also } & \widehat{\mu_{Z \times k}}[((\hbar_1, \eth_1) \circ_2 (\hbar_2, \eth_2))] \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1) \circ_2 (\hbar_2, \eth_2))} \\ \succeq & \min\{\widehat{\mu_{Z \times k}}((\hbar_1, \eth_1)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1))}, \widehat{\mu_{Z \times k}}((\hbar_2, \eth_2)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_2, \eth_2))}\} \end{aligned}$$

$$\begin{aligned} \text{and } & \widehat{\mu_{Z \times k}}[((\hbar_1, \eth_1) \circ_3 (\hbar_2, \eth_2))] \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1) \circ_3 (\hbar_2, \eth_2))} \\ \succeq & \min\{\widehat{\mu_{Z \times k}}((\hbar_1, \eth_1)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1))}, \widehat{\mu_{Z \times k}}((\hbar_2, \eth_2)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_2, \eth_2))}\}. \end{aligned}$$

Now,

$$\begin{aligned} & \widehat{\nu_{Z \times k}}[((\hbar_1, \eth_1) \circ_1 (\hbar_2, \eth_2))] \cdot e^{i2\pi\widehat{\gamma_{Z \times k}}((\hbar_1, \eth_1) \circ_1 (\hbar_2, \eth_2))} \\ = & \widehat{\nu_{Z \times k}}((\hbar_1 \circ_1 \hbar_2, \eth_1 \circ_1 \eth_2)) \cdot e^{i2\pi\widehat{\gamma_{Z \times k}}((\hbar_1 \circ_1 \hbar_2, \eth_1 \circ_1 \eth_2))} \\ = & \max\{\widehat{\nu_Z}((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\widehat{\gamma_Z}((\hbar_1 \circ_1 \hbar_2))}, \widehat{\nu_k}((\eth_1 \circ_1 \eth_2)) \cdot e^{i2\pi\widehat{\gamma_k}((\eth_1 \circ_1 \eth_2))}\} \\ \preceq & \max\{\max\{\widehat{\nu_Z}(\hbar_1) \cdot e^{i2\pi\widehat{\gamma_Z}(\hbar_1)}, \widehat{\nu_Z}(\hbar_2) \cdot e^{i2\pi\widehat{\gamma_Z}(\hbar_2)}\}, \max\{\widehat{\nu_k}(\eth_1) \cdot e^{i2\pi\widehat{\gamma_k}(\eth_1)}, \widehat{\nu_k}(\eth_2) \cdot e^{i2\pi\widehat{\gamma_k}(\eth_2)}\}\} \\ = & \max\{\max\{\widehat{\nu_Z}(\hbar_1) \cdot e^{i2\pi\widehat{\gamma_Z}(\hbar_1)}, \widehat{\nu_k}(\eth_1) \cdot e^{i2\pi\widehat{\gamma_k}(\eth_1)}\}, \max\{\widehat{\nu_Z}(\hbar_2) \cdot e^{i2\pi\widehat{\gamma_Z}(\hbar_2)}, \widehat{\nu_k}(\eth_2) \cdot e^{i2\pi\widehat{\gamma_k}(\eth_2)}\}\} \\ = & \max\{\widehat{\nu_{Z \times k}}((\hbar_1, \eth_1)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1))}, \widehat{\nu_{Z \times k}}((\hbar_2, \eth_2)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_2, \eth_2))}\} \end{aligned}$$

$$\begin{aligned} \text{Also } & \widehat{\nu_{Z \times k}}[((\hbar_1, \eth_1) \circ_2 (\hbar_2, \eth_2))] \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1) \circ_2 (\hbar_2, \eth_2))} \\ \preceq & \max\{\widehat{\nu_{Z \times k}}((\hbar_1, \eth_1)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1))}, \widehat{\nu_{Z \times k}}((\hbar_2, \eth_2)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_2, \eth_2))}\}, \\ & \widehat{\nu_{Z \times k}}[((\hbar_1, \eth_1) \circ_3 (\hbar_2, \eth_2))] \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1) \circ_3 (\hbar_2, \eth_2))} \\ \preceq & \max\{\widehat{\nu_{Z \times k}}((\hbar_1, \eth_1)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_1, \eth_1))}, \widehat{\nu_{Z \times k}}((\hbar_2, \eth_2)) \cdot e^{i2\pi\widehat{\beta_{Z \times k}}((\hbar_2, \eth_2))}\}. \end{aligned}$$

Let $\hbar_1, \hbar_2 \in \mathcal{B}_1$ and $\eth_1, \eth_2 \in \mathcal{B}_2$. Then (\hbar_1, \eth_1) and (\hbar_2, \eth_2) are in $\mathcal{B}_1 \times \mathcal{B}_2$. Now

$$\begin{aligned} & \mu_{Z \times k}[((\hbar_1, \eth_1) \circ_1 (\hbar_2, \eth_2))] \cdot e^{i2\pi\beta_{Z \times k}((\hbar_1, \eth_1) \circ_1 (\hbar_2, \eth_2))} \\ = & \mu_{Z \times k}((\hbar_1 \circ_1 \hbar_2, \eth_1 \circ_1 \eth_2)) \cdot e^{i2\pi\beta_{Z \times k}((\hbar_1 \circ_1 \hbar_2, \eth_1 \circ_1 \eth_2))} \\ = & \min\{\mu_Z((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\beta_Z((\hbar_1 \circ_1 \hbar_2))}, \mu_k((\eth_1 \circ_1 \eth_2)) \cdot e^{i2\pi\beta_k((\eth_1 \circ_1 \eth_2))}\} \\ \succeq & \min\{\min\{\mu_Z(\hbar_1) \cdot e^{i2\pi\beta_Z(\hbar_1)}, \mu_Z(\hbar_2) \cdot e^{i2\pi\beta_Z(\hbar_2)}\}, \min\{\mu_k(\eth_1) \cdot e^{i2\pi\beta_Z(\eth_1)}, \mu_k(\eth_2) \cdot e^{i2\pi\beta_Z(\eth_2)}\}\} \\ = & \min\{\min\{\mu_Z(\hbar_1) \cdot e^{i2\pi\beta_Z(\hbar_1)}, \mu_k(\eth_1) \cdot e^{i2\pi\beta_Z(\eth_1)}\}, \min\{\mu_Z(\hbar_2) \cdot e^{i2\pi\beta_Z(\hbar_2)}, \mu_k(\eth_2) \cdot e^{i2\pi\beta_Z(\eth_2)}\}\} \\ = & \min\{\mu_{Z \times k}((\hbar_1, \eth_1)) \cdot e^{i2\pi\beta_{Z \times k}((\hbar_1, \eth_1))}, \mu_{Z \times k}((\hbar_2, \eth_2)) \cdot e^{i2\pi\beta_{Z \times k}((\hbar_2, \eth_2))}\} \end{aligned}$$

$$\begin{aligned} \text{Also } & \mu_{Z \times k}[((\hbar_1, \eth_1) \circ_2 (\hbar_2, \eth_2))] \cdot e^{i2\pi\beta_{Z \times k}((\hbar_1, \eth_1) \circ_2 (\hbar_2, \eth_2))} \\ \succeq & \min\{\mu_{Z \times k}((\hbar_1, \eth_1)) \cdot e^{i2\pi\beta_{Z \times k}((\hbar_1, \eth_1))}, \mu_{Z \times k}((\hbar_2, \eth_2)) \cdot e^{i2\pi\beta_{Z \times k}((\hbar_2, \eth_2))}\} \end{aligned}$$

and $\mu_{Z \times \mathbb{k}}[((\hbar_1, \check{\delta}_1) \odot_3 (\hbar_2, \check{\delta}_2))] \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1) \odot_3 (\hbar_2, \check{\delta}_2))}$
 $\succeq \min\{\mu_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1)) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1))}, \mu_{Z \times \mathbb{k}}((\hbar_2, \check{\delta}_2)) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_2, \check{\delta}_2))}\}.$

Now,

$$\begin{aligned} & \nu_{Z \times \mathbb{k}}[((\hbar_1, \check{\delta}_1) \odot_1 (\hbar_2, \check{\delta}_2))] \cdot e^{i2\pi\gamma_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1) \odot_1 (\hbar_2, \check{\delta}_2))} \\ &= \nu_{Z \times \mathbb{k}}((\hbar_1 \odot_1 \hbar_2, \check{\delta}_1 \odot_1 \check{\delta}_2)) \cdot e^{i2\pi\gamma_{Z \times \mathbb{k}}((\hbar_1 \odot_1 \hbar_2, \check{\delta}_1 \odot_1 \check{\delta}_2))} \\ &= \max\{\nu_Z((\hbar_1 \odot_1 \hbar_2)) \cdot e^{i2\pi\gamma_{Z \times \mathbb{k}}((\hbar_1 \odot_1 \hbar_2))}, \nu_{\mathbb{k}}((\check{\delta}_1 \odot_1 \check{\delta}_2)) \cdot e^{i2\pi\gamma_{Z \times \mathbb{k}}((\check{\delta}_1 \odot_1 \check{\delta}_2))}\} \\ &\preceq \max\{\max\{\nu_Z(\hbar_1) \cdot e^{i2\pi\gamma_Z(\hbar_1)}, \nu_Z(\hbar_2) \cdot e^{i2\pi\gamma_Z(\hbar_2)}\}, \max\{\nu_{\mathbb{k}}(\check{\delta}_1) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\check{\delta}_1)}, \nu_{\mathbb{k}}(\check{\delta}_2) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\check{\delta}_2)}\}\} \\ &= \max\{\max\{\nu_Z(\hbar_1) \cdot e^{i2\pi\gamma_Z(\hbar_1)}, \nu_{\mathbb{k}}(\check{\delta}_1) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\check{\delta}_1)}\}, \max\{\nu_Z(\hbar_2) \cdot e^{i2\pi\gamma_Z(\hbar_2)}, \nu_{\mathbb{k}}(\check{\delta}_2) \cdot e^{i2\pi\gamma_{\mathbb{k}}(\check{\delta}_2)}\}\} \\ &= \max\{\nu_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1)) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1))}, \nu_{Z \times \mathbb{k}}((\hbar_2, \check{\delta}_2)) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_2, \check{\delta}_2))}\} \end{aligned}$$

Also $\nu_{Z \times \mathbb{k}}[((\hbar_1, \check{\delta}_1) \odot_2 (\hbar_2, \check{\delta}_2))] \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1) \odot_2 (\hbar_2, \check{\delta}_2))}$
 $\preceq \max\{\nu_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1)) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1))}, \nu_{Z \times \mathbb{k}}((\hbar_2, \check{\delta}_2)) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_2, \check{\delta}_2))}\},$
 $\nu_{Z \times \mathbb{k}}[((\hbar_1, \check{\delta}_1) \odot_3 (\hbar_2, \check{\delta}_2))] \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1) \odot_3 (\hbar_2, \check{\delta}_2))}$
 $\preceq \max\{\nu_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1)) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_1, \check{\delta}_1))}, \nu_{Z \times \mathbb{k}}((\hbar_2, \check{\delta}_2)) \cdot e^{i2\pi\beta_{Z \times \mathbb{k}}((\hbar_2, \check{\delta}_2))}\}.$

Thus, $Z \times \mathbb{k}$ is a ComCIFSBS of \mathcal{B} .

Corollary 3.9. *If Z_1, Z_2, \dots, Z_n be the finite collection of ComCIFSBSs of $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n$ respectively. Then $Z_1 \times Z_2 \times \dots \times Z_n$ is a ComCIFSBS of $\mathcal{B}_1 \times \mathcal{B}_2 \times \dots \times \mathcal{B}_n$.*

Definition 3.10. Let $Z \subseteq \mathcal{B}$, the strongest ComCN relation on \mathcal{B} is

$$\left\{ \begin{aligned} \widehat{\mu}_{\Upsilon}((\hbar, \check{\delta})) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}((\hbar, \check{\delta}))} &= \min\{\widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar)}, \widehat{\mu}_Z(\check{\delta}) \cdot e^{i2\pi\widehat{\beta}_Z(\check{\delta})}\} \\ \widehat{\nu}_{\Upsilon}((\hbar, \check{\delta})) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon}((\hbar, \check{\delta}))} &= \max\{\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)}, \widehat{\nu}_Z(\check{\delta}) \cdot e^{i2\pi\widehat{\gamma}_Z(\check{\delta})}\} \end{aligned} \right\}$$

Theorem 3.11. *Let Z be a ComCIFSBS of \mathcal{B} and Υ be a strongest complex cubic neutrosophic relation of \mathcal{B} . Then Z is a ComCIFSBS of $\mathcal{B} \times \mathcal{B}$ if and only if Υ is a ComCIFSBS of $\mathcal{B} \times \mathcal{B}$.*

Proof. Suppose Z is a ComCIFSBS of $\mathcal{B} \times \mathcal{B}$ and $\widehat{\Upsilon}$ be the strongest complex cubic neutrosophic relation of \mathcal{B} .

For any $\hbar = (\hbar_1, \hbar_2), \check{\delta} = (\check{\delta}_1, \check{\delta}_2) \in S \times \mathcal{B}$. Now,

$$\begin{aligned} & \widehat{\mu}_{\Upsilon}((\hbar \odot_1 \check{\delta})) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}((\hbar \odot_1 \check{\delta}))} \\ &= \widehat{\mu}_{\Upsilon}((((\hbar_1, \hbar_2) \odot_1 ((\check{\delta}_1, \check{\delta}_2)))) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}((((\hbar_1, \hbar_2) \odot_1 ((\check{\delta}_1, \check{\delta}_2))))} \\ &= \widehat{\mu}_{\Upsilon}(\hbar_1 \odot_1 \check{\delta}_1, \hbar_2 \odot_1 \check{\delta}_2) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}(\hbar_1 \odot_1 \check{\delta}_1, \hbar_2 \odot_1 \check{\delta}_2)} \\ &= \min\{\widehat{\mu}_Z((\hbar_1 \odot_1 \check{\delta}_1)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar_1 \odot_1 \check{\delta}_1))}, \widehat{\mu}_Z((\hbar_2 \odot_1 \check{\delta}_2)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar_2 \odot_1 \check{\delta}_2))}\} \\ &\succeq \min\{\min\{\widehat{\mu}_Z(\hbar_1) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar_1)}, \widehat{\mu}_Z(\check{\delta}_1) \cdot e^{i2\pi\widehat{\beta}_Z(\check{\delta}_1)}\}, \\ &\quad \min\{\widehat{\mu}_Z(\hbar_2) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar_2)}, \widehat{\mu}_Z(\check{\delta}_2) \cdot e^{i2\pi\widehat{\beta}_Z(\check{\delta}_2)}\}\} \\ &= \min\{\min\{\widehat{\mu}_Z(\hbar_1) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar_1)}, \widehat{\mu}_Z(\hbar_2) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar_2)}\}, \\ &\quad \min\{\widehat{\mu}_Z(\check{\delta}_1) \cdot e^{i2\pi\widehat{\beta}_Z(\check{\delta}_1)}, \widehat{\mu}_Z(\check{\delta}_2) \cdot e^{i2\pi\widehat{\beta}_Z(\check{\delta}_2)}\}\} \\ &= \min\{\widehat{\mu}_{\Upsilon}((\hbar_1, \hbar_2)) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}((\hbar_1, \hbar_2))}, \widehat{\mu}_{\Upsilon}((\check{\delta}_1, \check{\delta}_2)) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}((\check{\delta}_1, \check{\delta}_2))}\} \\ &= \min\{\widehat{\mu}_{\Upsilon}(\hbar) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}(\hbar)}, \widehat{\mu}_{\Upsilon}(\check{\delta}) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}(\check{\delta})}\} \end{aligned}$$

Also $\widehat{\mu}_\Upsilon((\hbar \circledast_2 \bar{\delta})) \cdot e^{i2\pi\widehat{\beta}_\Upsilon((\hbar \circledast_2 \bar{\delta}))} \succeq \min\{\widehat{\mu}_\Upsilon(\hbar) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\hbar)}, \widehat{\mu}_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\bar{\delta})}\}$,
 $\widehat{\mu}_\Upsilon((\hbar \circledast_3 \bar{\delta})) \cdot e^{i2\pi\widehat{\beta}_\Upsilon((\hbar \circledast_3 \bar{\delta}))} \succeq \min\{\widehat{\mu}_\Upsilon(\hbar) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\hbar)}, \widehat{\mu}_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\bar{\delta})}\}.$

Similarly, $\widehat{\nu}_\Upsilon((\hbar \circledast_1 \bar{\delta})) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon((\hbar \circledast_1 \bar{\delta}))} \preceq \max\{\widehat{\nu}_\Upsilon(\hbar) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon(\hbar)}, \widehat{\nu}_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon(\bar{\delta})}\}$,
 $\widehat{\nu}_\Upsilon((\hbar \circledast_2 \bar{\delta})) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon((\hbar \circledast_2 \bar{\delta}))} \preceq \max\{\widehat{\nu}_\Upsilon(\hbar) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon(\hbar)}, \widehat{\nu}_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon(\bar{\delta})}\}$ and
 $\widehat{\nu}_\Upsilon((\hbar \circledast_3 \bar{\delta})) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon((\hbar \circledast_3 \bar{\delta}))} \preceq \max\{\widehat{\nu}_\Upsilon(\hbar) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon(\hbar)}, \widehat{\nu}_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon(\bar{\delta})}\}.$

For any $\hbar = (\hbar_1, \hbar_2), \bar{\delta} = (\bar{\delta}_1, \bar{\delta}_2) \in S \times \mathcal{B}$. Now,

$$\begin{aligned} & \mu_\Upsilon((\hbar \circledast_1 \bar{\delta})) \cdot e^{i2\pi\beta_\Upsilon((\hbar \circledast_1 \bar{\delta}))} \\ &= \mu_\Upsilon[(((\hbar_1, \hbar_2)) \circledast_1 ((\bar{\delta}_1, \bar{\delta}_2)))] \cdot e^{i2\pi\beta_\Upsilon[(((\hbar_1, \hbar_2)) \circledast_1 ((\bar{\delta}_1, \bar{\delta}_2)))]} \\ &= \mu_\Upsilon(\hbar_1 \circledast_1 \bar{\delta}_1, \hbar_2 \circledast_1 \bar{\delta}_2) \cdot e^{i2\pi\beta_\Upsilon(\hbar_1 \circledast_1 \bar{\delta}_1, \hbar_2 \circledast_1 \bar{\delta}_2)} \\ &= \min\{\mu_Z((\hbar_1 \circledast_1 \bar{\delta}_1)) \cdot e^{i2\pi\beta_Z((\hbar_1 \circledast_1 \bar{\delta}_1))}, \mu_Z((\hbar_2 \circledast_1 \bar{\delta}_2)) \cdot e^{i2\pi\beta_Z((\hbar_2 \circledast_1 \bar{\delta}_2))}\} \\ &\succeq \min\{\min\{\mu_Z(\hbar_1) \cdot e^{i2\pi\beta_Z(\hbar_1)}, \mu_Z(\bar{\delta}_1) \cdot e^{i2\pi\beta_Z(\bar{\delta}_1)}\}, \\ &\quad \min\{\mu_Z(\hbar_2) \cdot e^{i2\pi\beta_Z(\hbar_2)}, \mu_Z(\bar{\delta}_2) \cdot e^{i2\pi\beta_Z(\bar{\delta}_2)}\}\} \\ &= \min\{\min\{\mu_Z(\hbar_1) \cdot e^{i2\pi\beta_Z(\hbar_1)}, \mu_Z(\hbar_2) \cdot e^{i2\pi\beta_Z(\hbar_2)}\}, \\ &\quad \min\{\mu_Z(\bar{\delta}_1) \cdot e^{i2\pi\beta_Z(\bar{\delta}_1)}, \mu_Z(\bar{\delta}_2) \cdot e^{i2\pi\beta_Z(\bar{\delta}_2)}\}\} \\ &= \min\{\mu_\Upsilon((\hbar_1, \hbar_2)) \cdot e^{i2\pi\beta_\Upsilon((\hbar_1, \hbar_2))}, \mu_\Upsilon((\bar{\delta}_1, \bar{\delta}_2)) \cdot e^{i2\pi\beta_\Upsilon((\bar{\delta}_1, \bar{\delta}_2))}\} \\ &= \min\{\mu_\Upsilon(\hbar) \cdot e^{i2\pi\beta_\Upsilon(\hbar)}, \mu_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\beta_\Upsilon(\bar{\delta})}\} \end{aligned}$$

Also $\mu_\Upsilon((\hbar \circledast_2 \bar{\delta})) \cdot e^{i2\pi\beta_\Upsilon((\hbar \circledast_2 \bar{\delta}))} \succeq \min\{\mu_\Upsilon(\hbar) \cdot e^{i2\pi\beta_\Upsilon(\hbar)}, \mu_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\beta_\Upsilon(\bar{\delta})}\}$,
 $\mu_\Upsilon((\hbar \circledast_3 \bar{\delta})) \cdot e^{i2\pi\beta_\Upsilon((\hbar \circledast_3 \bar{\delta}))} \succeq \min\{\mu_\Upsilon(\hbar) \cdot e^{i2\pi\beta_\Upsilon(\hbar)}, \mu_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\beta_\Upsilon(\bar{\delta})}\}.$

Similarly, $\nu_\Upsilon((\hbar \circledast_1 \bar{\delta})) \cdot e^{i2\pi\gamma_\Upsilon((\hbar \circledast_1 \bar{\delta}))} \preceq \max\{\nu_\Upsilon(\hbar) \cdot e^{i2\pi\gamma_\Upsilon(\hbar)}, \nu_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\gamma_\Upsilon(\bar{\delta})}\}$,
 $\nu_\Upsilon((\hbar \circledast_2 \bar{\delta})) \cdot e^{i2\pi\gamma_\Upsilon((\hbar \circledast_2 \bar{\delta}))} \preceq \max\{\nu_\Upsilon(\hbar) \cdot e^{i2\pi\gamma_\Upsilon(\hbar)}, \nu_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\gamma_\Upsilon(\bar{\delta})}\}$ and
 $\nu_\Upsilon((\hbar \circledast_3 \bar{\delta})) \cdot e^{i2\pi\gamma_\Upsilon((\hbar \circledast_3 \bar{\delta}))} \preceq \max\{\nu_\Upsilon(\hbar) \cdot e^{i2\pi\gamma_\Upsilon(\hbar)}, \nu_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\gamma_\Upsilon(\bar{\delta})}\}.$

Therefore, Υ is a ComCIFSBS of $\mathcal{B} \times \mathcal{B}$.

Conversely, suppose that Υ is a ComCIFSBS of $\mathcal{B} \times \mathcal{B}$. Let $\hbar = ((\hbar_1, \hbar_2)), \bar{\delta} = ((\bar{\delta}_1, \bar{\delta}_2)) \in S \times \mathcal{B}$. Now,

$$\begin{aligned} & \min\{\widehat{\mu}_Z((\hbar_1 \circledast_1 \bar{\delta}_1)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar_1 \circledast_1 \bar{\delta}_1))}, \widehat{\mu}_Z((\hbar_2 \circledast_1 \bar{\delta}_2)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar_2 \circledast_1 \bar{\delta}_2))}\} \\ &= \widehat{\mu}_\Upsilon(\hbar_1 \circledast_1 \bar{\delta}_1, \hbar_2 \circledast_1 \bar{\delta}_2) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\hbar_1 \circledast_1 \bar{\delta}_1, \hbar_2 \circledast_1 \bar{\delta}_2)} \\ &= \widehat{\mu}_\Upsilon[(((\hbar_1, \hbar_2)) \circledast_1 ((\bar{\delta}_1, \bar{\delta}_2)))] \cdot e^{i2\pi\widehat{\beta}_\Upsilon[(((\hbar_1, \hbar_2)) \circledast_1 ((\bar{\delta}_1, \bar{\delta}_2)))]} \\ &= \widehat{\mu}_\Upsilon(\hbar \circledast_1 \bar{\delta}) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\hbar \circledast_1 \bar{\delta})} \\ &\succeq \min\{\widehat{\mu}_\Upsilon(\hbar) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\hbar)}, \widehat{\mu}_\Upsilon(\bar{\delta}) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\bar{\delta})}\} \\ &= \min\{\widehat{\mu}_\Upsilon((\hbar_1, \hbar_2)) \cdot e^{i2\pi\widehat{\beta}_\Upsilon((\hbar_1, \hbar_2))}, \widehat{\mu}_\Upsilon((\bar{\delta}_1, \bar{\delta}_2)) \cdot e^{i2\pi\widehat{\beta}_\Upsilon((\bar{\delta}_1, \bar{\delta}_2))}\} \\ &= \min\{\min\{\widehat{\mu}_Z(\hbar_1) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar_1)}, \widehat{\mu}_Z(\hbar_2) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar_2)}\}, \min\{\widehat{\mu}_Z(\bar{\delta}_1) \cdot e^{i2\pi\widehat{\beta}_Z(\bar{\delta}_1)}, \widehat{\mu}_Z(\bar{\delta}_2) \cdot e^{i2\pi\widehat{\beta}_Z(\bar{\delta}_2)}\}\} \end{aligned}$$

If $\widehat{\mu}_Z((\hbar_1 \circledast_1 \bar{\delta}_1)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar_1 \circledast_1 \bar{\delta}_1))} \preceq \widehat{\mu}_Z((\hbar_2 \circledast_1 \bar{\delta}_2)) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar_2 \circledast_1 \bar{\delta}_2))}$, then $\widehat{\mu}_Z(\hbar_1) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar_1)} \preceq \widehat{\mu}_Z(\hbar_2) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar_2)}$ and $\widehat{\mu}_Z(\bar{\delta}_1) \cdot e^{i2\pi\widehat{\beta}_Z(\bar{\delta}_1)} \preceq \widehat{\mu}_Z(\bar{\delta}_2) \cdot e^{i2\pi\widehat{\beta}_Z(\bar{\delta}_2)}$. We get $\widehat{\mu}_Z((\hbar_1 \circledast_1 \bar{\delta}_1)) \cdot$

$e^{i2\pi\beta_Z(\widehat{(h_1 \odot_1 \delta_1)})} \succeq \min\{\widehat{\mu_Z}(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \widehat{\mu_Z}(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}\}$ for all $h_1, \delta_1 \in \mathcal{B}$, and
 $\min\{\widehat{\mu_Z}(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_2 \delta_1)})}, \widehat{\mu_Z}(\widehat{(h_2 \odot_2 \delta_2)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_2 \odot_2 \delta_2)})}\} \succeq \min\{\min\{\widehat{\mu_Z}(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \widehat{\mu_Z}(h_2) \cdot e^{i2\pi\beta_Z(h_2)}\}, \min\{\widehat{\mu_Z}(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}, \widehat{\mu_Z}(\delta_2) \cdot e^{i2\pi\beta_Z(\delta_2)}\}\}$
 If $\widehat{\mu_Z}(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_2 \delta_1)})} \preceq \widehat{\mu_Z}(\widehat{(h_2 \odot_2 \delta_2)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_2 \odot_2 \delta_2)})}$, then $\widehat{\mu_Z}(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_2 \delta_1)})} \succeq \min\{\widehat{\mu_Z}(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \widehat{\mu_Z}(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}\}$.
 $\min\{\widehat{\mu_Z}(\widehat{(h_1 \odot_3 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_3 \delta_1)})}, \widehat{\mu_Z}(\widehat{(h_2 \odot_3 \delta_2)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_2 \odot_3 \delta_2)})}\} \succeq \min\{\min\{\widehat{\mu_Z}(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \widehat{\mu_Z}(h_2) \cdot e^{i2\pi\beta_Z(h_2)}\}, \min\{\widehat{\mu_Z}(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}, \widehat{\mu_Z}(\delta_2) \cdot e^{i2\pi\beta_Z(\delta_2)}\}\}$
 If $\widehat{\mu_Z}(\widehat{(h_1 \odot_3 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_3 \delta_1)})} \preceq \widehat{\mu_Z}(\widehat{(h_2 \odot_3 \delta_2)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_2 \odot_3 \delta_2)})}$, then $\widehat{\mu_Z}(\widehat{(h_1 \odot_3 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_3 \delta_1)})} \succeq \min\{\widehat{\mu_Z}(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \widehat{\mu_Z}(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}\}$.

Similarly to prove that $\max\{\widehat{\nu_Z}(\widehat{(h_1 \odot_1 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_1 \delta_1)})}, \widehat{\nu_Z}(\widehat{(h_2 \odot_1 \delta_2)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_2 \odot_1 \delta_2)})}\} \preceq \max\{\max\{\widehat{\nu_Z}(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \widehat{\nu_Z}(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}\}, \max\{\widehat{\nu_Z}(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}, \widehat{\nu_Z}(\delta_2) \cdot e^{i2\pi\gamma_Z(\delta_2)}\}\}$
 If $\widehat{\nu_Z}(\widehat{(h_1 \odot_1 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_1 \delta_1)})} \succeq \widehat{\nu_Z}(\widehat{(h_2 \odot_1 \delta_2)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_2 \odot_1 \delta_2)})}$, then $\widehat{\nu_Z}(h_1) \cdot e^{i2\pi\gamma_Z(h_1)} \succeq \widehat{\nu_Z}(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}$ and $\widehat{\nu_Z}(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)} \succeq \widehat{\nu_Z}(\delta_2) \cdot e^{i2\pi\gamma_Z(\delta_2)}$.
 We get $\widehat{\nu_Z}(\widehat{(h_1 \odot_1 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_1 \delta_1)})} \preceq \max\{\widehat{\nu_Z}(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \widehat{\nu_Z}(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}\}$.
 $\max\{\widehat{\nu_Z}(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_2 \delta_1)})}, \widehat{\nu_Z}(\widehat{(h_2 \odot_2 \delta_2)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_2 \odot_2 \delta_2)})}\} \preceq \max\{\max\{\widehat{\nu_Z}(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \widehat{\nu_Z}(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}\}, \max\{\widehat{\nu_Z}(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}, \widehat{\nu_Z}(\delta_2) \cdot e^{i2\pi\gamma_Z(\delta_2)}\}\}$
 If $\widehat{\nu_Z}(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_2 \delta_1)})} \succeq \widehat{\nu_Z}(\widehat{(h_2 \odot_2 \delta_2)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_2 \odot_2 \delta_2)})}$, then $\widehat{\nu_Z}(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_2 \delta_1)})} \preceq \max\{\widehat{\nu_Z}(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \widehat{\nu_Z}(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}\}$.
 $\max\{\widehat{\nu_Z}(\widehat{(h_1 \odot_3 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_3 \delta_1)})}, \widehat{\nu_Z}(\widehat{(h_2 \odot_3 \delta_2)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_2 \odot_3 \delta_2)})}\} \preceq \max\{\max\{\widehat{\nu_Z}(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \widehat{\nu_Z}(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}\}, \max\{\widehat{\nu_Z}(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}, \widehat{\nu_Z}(\delta_2) \cdot e^{i2\pi\gamma_Z(\delta_2)}\}\}$
 If $\widehat{\nu_Z}(\widehat{(h_1 \odot_3 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_3 \delta_1)})} \succeq \widehat{\nu_Z}(\widehat{(h_2 \odot_3 \delta_2)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_2 \odot_3 \delta_2)})}$, then $\widehat{\nu_Z}(\widehat{(h_1 \odot_3 \delta_1)}) \cdot e^{i2\pi\gamma_Z(\widehat{(h_1 \odot_3 \delta_1)})} \preceq \max\{\widehat{\nu_Z}(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \widehat{\nu_Z}(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}\}$.

Let $h = ((h_1, h_2))$, $\delta = ((\delta_1, \delta_2)) \in S \times \mathcal{B}$. Now,

$$\begin{aligned}
 & \min\{\mu_Z(\widehat{(h_1 \odot_1 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_1 \delta_1)})}, \mu_Z(\widehat{(h_2 \odot_1 \delta_2)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_2 \odot_1 \delta_2)})}\} \\
 &= \mu_{\Upsilon}(\widehat{(h_1 \odot_1 \delta_1)}, \widehat{(h_2 \odot_1 \delta_2)}) \cdot e^{i2\pi\beta_{\Upsilon}(\widehat{(h_1 \odot_1 \delta_1)}, \widehat{(h_2 \odot_1 \delta_2)})} \\
 &= \mu_{\Upsilon}[\widehat{((h_1, h_2)) \odot_1 ((\delta_1, \delta_2))}] \cdot e^{i2\pi\mathcal{I}_v[\widehat{((h_1, h_2)) \odot_1 ((\delta_1, \delta_2))}]} \\
 &= \mu_{\Upsilon}(\widehat{h \odot_1 \delta}) \cdot e^{i2\pi\beta_{\Upsilon}(\widehat{h \odot_1 \delta})} \\
 &\succeq \min\{\mu_{\Upsilon}(h) \cdot e^{i2\pi\beta_{\Upsilon}(h)}, \mu_{\Upsilon}(\delta) \cdot e^{i2\pi\beta_{\Upsilon}(\delta)}\} \\
 &= \min\{\mu_{\Upsilon}(\widehat{(h_1, h_2)}) \cdot e^{i2\pi\beta_{\Upsilon}(\widehat{(h_1, h_2)})}, \mu_{\Upsilon}(\widehat{(\delta_1, \delta_2)}) \cdot e^{i2\pi\beta_{\Upsilon}(\widehat{(\delta_1, \delta_2)})}\} \\
 &= \min\{\min\{\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \mu_Z(h_2) \cdot e^{i2\pi\beta_Z(h_2)}\}, \min\{\mu_Z(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}, \mu_Z(\delta_2) \cdot e^{i2\pi\beta_Z(\delta_2)}\}\}
 \end{aligned}$$

If $\mu_Z(\widehat{(h_1 \odot_1 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_1 \delta_1)})} \preceq \mu_Z(\widehat{(h_2 \odot_1 \delta_2)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_2 \odot_1 \delta_2)})}$, then $\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)} \preceq \mu_Z(h_2) \cdot e^{i2\pi\beta_Z(h_2)}$ and $\mu_Z(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)} \preceq \mu_Z(\delta_2) \cdot e^{i2\pi\beta_Z(\delta_2)}$. We get $\mu_Z(\widehat{(h_1 \odot_1 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_1 \delta_1)})} \succeq \min\{\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \mu_Z(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}\}$ for all $h_1, \delta_1 \in \mathcal{B}$, and
 $\min\{\mu_Z(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_2 \delta_1)})}, \mu_Z(\widehat{(h_2 \odot_2 \delta_2)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_2 \odot_2 \delta_2)})}\} \succeq \min\{\min\{\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \mu_Z(h_2) \cdot e^{i2\pi\beta_Z(h_2)}\}, \min\{\mu_Z(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}, \mu_Z(\delta_2) \cdot e^{i2\pi\beta_Z(\delta_2)}\}\}$
 If $\mu_Z(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_2 \delta_1)})} \preceq \mu_Z(\widehat{(h_2 \odot_2 \delta_2)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_2 \odot_2 \delta_2)})}$, then $\mu_Z(\widehat{(h_1 \odot_2 \delta_1)}) \cdot e^{i2\pi\beta_Z(\widehat{(h_1 \odot_2 \delta_1)})} \succeq \min\{\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \mu_Z(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}\}$.

$\min\{\mu_Z((h_1 \odot_3 \delta_1)) \cdot e^{i2\pi\beta_Z((h_1 \odot_3 \delta_1))}, \mu_Z((h_2 \odot_3 \delta_2)) \cdot e^{i2\pi\beta_Z((h_2 \odot_3 \delta_2))}\} \succeq \min\{\min\{\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \mu_Z(h_2) \cdot e^{i2\pi\beta_Z(h_2)}\}, \min\{\mu_Z(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}, \mu_Z(\delta_2) \cdot e^{i2\pi\beta_Z(\delta_2)}\}\}$

If $\mu_Z((h_1 \odot_3 \delta_1)) \cdot e^{i2\pi\beta_Z((h_1 \odot_3 \delta_1))} \preceq \mu_Z((h_2 \odot_3 \delta_2)) \cdot e^{i2\pi\beta_Z((h_2 \odot_3 \delta_2))}$, then $\mu_Z((h_1 \odot_3 \delta_1)) \cdot e^{i2\pi\beta_Z((h_1 \odot_3 \delta_1))} \succeq \min\{\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \mu_Z(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}\}$.

Similarly to prove that $\max\{\nu_Z((h_1 \odot_1 \delta_1)) \cdot e^{i2\pi\gamma_Z((h_1 \odot_1 \delta_1))}, \nu_Z((h_2 \odot_1 \delta_2)) \cdot e^{i2\pi\gamma_Z((h_2 \odot_1 \delta_2))}\} \preceq \max\{\max\{\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \nu_Z(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}\}, \max\{\nu_Z(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}, \nu_Z(\delta_2) \cdot e^{i2\pi\gamma_Z(\delta_2)}\}\}$

If $\nu_Z((h_1 \odot_1 \delta_1)) \cdot e^{i2\pi\gamma_Z((h_1 \odot_1 \delta_1))} \succeq \nu_Z((h_2 \odot_1 \delta_2)) \cdot e^{i2\pi\gamma_Z((h_2 \odot_1 \delta_2))}$, then $\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)} \succeq \nu_Z(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}$ and $\nu_Z(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)} \succeq \nu_Z(\delta_2) \cdot e^{i2\pi\gamma_Z(\delta_2)}$.

We get $\nu_Z((h_1 \odot_1 \delta_1)) \cdot e^{i2\pi\gamma_Z((h_1 \odot_1 \delta_1))} \preceq \max\{\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \nu_Z(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}\}$.

$\max\{\nu_Z((h_1 \odot_2 \delta_1)) \cdot e^{i2\pi\gamma_Z((h_1 \odot_2 \delta_1))}, \nu_Z((h_2 \odot_2 \delta_2)) \cdot e^{i2\pi\gamma_Z((h_2 \odot_2 \delta_2))}\} \preceq \max\{\max\{\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \nu_Z(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}\}, \max\{\nu_Z(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}, \nu_Z(\delta_2) \cdot e^{i2\pi\gamma_Z(\delta_2)}\}\}$

If $\nu_Z((h_1 \odot_2 \delta_1)) \cdot e^{i2\pi\gamma_Z((h_1 \odot_2 \delta_1))} \succeq \nu_Z((h_2 \odot_2 \delta_2)) \cdot e^{i2\pi\gamma_Z((h_2 \odot_2 \delta_2))}$, then $\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)} \preceq \max\{\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \nu_Z(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}\}$.

$\max\{\nu_Z((h_1 \odot_3 \delta_1)) \cdot e^{i2\pi\gamma_Z((h_1 \odot_3 \delta_1))}, \nu_Z((h_2 \odot_3 \delta_2)) \cdot e^{i2\pi\gamma_Z((h_2 \odot_3 \delta_2))}\} \preceq \max\{\max\{\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \nu_Z(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}\}, \max\{\nu_Z(\delta_1) \cdot e^{i2\pi\gamma_Z(\delta_1)}, \nu_Z(\delta_2) \cdot e^{i2\pi\gamma_Z(\delta_2)}\}\}$

If $\nu_Z((h_1 \odot_3 \delta_1)) \cdot e^{i2\pi\beta_Z((h_1 \odot_3 \delta_1))} \succeq \nu_Z((h_2 \odot_3 \delta_2)) \cdot e^{i2\pi\beta_Z((h_2 \odot_3 \delta_2))}$, then $\nu_Z((h_1 \odot_3 \delta_1)) \cdot e^{i2\pi\beta_Z((h_1 \odot_3 \delta_1))} \preceq \max\{\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \nu_Z(\delta_1) \cdot e^{i2\pi\beta_Z(\delta_1)}\}$. Therefore, Z is a ComCIFSBS of \mathcal{B} .

Theorem 3.12. Suppose that Z is a subset of \mathcal{B} . Then $R = (\widehat{\mu}_Z \cdot e^{i2\pi\widehat{\beta}_Z}, \widehat{\nu}_Z \cdot e^{i2\pi\widehat{\gamma}_Z}, \mu_Z \cdot e^{i2\pi\beta_Z}, \nu_Z \cdot e^{i2\pi\gamma_Z})$ is a ComCIFSBS of \mathcal{B} if and only if $\mathcal{R}^{(t,s)}$ is a subbisemiring of \mathcal{B} for all $t, s \in D[0, 1]$.

Proof. Assume that $\widehat{\mathcal{R}}$ is a ComCIFSBS of \mathcal{B} . For each $t, s \in D[0, 1]$ and $h_1, h_2 \in \widehat{\mathcal{R}}^{(t,s)}$. Now, $\widehat{\mu}_Z(h_1) \cdot e^{i2\pi\widehat{\beta}_Z(h_1)} \succeq t, \widehat{\mu}_Z(h_2) \cdot e^{i2\pi\widehat{\beta}_Z(h_2)} \succeq t$ and $\widehat{\nu}_Z(h_1) \cdot e^{i2\pi\widehat{\gamma}_Z(h_1)} \preceq s, \widehat{\nu}_Z(h_2) \cdot e^{i2\pi\widehat{\gamma}_Z(h_2)} \preceq s$. Now, $\widehat{\mu}_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\widehat{\beta}_Z((h_1 \odot_1 h_2))} \succeq \min\{\widehat{\mu}_Z(h_1) \cdot e^{i2\pi\widehat{\beta}_Z(h_1)}, \widehat{\mu}_Z(h_2) \cdot e^{i2\pi\widehat{\beta}_Z(h_2)}\} \succeq t$ and $\widehat{\nu}_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\widehat{\gamma}_Z((h_1 \odot_1 h_2))} \preceq \max\{\widehat{\nu}_Z(h_1) \cdot e^{i2\pi\widehat{\gamma}_Z(h_1)}, \widehat{\nu}_Z(h_2) \cdot e^{i2\pi\widehat{\gamma}_Z(h_2)}\} \preceq s$. This implies that $h_1 \odot_1 h_2 \in \widehat{\mathcal{R}}^{(t,s)}$. Similarly, $h_1 \odot_2 h_2 \in \widehat{\mathcal{R}}^{(t,s)}$ and $h_1 \odot_3 h_2 \in \widehat{\mathcal{R}}^{(t,s)}$. Hence, $\widehat{\mathcal{R}}^{(t,s)}$ is a subbisemiring of \mathcal{B} , for all $t, s \in D[0, 1]$.

For each $t, s \in [0, 1]$ and $h_1, h_2 \in \mathcal{R}^{(t,s)}$. Now, $\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)} \succeq t, \mu_Z(h_2) \cdot e^{i2\pi\beta_Z(h_2)} \succeq t$ and $\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)} \preceq s, \nu_Z(h_2) \cdot e^{i2\pi\gamma_Z(h_2)} \preceq s$. Now, $\mu_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\beta_Z((h_1 \odot_1 h_2))} \succeq \min\{\mu_Z(h_1) \cdot e^{i2\pi\beta_Z(h_1)}, \mu_Z(h_2) \cdot e^{i2\pi\beta_Z(h_2)}\} \succeq t$ and $\nu_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\gamma_Z((h_1 \odot_1 h_2))} \preceq \max\{\nu_Z(h_1) \cdot e^{i2\pi\gamma_Z(h_1)}, \nu_Z(h_2) \cdot e^{i2\pi\gamma_Z(h_2)}\} \preceq s$. This implies that $h_1 \odot_1 h_2 \in \mathcal{R}^{(t,s)}$. Similarly, $h_1 \odot_2 h_2 \in \mathcal{R}^{(t,s)}$ and $h_1 \odot_3 h_2 \in \mathcal{R}^{(t,s)}$. Hence, $\mathcal{R}^{(t,s)}$ is a subbisemiring of \mathcal{B} , for all $t, s \in D[0, 1]$.

Conversely, assume that $\widehat{\mathcal{R}}^{(t,s)}$ is a subbisemiring of \mathcal{B} and $t, s \in D[0, 1]$. Suppose if there exist $h_1, h_2 \in \mathcal{B}$ such that $\widehat{\mu}_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\widehat{\beta}_Z((h_1 \odot_1 h_2))} \prec \min\{\widehat{\mu}_Z(h_1) \cdot e^{i2\pi\widehat{\beta}_Z(h_1)}, \widehat{\mu}_Z(h_2) \cdot e^{i2\pi\widehat{\beta}_Z(h_2)}\}$ and $\widehat{\nu}_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\widehat{\gamma}_Z((h_1 \odot_1 h_2))} \succ \max\{\widehat{\nu}_Z(h_1) \cdot e^{i2\pi\widehat{\gamma}_Z(h_1)}, \widehat{\nu}_Z(h_2) \cdot e^{i2\pi\widehat{\gamma}_Z(h_2)}\}$. For $t, s \in D[0, 1]$ such that $\widehat{\mu}_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\widehat{\beta}_Z((h_1 \odot_1 h_2))} \prec t \preceq \min\{\widehat{\mu}_Z(h_1) \cdot e^{i2\pi\widehat{\beta}_Z(h_1)}, \widehat{\mu}_Z(h_2) \cdot e^{i2\pi\widehat{\beta}_Z(h_2)}\}$ and $\widehat{\nu}_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\widehat{\gamma}_Z((h_1 \odot_1 h_2))} \succ s \succeq \max\{\widehat{\nu}_Z(h_1) \cdot e^{i2\pi\widehat{\gamma}_Z(h_1)}, \widehat{\nu}_Z(h_2) \cdot e^{i2\pi\widehat{\gamma}_Z(h_2)}\}$. Thus, $h_1, h_2 \in \widehat{\mathcal{R}}^{(t,s)}$, but $h_1 \odot_1 h_2 \notin \widehat{\mathcal{R}}^{(t,s)}$. This contradicts, $\widehat{\mathcal{R}}^{(t,s)}$ is a SBS of \mathcal{B} . Therefore $\widehat{\mu}_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\widehat{\beta}_Z((h_1 \odot_1 h_2))} \succeq \min\{\widehat{\mu}_Z(h_1) \cdot e^{i2\pi\widehat{\beta}_Z(h_1)}, \widehat{\mu}_Z(h_2) \cdot e^{i2\pi\widehat{\beta}_Z(h_2)}\}$ and $\widehat{\nu}_Z((h_1 \odot_1 h_2)) \cdot e^{i2\pi\widehat{\gamma}_Z((h_1 \odot_1 h_2))} \preceq \max\{\widehat{\nu}_Z(h_1) \cdot e^{i2\pi\widehat{\gamma}_Z(h_1)}, \widehat{\nu}_Z(h_2) \cdot e^{i2\pi\widehat{\gamma}_Z(h_2)}\}$. Similarly, \odot_2 and \odot_3 cases. Hence $\widehat{\mathcal{R}} = (\widehat{\mu}_Z \cdot e^{i2\pi\widehat{\beta}_Z}, \widehat{\mathcal{R}}_Z^I \cdot e^{i2\pi\widehat{\mathcal{I}}_Z}, \widehat{\nu}_Z \cdot e^{i2\pi\widehat{\gamma}_Z})$ is a ComCIFSBS of \mathcal{B} .

Let us assume that $\mathcal{R}^{(t,s)}$ is a subbisemiring of \mathcal{B} and $t, s \in [0, 1]$. Suppose if there exist

$\hbar_1, \hbar_2 \in \mathcal{B}$ such that $\mu_Z((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\beta_Z((\hbar_1 \circ_1 \hbar_2))} \prec \min\{\mu_Z(\hbar_1) \cdot e^{i2\pi\beta_Z(\hbar_1)}, \mu_Z(\hbar_2) \cdot e^{i2\pi\beta_Z(\hbar_2)}\}$ and $\nu_Z((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\gamma_Z((\hbar_1 \circ_1 \hbar_2))} \succ \max\{\nu_Z(\hbar_1) \cdot e^{i2\pi\gamma_Z(\hbar_1)}, \nu_Z(\hbar_2) \cdot e^{i2\pi\gamma_Z(\hbar_2)}\}$. For $t, s \in D[0, 1]$ such that $\mu_Z((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\beta_Z((\hbar_1 \circ_1 \hbar_2))} \prec t \preceq \min\{\mu_Z(\hbar_1) \cdot e^{i2\pi\beta_Z(\hbar_1)}, \mu_Z(\hbar_2) \cdot e^{i2\pi\beta_Z(\hbar_2)}\}$ and $\mathcal{R}_Z^I((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\mathcal{I}_Z^I((\hbar_1 \circ_1 \hbar_2))} \prec t \preceq \frac{\mathcal{R}_Z^I(\hbar_1) \cdot e^{i2\pi\mathcal{I}_Z^I(\hbar_1)} + \mathcal{R}_Z^I(\hbar_2) \cdot e^{i2\pi\mathcal{I}_Z^I(\hbar_2)}}{2}$ and $\nu_Z((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\gamma_Z((\hbar_1 \circ_1 \hbar_2))} \succ s \succeq \max\{\nu_Z(\hbar_1) \cdot e^{i2\pi\gamma_Z(\hbar_1)}, \nu_Z(\hbar_2) \cdot e^{i2\pi\gamma_Z(\hbar_2)}\}$. Thus, $\hbar_1, \hbar_2 \in \mathcal{R}^{(t,s)}$, but $\hbar_1 \circ_1 \hbar_2 \notin \mathcal{R}^{(t,s)}$. This contradicts, $\mathcal{R}^{(t,s)}$ is a SBS of \mathcal{B} . Therefore $\mu_Z((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\beta_Z((\hbar_1 \circ_1 \hbar_2))} \succeq \min\{\mu_Z(\hbar_1) \cdot e^{i2\pi\beta_Z(\hbar_1)}, \mu_Z(\hbar_2) \cdot e^{i2\pi\beta_Z(\hbar_2)}\}$ and $\nu_Z((\hbar_1 \circ_1 \hbar_2)) \cdot e^{i2\pi\gamma_Z((\hbar_1 \circ_1 \hbar_2))} \preceq \max\{\nu_Z(\hbar_1) \cdot e^{i2\pi\gamma_Z(\hbar_1)}, \nu_Z(\hbar_2) \cdot e^{i2\pi\gamma_Z(\hbar_2)}\}$. Similarly, \circ_2 and \circ_3 cases. Hence $\mathcal{R} = (\mu_Z \cdot e^{i2\pi\beta_Z}, \nu_Z \cdot e^{i2\pi\gamma_Z})$ is a ComCIFSBS of \mathcal{B} .

Definition 3.13. Let $(\mathcal{B}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{B}_2, \wedge_1, \wedge_2, \wedge_3)$ be any two bisemirings. The mapping $\downarrow : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ and Z be any ComCIFSBS in \mathcal{B}_1 , Υ be any ComCIFSBS in $\downarrow(\mathcal{B}_1) = \mathcal{B}_2$. If $\mu_Z \cdot e^{i2\pi\beta_Z} = [\widehat{\mu}_Z \cdot e^{i2\pi\widehat{\beta}_Z}, \widehat{\nu}_Z \cdot e^{i2\pi\widehat{\gamma}_Z}, \mu_Z \cdot e^{i2\pi\beta_Z}, \mu_Z \cdot e^{i2\pi\beta_Z}, \nu_Z \cdot e^{i2\pi\gamma_Z}]$ is a ComCIF in \mathcal{B}_1 , then \mathcal{R}_Υ is a ComCIF in \mathcal{B}_2 , defined by

$$\begin{aligned} \widehat{\mu}_\Upsilon(\check{\delta}) \cdot e^{i2\pi\widehat{\beta}_Z(\check{\delta})} &= \begin{cases} \sup \widehat{\mu}_Z(\check{h}) \cdot e^{i2\pi\widehat{\beta}_Z(\check{h})} & \text{if } \check{h} \in \downarrow^{-1}\check{\delta} \\ 0 & \text{otherwise} \end{cases} \\ \widehat{\nu}_\Upsilon(\check{\delta}) \cdot e^{i2\pi\widehat{\gamma}_Z(\check{\delta})} &= \begin{cases} \inf \widehat{\nu}_Z(\check{h}) \cdot e^{i2\pi\widehat{\gamma}_Z(\check{h})} & \text{if } \check{h} \in \downarrow^{-1}\check{\delta} \\ 1 & \text{otherwise} \end{cases} \\ \mu_\Upsilon(\check{\delta}) \cdot e^{i2\pi\beta_Z(\check{\delta})} &= \begin{cases} \sup \mu_Z(\check{h}) \cdot e^{i2\pi\beta_Z(\check{h})} & \text{if } \check{h} \in \downarrow^{-1}\check{\delta} \\ 0 & \text{otherwise} \end{cases} \\ \nu_\Upsilon(\check{\delta}) \cdot e^{i2\pi\gamma_Z(\check{\delta})} &= \begin{cases} \inf \nu_Z(\check{h}) \cdot e^{i2\pi\gamma_Z(\check{h})} & \text{if } \check{h} \in \downarrow^{-1}\check{\delta} \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

for all $\check{h} \in \mathcal{B}_1$ and $\check{\delta} \in \mathcal{B}_2$ is represents the image of R_Z under \downarrow .

Similarly, If $\mathcal{R}_\Upsilon \cdot e^{i2\pi\mathcal{I}_\Upsilon} = [\widehat{\mu}_\Upsilon \cdot e^{i2\pi\widehat{\beta}_Z}, \widehat{\nu}_\Upsilon \cdot e^{i2\pi\widehat{\gamma}_Z}, \mathcal{R}_\Upsilon \cdot e^{i2\pi\mathcal{I}_\Upsilon}, \mu_\Upsilon \cdot e^{i2\pi\beta_Z}, \nu_\Upsilon \cdot e^{i2\pi\gamma_Z}]$ is a ComCIF in \mathcal{B}_2 , then ComCIF $\mathcal{R}_Z = \downarrow \circ \mathcal{R}_\Upsilon$ in \mathcal{B}_1 ie, the ComCIF defined by $\mathcal{R}_Z(\check{h}) \cdot e^{i2\pi\mathcal{I}_Z(\check{h})}, \mathcal{R}_\Upsilon(\downarrow(\check{h})) \cdot e^{i2\pi\mathcal{I}_Z(\downarrow(\check{h}))}, \mathcal{R}_Z = \downarrow \circ \mathcal{R}_\Upsilon$ in \mathcal{B}_1 [ie, the ComCIF defined by $\mathcal{R}_Z(\check{h}) \cdot e^{i2\pi\mathcal{I}_Z(\check{h})} = \mathcal{R}_\Upsilon(\downarrow(\check{h})) \cdot e^{i2\pi\mathcal{I}_Z(\downarrow(\check{h}))}$ is represents the preimage of \mathcal{R}_Υ under \downarrow .

Theorem 3.14. The homomorphic image of every ComCIFSBS is a ComCIFSBS.

Proof. The mapping $\downarrow : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be any homomorphism.

Now, $\downarrow((\check{h} \vee_1 \check{\delta})) = \downarrow(\check{h}) \wedge_1 \downarrow(\check{\delta}), \downarrow((\check{h} \vee_2 \check{\delta})) = \downarrow(\check{h}) \wedge_2 \downarrow(\check{\delta})$ and $\downarrow((\check{h} \vee_3 \check{\delta})) = \downarrow(\check{h}) \wedge_3 \downarrow(\check{\delta})$

for all $\check{h}, \check{\delta} \in \mathcal{B}_1$. Let $\widehat{\Upsilon} = \downarrow(Z), Z$ is any ComCIFSBS of \mathcal{B}_1 . Let $\downarrow(\check{h}), \downarrow(\check{\delta}) \in \mathcal{B}_2$. Let

$\check{h} \in u\downarrow^{-1}(\downarrow(\check{h}))$ and $\check{\delta} \in \downarrow^{-1}(\downarrow(\check{\delta}))$ be such that $\widehat{\mu}_Z(\check{h}) \cdot e^{i2\pi\widehat{\beta}_Z(\check{h})} = \sup_{h \in \downarrow^{-1}(\downarrow(\check{h}))} \widehat{\mu}_Z(h) \cdot$

$e^{i2\pi\widehat{\beta}_Z(h)}$ and $\widehat{\mu}_Z(\check{\delta}) \cdot e^{i2\pi\widehat{\beta}_Z(\check{\delta})} = \sup_{h \in \downarrow^{-1}(\downarrow(\check{\delta}))} \widehat{\mu}_Z(h) \cdot e^{i2\pi\widehat{\beta}_Z(h)}$.

Now, $\widehat{\mu}_\Upsilon((\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta}))) \cdot e^{i2\pi\widehat{\beta}_Z((\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta})))}$

$$\begin{aligned} &= \sup_{(h') \in \downarrow^{-1}(\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta}))} \widehat{\mu}_Z(h') \cdot e^{i2\pi\widehat{\beta}_Z(h')} \\ &= \sup_{(h') \in \downarrow^{-1}(\downarrow(\hbar \vee_1 \bar{\theta}))} \widehat{\mu}_Z(h') \cdot e^{i2\pi\widehat{\beta}_Z(h')} \\ &= \widehat{\mu}_Z(\hbar \vee_1 \bar{\theta}) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar \vee_1 \bar{\theta})} \\ &\succeq \min\{\widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar)}, \widehat{\mu}_Z(\bar{\theta}) \cdot e^{i2\pi\widehat{\beta}_Z(\bar{\theta})}\} \\ &= \min\{\widehat{\mu}_\Upsilon \downarrow(\hbar) \cdot e^{i2\pi\widehat{\beta}_\Upsilon \downarrow(\hbar)}, \widehat{\mu}_\Upsilon \downarrow(\bar{\theta}) \cdot e^{i2\pi\widehat{\beta}_\Upsilon \downarrow(\bar{\theta})}\}. \end{aligned}$$

Thus, $\widehat{\mu}_\Upsilon((\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta}))) \cdot e^{i2\pi\widehat{\beta}_\Upsilon((\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta})))} \succeq \min\{\widehat{\mu}_\Upsilon \downarrow(\hbar) \cdot e^{i2\pi\widehat{\beta}_\Upsilon \downarrow(\hbar)}, \widehat{\mu}_\Upsilon \downarrow(\bar{\theta}) \cdot e^{i2\pi\widehat{\beta}_\Upsilon \downarrow(\bar{\theta})}\}$.

Similarly, $\widehat{\mu}_\Upsilon((\downarrow(\hbar) \wedge_2 \downarrow(\bar{\theta}))) \cdot e^{i2\pi\widehat{\beta}_\Upsilon((\downarrow(\hbar) \wedge_2 \downarrow(\bar{\theta})))} \succeq \min\{\widehat{\mu}_\Upsilon \downarrow(\hbar) \cdot e^{i2\pi\widehat{\beta}_\Upsilon \downarrow(\hbar)}, \widehat{\mu}_\Upsilon \downarrow(\bar{\theta}) \cdot e^{i2\pi\widehat{\beta}_\Upsilon \downarrow(\bar{\theta})}\}$

and

$\widehat{\mu}_\Upsilon((\downarrow(\hbar) \wedge_3 \downarrow(\bar{\theta}))) \cdot e^{i2\pi\widehat{\beta}_\Upsilon((\downarrow(\hbar) \wedge_3 \downarrow(\bar{\theta})))} \succeq \min\{\widehat{\mu}_\Upsilon \downarrow(\hbar) \cdot e^{i2\pi\widehat{\beta}_\Upsilon \downarrow(\hbar)}, \widehat{\mu}_\Upsilon \downarrow(\bar{\theta}) \cdot e^{i2\pi\widehat{\beta}_\Upsilon \downarrow(\bar{\theta})}\}$.

Let $\downarrow(\hbar), \downarrow(\bar{\theta}) \in \mathcal{B}_2$. Let $\hbar \in \downarrow^{-1}(\downarrow(\hbar))$ and $\bar{\theta} \in \downarrow^{-1}(\downarrow(\bar{\theta}))$ be such that $\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)} = \inf_{h \in \downarrow^{-1}(\downarrow(\hbar))} \widehat{\nu}_Z(h) \cdot e^{i2\pi\widehat{\gamma}_Z(h)}$ and $\widehat{\nu}_Z(\bar{\theta}) \cdot e^{i2\pi\widehat{\gamma}_Z(\bar{\theta})} = \inf_{h \in \downarrow^{-1}(\downarrow(\bar{\theta}))} \widehat{\nu}_Z(h) \cdot e^{i2\pi\widehat{\gamma}_Z(h)}$.

Now, $\widehat{\nu}_\Upsilon((\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta}))) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon((\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta})))}$

$$\begin{aligned} &= \inf_{(h') \in \downarrow^{-1}(\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta}))} \widehat{\nu}_Z(h') \cdot e^{i2\pi\widehat{\gamma}_Z(h')} \\ &= \inf_{(h') \in \downarrow^{-1}(\downarrow(\hbar \vee_1 \bar{\theta}))} \widehat{\nu}_Z(h') \cdot e^{i2\pi\widehat{\gamma}_Z(h')} \\ &= \widehat{\nu}_Z(\hbar \vee_1 \bar{\theta}) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar \vee_1 \bar{\theta})} \\ &\preceq \max\{\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)}, \widehat{\nu}_Z(\bar{\theta}) \cdot e^{i2\pi\widehat{\gamma}_Z(\bar{\theta})}\} \\ &= \max\{\widehat{\nu}_\Upsilon \downarrow(\hbar) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon \downarrow(\hbar)}, \widehat{\nu}_\Upsilon \downarrow(\bar{\theta}) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon \downarrow(\bar{\theta})}\}. \end{aligned}$$

Thus, $\widehat{\nu}_\Upsilon((\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta}))) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon((\downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta})))} \preceq \max\{\widehat{\nu}_\Upsilon \downarrow(\hbar) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon \downarrow(\hbar)}, \widehat{\nu}_\Upsilon \downarrow(\bar{\theta}) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon \downarrow(\bar{\theta})}\}$.

Similarly, $\widehat{\nu}_\Upsilon((\downarrow(\hbar) \wedge_2 \downarrow(\bar{\theta}))) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon((\downarrow(\hbar) \wedge_2 \downarrow(\bar{\theta})))} \preceq \max\{\widehat{\nu}_\Upsilon \downarrow(\hbar) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon \downarrow(\hbar)}, \widehat{\nu}_\Upsilon \downarrow(\bar{\theta}) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon \downarrow(\bar{\theta})}\}$

and

$\widehat{\nu}_\Upsilon((\downarrow(\hbar) \wedge_3 \downarrow(\bar{\theta}))) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon((\downarrow(\hbar) \wedge_3 \downarrow(\bar{\theta})))} \preceq \max\{\widehat{\nu}_\Upsilon \downarrow(\hbar) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon \downarrow(\hbar)}, \widehat{\nu}_\Upsilon \downarrow(\bar{\theta}) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon \downarrow(\bar{\theta})}\}$.

The mapping $\downarrow : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be any homomorphism.

Now, $\downarrow(\hbar \vee_1 \bar{\theta}) = \downarrow(\hbar) \wedge_1 \downarrow(\bar{\theta})$, $\downarrow(\hbar \vee_2 \bar{\theta}) = \downarrow(\hbar) \wedge_2 \downarrow(\bar{\theta})$ and $\downarrow(\hbar \vee_3 \bar{\theta}) = \downarrow(\hbar) \wedge_3 \downarrow(\bar{\theta})$

for all $\hbar, \bar{\theta} \in \mathcal{B}_1$. Let $\Upsilon = \downarrow(Z)$, Z is any ComCIFSBS of \mathcal{B}_1 . Let $\downarrow(\hbar), \downarrow(\bar{\theta}) \in \mathcal{B}_2$. Let

$\hbar \in \downarrow^{-1}(\downarrow(\hbar))$ and $\bar{\theta} \in \downarrow^{-1}(\downarrow(\bar{\theta}))$ be such that $\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)} = \sup_{h \in \downarrow^{-1}(\downarrow(\hbar))} \mu_Z(h) \cdot e^{i2\pi\beta_Z(h)}$

and $\mu_Z(\bar{\theta}) \cdot e^{i2\pi\beta_Z(\bar{\theta})} = \sup_{h \in \downarrow^{-1}(\downarrow(\bar{\theta}))} \mu_Z(h) \cdot e^{i2\pi\beta_Z(h)}$.

Now, $\mu_{\Upsilon}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta)) \cdot e^{i2\pi\beta_Z(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta))}$

$$\begin{aligned} &= \sup_{(h') \in \lrcorner^{-1}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta))} \mu_Z(h') \cdot e^{i2\pi\beta_Z(h')} \\ &= \sup_{(h') \in \lrcorner^{-1}(\lrcorner(\hbar \vee_1 \delta))} \mu_Z(h') \cdot e^{i2\pi\beta_Z(h')} \\ &= \mu_Z(\lrcorner(\hbar \vee_1 \delta)) \cdot e^{i2\pi\beta_Z(\lrcorner(\hbar \vee_1 \delta))} \\ &\succeq \min\{\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \mu_Z(\delta) \cdot e^{i2\pi\beta_Z(\delta)}\} \\ &= \min\{\mu_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\beta_{\Upsilon}\lrcorner(\hbar)}, \mu_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\beta_{\Upsilon}\lrcorner(\delta)}\}. \end{aligned}$$

Thus, $\mu_{\Upsilon}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta)) \cdot e^{i2\pi\beta_{\Upsilon}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta))} \succeq \min\{\mu_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\beta_{\Upsilon}\lrcorner(\hbar)}, \mu_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\beta_{\Upsilon}\lrcorner(\delta)}\}$.

Similarly, $\mu_{\Upsilon}(\lrcorner(\hbar) \wedge_2 \lrcorner(\delta)) \cdot e^{i2\pi\beta_{\Upsilon}(\lrcorner(\hbar) \wedge_2 \lrcorner(\delta))} \succeq \min\{\mu_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\beta_{\Upsilon}\lrcorner(\hbar)}, \mu_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\beta_{\Upsilon}\lrcorner(\delta)}\}$ and

$\mu_{\Upsilon}(\lrcorner(\hbar) \wedge_3 \lrcorner(\delta)) \cdot e^{i2\pi\beta_{\Upsilon}(\lrcorner(\hbar) \wedge_3 \lrcorner(\delta))} \succeq \min\{\mu_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\beta_{\Upsilon}\lrcorner(\hbar)}, \mu_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\beta_{\Upsilon}\lrcorner(\delta)}\}$.

Let $\lrcorner(\hbar), \lrcorner(\delta) \in \mathcal{B}_2$. Let $\hbar \in \lrcorner^{-1}(\lrcorner(\hbar))$ and $\delta \in \lrcorner^{-1}(\lrcorner(\delta))$ be such that $\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)} = \inf_{h \in \lrcorner^{-1}(\lrcorner(\hbar))} \nu_Z(h) \cdot e^{i2\pi\gamma_Z(h)}$ and $\nu_Z(\delta) \cdot e^{i2\pi\gamma_Z(\delta)} = \inf_{h \in \lrcorner^{-1}(\lrcorner(\delta))} \nu_Z(h) \cdot e^{i2\pi\gamma_Z(h)}$.

Now, $\nu_{\Upsilon}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta)) \cdot e^{i2\pi\gamma_{\Upsilon}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta))}$

$$\begin{aligned} &= \inf_{(h') \in \lrcorner^{-1}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta))} \nu_Z(h') \cdot e^{i2\pi\gamma_Z(h')} \\ &= \inf_{(h') \in \lrcorner^{-1}(\lrcorner(\hbar \vee_1 \delta))} \nu_Z(h') \cdot e^{i2\pi\gamma_Z(h')} \\ &= \nu_Z(\lrcorner(\hbar \vee_1 \delta)) \cdot e^{i2\pi\gamma_Z(\lrcorner(\hbar \vee_1 \delta))} \\ &\preceq \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\delta) \cdot e^{i2\pi\gamma_Z(\delta)}\} \\ &= \max\{\nu_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\gamma_{\Upsilon}\lrcorner(\hbar)}, \nu_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\gamma_{\Upsilon}\lrcorner(\delta)}\}. \end{aligned}$$

Thus, $\nu_{\Upsilon}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta)) \cdot e^{i2\pi\gamma_{\Upsilon}(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta))} \preceq \max\{\nu_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\gamma_{\Upsilon}\lrcorner(\hbar)}, \nu_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\gamma_{\Upsilon}\lrcorner(\delta)}\}$.

Similarly, $\nu_{\Upsilon}(\lrcorner(\hbar) \wedge_2 \lrcorner(\delta)) \cdot e^{i2\pi\gamma_{\Upsilon}(\lrcorner(\hbar) \wedge_2 \lrcorner(\delta))} \preceq \max\{\nu_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\gamma_{\Upsilon}\lrcorner(\hbar)}, \nu_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\gamma_{\Upsilon}\lrcorner(\delta)}\}$ and

$\nu_{\Upsilon}(\lrcorner(\hbar) \wedge_3 \lrcorner(\delta)) \cdot e^{i2\pi\gamma_{\Upsilon}(\lrcorner(\hbar) \wedge_3 \lrcorner(\delta))} \preceq \max\{\nu_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\gamma_{\Upsilon}\lrcorner(\hbar)}, \nu_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\gamma_{\Upsilon}\lrcorner(\delta)}\}$.

Thus, Υ is a ComCIFSBS of \mathcal{B}_2 .

Theorem 3.15. *The homomorphic preimage of every ComCIFSBS is a ComCIFSBS.*

Proof. The mapping $\lrcorner : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a homomorphism.

Now, $\lrcorner(\hbar \vee_1 \delta) = \lrcorner(\hbar) \wedge_1 \lrcorner(\delta)$, $\lrcorner(\hbar \vee_2 \delta) = \lrcorner(\hbar) \wedge_2 \lrcorner(\delta)$ and $\lrcorner(\hbar \vee_3 \delta) = \lrcorner(\hbar) \wedge_3 \lrcorner(\delta)$

for all $\hbar, \delta \in \mathcal{B}_1$. Let $\widehat{\Upsilon} = \lrcorner(Z)$, $\widehat{\Upsilon}$ is a ComCIFSBS of \mathcal{B}_2 . Let $\hbar, \delta \in \mathcal{B}_1$. Now, $\widehat{\mu}_Z(\lrcorner(\hbar \vee_1 \delta)) \cdot e^{i2\pi\widehat{\beta}_Z(\lrcorner(\hbar \vee_1 \delta))} = \widehat{\mu}_{\Upsilon}(\lrcorner(\lrcorner(\hbar \vee_1 \delta))) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}(\lrcorner(\lrcorner(\hbar \vee_1 \delta)))} = \widehat{\mu}_{\Upsilon}(\lrcorner(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta))) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}(\lrcorner(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta)))} \succeq$

$\min\{\widehat{\mu}_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}\lrcorner(\hbar)}, \widehat{\mu}_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\widehat{\beta}_{\Upsilon}\lrcorner(\delta)}\} = \min\{\widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar)}, \widehat{\mu}_Z(\delta) \cdot e^{i2\pi\widehat{\beta}_Z(\delta)}\}$.

Thus, $\widehat{\mu}_Z(\lrcorner(\hbar \vee_1 \delta)) \cdot e^{i2\pi\widehat{\beta}_Z(\lrcorner(\hbar \vee_1 \delta))} \succeq \min\{\widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar)}, \widehat{\mu}_Z(\delta) \cdot e^{i2\pi\widehat{\beta}_Z(\delta)}\}$.

Now, $\widehat{\nu}_Z(\lrcorner(\hbar \vee_1 \delta)) \cdot e^{i2\pi\widehat{\gamma}_Z(\lrcorner(\hbar \vee_1 \delta))} = \widehat{\nu}_{\Upsilon}(\lrcorner(\lrcorner(\hbar \vee_1 \delta))) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon}(\lrcorner(\lrcorner(\hbar \vee_1 \delta)))} = \widehat{\nu}_{\Upsilon}(\lrcorner(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta))) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon}(\lrcorner(\lrcorner(\hbar) \wedge_1 \lrcorner(\delta)))}$

$\preceq \max\{\widehat{\nu}_{\Upsilon}\lrcorner(\hbar) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon}\lrcorner(\hbar)}, \widehat{\nu}_{\Upsilon}\lrcorner(\delta) \cdot e^{i2\pi\widehat{\gamma}_{\Upsilon}\lrcorner(\delta)}\} = \max\{\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)}, \widehat{\nu}_Z(\delta) \cdot e^{i2\pi\widehat{\gamma}_Z(\delta)}\}$.

Thus, $\widehat{\nu}_Z(\lrcorner(\hbar \vee_1 \delta)) \cdot e^{i2\pi\widehat{\gamma}_Z(\lrcorner(\hbar \vee_1 \delta))} \preceq \max\{\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)}, \widehat{\nu}_Z(\delta) \cdot e^{i2\pi\widehat{\gamma}_Z(\delta)}\}$.

The mapping $\lrcorner : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a homomorphism.

Now, $\lrcorner((\hbar \vee_1 \bar{\delta})) = \lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})$, $\lrcorner((\hbar \vee_2 \bar{\delta})) = \lrcorner(\hbar) \wedge_2 \lrcorner(\bar{\delta})$ and $\lrcorner((\hbar \vee_3 \bar{\delta})) = \lrcorner(\hbar) \wedge_3 \lrcorner(\bar{\delta})$ for all $\hbar, \bar{\delta} \in \mathcal{B}_1$. Let $\Upsilon = \lrcorner(Z)$, Υ is a ComCIFSBS of \mathcal{B}_2 . Let $\hbar, \bar{\delta} \in \mathcal{B}_1$. Now, $\mu_Z((\hbar \vee_1 \bar{\delta})) \cdot e^{i2\pi\beta_Z((\hbar \vee_1 \bar{\delta}))} = \mu_\Upsilon(\lrcorner((\hbar \vee_1 \bar{\delta}))) \cdot e^{i2\pi\beta_\Upsilon(\lrcorner((\hbar \vee_1 \bar{\delta})))} = \mu_\Upsilon((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta}))) \cdot e^{i2\pi\beta_\Upsilon((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})))} \succeq \min\{\mu_\Upsilon \lrcorner(\hbar) \cdot e^{i2\pi\beta_\Upsilon \lrcorner(\hbar)}, \mu_\Upsilon \lrcorner(\bar{\delta}) \cdot e^{i2\pi\beta_\Upsilon \lrcorner(\bar{\delta})}\} = \min\{\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \mu_Z(\bar{\delta}) \cdot e^{i2\pi\beta_Z(\bar{\delta})}\}$. Thus, $\mu_Z((\hbar \vee_1 \bar{\delta})) \cdot e^{i2\pi\beta_Z((\hbar \vee_1 \bar{\delta}))} \succeq \min\{\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \mu_Z(\bar{\delta}) \cdot e^{i2\pi\beta_Z(\bar{\delta})}\}$. Now, $\nu_Z((\hbar \vee_1 \bar{\delta})) \cdot e^{i2\pi\gamma_Z((\hbar \vee_1 \bar{\delta}))} = \nu_\Upsilon(\lrcorner((\hbar \vee_1 \bar{\delta}))) \cdot e^{i2\pi\gamma_\Upsilon(\lrcorner((\hbar \vee_1 \bar{\delta})))} = \nu_\Upsilon((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta}))) \cdot e^{i2\pi\gamma_\Upsilon((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})))} \preceq \max\{\nu_\Upsilon \lrcorner(\hbar) \cdot e^{i2\pi\gamma_\Upsilon \lrcorner(\hbar)}, \nu_\Upsilon \lrcorner(\bar{\delta}) \cdot e^{i2\pi\gamma_\Upsilon \lrcorner(\bar{\delta})}\} = \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\bar{\delta}) \cdot e^{i2\pi\gamma_Z(\bar{\delta})}\}$. Thus, $\nu_Z((\hbar \vee_1 \bar{\delta})) \cdot e^{i2\pi\gamma_Z((\hbar \vee_1 \bar{\delta}))} \preceq \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\bar{\delta}) \cdot e^{i2\pi\gamma_Z(\bar{\delta})}\}$.

Theorem 3.16. If $\lrcorner : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a homomorphism, then $\lrcorner(\widehat{Z}_{(t,s)})$ is a subbisemiring of ComCIFSBS $\widehat{\Upsilon}$ of \mathcal{B}_2 .

Proof. The mapping $\lrcorner : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a homomorphism.

Now, $\lrcorner((\hbar \vee_1 \bar{\delta})) = \lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})$, $\lrcorner((\hbar \vee_2 \bar{\delta})) = \lrcorner(\hbar) \wedge_2 \lrcorner(\bar{\delta})$ and $\lrcorner((\hbar \vee_3 \bar{\delta})) = \lrcorner(\hbar) \wedge_3 \lrcorner(\bar{\delta})$ for all $\hbar, \bar{\delta} \in \mathcal{B}_1$. Let $\widehat{\Upsilon} = \lrcorner(\widehat{Z})$, $\widehat{\Upsilon}$ is a ComCIFSBS of \mathcal{B}_2 . By Theorem 3.14, $\widehat{\Upsilon}$ is a ComCIFSBS of \mathcal{B}_2 . Let $\widehat{Z}_{(t,s)}$ be any subbisemiring of \widehat{Z} . Suppose that $\hbar, \bar{\delta} \in \widehat{Z}_{(t,s)}$. Then $\hbar \vee_1 \bar{\delta}, \hbar \vee_2 \bar{\delta}$ and $\hbar \vee_3 \bar{\delta} \in \widehat{Z}_{(t,s)}$. Now, $\widehat{\mu}_{\widehat{\Upsilon}}(\lrcorner(\hbar)) \cdot e^{i2\pi\widehat{\beta}_{\widehat{\Upsilon}}(\lrcorner(\hbar))} = \widehat{\mu}_{\widehat{Z}}(\hbar) \cdot e^{i2\pi\widehat{\beta}_{\widehat{Z}}(\hbar)} \succeq t$, $\widehat{\mu}_{\widehat{\Upsilon}}(\lrcorner(\bar{\delta})) \cdot e^{i2\pi\widehat{\beta}_{\widehat{\Upsilon}}(\lrcorner(\bar{\delta}))} = \widehat{\mu}_{\widehat{Z}}(\bar{\delta}) \cdot e^{i2\pi\widehat{\beta}_{\widehat{Z}}(\bar{\delta})} \succeq t$. Thus, $\widehat{\mu}_{\widehat{\Upsilon}}((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta}))) \cdot e^{i2\pi\widehat{\beta}_{\widehat{\Upsilon}}((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})))} \succeq \widehat{\mu}_{\widehat{Z}}((\hbar \vee_1 \bar{\delta})) \cdot e^{i2\pi\widehat{\beta}_{\widehat{Z}}((\hbar \vee_1 \bar{\delta}))} \succeq t$.

Now, $\widehat{\nu}_{\widehat{\Upsilon}}(\lrcorner(\hbar)) \cdot e^{i2\pi\widehat{\gamma}_{\widehat{\Upsilon}}(\lrcorner(\hbar))} = \widehat{\nu}_{\widehat{Z}}(\hbar) \cdot e^{i2\pi\widehat{\gamma}_{\widehat{Z}}(\hbar)} \preceq s$, $\widehat{\nu}_{\widehat{\Upsilon}}(\lrcorner(\bar{\delta})) \cdot e^{i2\pi\widehat{\gamma}_{\widehat{\Upsilon}}(\lrcorner(\bar{\delta}))} = \widehat{\nu}_{\widehat{Z}}(\bar{\delta}) \cdot e^{i2\pi\widehat{\gamma}_{\widehat{Z}}(\bar{\delta})} \preceq s$.

Thus, $\widehat{\nu}_{\widehat{\Upsilon}}((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta}))) \cdot e^{i2\pi\widehat{\gamma}_{\widehat{\Upsilon}}((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})))} \preceq \widehat{\nu}_{\widehat{Z}}((\hbar \vee_1 \bar{\delta})) \cdot e^{i2\pi\widehat{\gamma}_{\widehat{Z}}((\hbar \vee_1 \bar{\delta}))} \preceq s$, for all $\lrcorner(\hbar), \lrcorner(\bar{\delta}) \in \mathcal{B}_2$. Similarly other operations, $\lrcorner(\widehat{Z}_{(t,s)})$ is a subbisemiring of ComCIFSBS $\widehat{\Upsilon}$ of \mathcal{B}_2 .

The mapping $\lrcorner : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a homomorphism.

Now, $\lrcorner((\hbar \vee_1 \bar{\delta})) = \lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})$, $\lrcorner((\hbar \vee_2 \bar{\delta})) = \lrcorner(\hbar) \wedge_2 \lrcorner(\bar{\delta})$ and $\lrcorner((\hbar \vee_3 \bar{\delta})) = \lrcorner(\hbar) \wedge_3 \lrcorner(\bar{\delta})$ for all $\hbar, \bar{\delta} \in \mathcal{B}_1$. Let $\Upsilon = \lrcorner(Z)$, Υ is a ComCIFSBS of \mathcal{B}_2 . By Theorem 3.14, Υ is a ComCIFSBS of \mathcal{B}_2 . Let $Z_{(t,s)}$ be any subbisemiring of Z . Suppose that $\hbar, \bar{\delta} \in Z_{(t,s)}$. Then $\hbar \vee_1 \bar{\delta}, \hbar \vee_2 \bar{\delta}$ and $\hbar \vee_3 \bar{\delta} \in Z_{(t,s)}$. Now, $\mu_\Upsilon(\lrcorner(\hbar)) \cdot e^{i2\pi\beta_\Upsilon(\lrcorner(\hbar))} = \mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)} \succeq t$, $\mu_\Upsilon(\lrcorner(\bar{\delta})) \cdot e^{i2\pi\beta_\Upsilon(\lrcorner(\bar{\delta}))} = \mu_Z(\bar{\delta}) \cdot e^{i2\pi\beta_Z(\bar{\delta})} \succeq t$. Thus, $\mu_\Upsilon((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta}))) \cdot e^{i2\pi\beta_\Upsilon((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})))} \succeq \mu_Z((\hbar \vee_1 \bar{\delta})) \cdot e^{i2\pi\beta_Z((\hbar \vee_1 \bar{\delta}))} \succeq t$. Now, $\nu_\Upsilon(\lrcorner(\hbar)) \cdot e^{i2\pi\gamma_\Upsilon(\lrcorner(\hbar))} = \nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)} \preceq s$, $\nu_\Upsilon(\lrcorner(\bar{\delta})) \cdot e^{i2\pi\gamma_\Upsilon(\lrcorner(\bar{\delta}))} = \nu_Z(\bar{\delta}) \cdot e^{i2\pi\gamma_Z(\bar{\delta})} \preceq s$. Thus, $\nu_\Upsilon((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta}))) \cdot e^{i2\pi\gamma_\Upsilon((\lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})))} \preceq \nu_Z((\hbar \vee_1 \bar{\delta})) \cdot e^{i2\pi\gamma_Z((\hbar \vee_1 \bar{\delta}))} \preceq s$, for all $\lrcorner(\hbar), \lrcorner(\bar{\delta}) \in \mathcal{B}_2$. Similarly other operations, $\lrcorner(Z_{(t,s)})$ is a subbisemiring of ComCIFSBS Υ of \mathcal{B}_2 .

Theorem 3.17. If $\lrcorner : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is any homomorphism, then $\lrcorner(\widehat{Z}_{(t,s)})$ is a subbisemiring of ComCIFSBS Z of \mathcal{B}_1 .

Proof. The mapping $\lrcorner : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be any homomorphism.

Now $\lrcorner((\hbar \vee_1 \bar{\delta})) = \lrcorner(\hbar) \wedge_1 \lrcorner(\bar{\delta})$, $\lrcorner((\hbar \vee_2 \bar{\delta})) = \lrcorner(\hbar) \wedge_2 \lrcorner(\bar{\delta})$ and $\lrcorner((\hbar \vee_3 \bar{\delta})) = \lrcorner(\hbar) \wedge_3 \lrcorner(\bar{\delta})$ for all $\hbar, \bar{\delta} \in \mathcal{B}_1$. Let $\widehat{\Upsilon} = \lrcorner(\widehat{Z})$, $\widehat{\Upsilon}$ is a ComCIFSBS of \mathcal{B}_2 . By Theorem 3.15, \widehat{Z} is a ComCIFSBS of \mathcal{B}_1 . Let $\lrcorner(\widehat{Z}_{(t,s)})$ be a subbisemiring of $\widehat{\Upsilon}$. Suppose that $\lrcorner(\hbar), \lrcorner(\bar{\delta}) \in \lrcorner(\widehat{Z}_{(t,s)})$. Now, $\lrcorner((\hbar \vee_1 \bar{\delta}))$, $\lrcorner((\hbar \vee_2 \bar{\delta}))$ and $\lrcorner((\hbar \vee_3 \bar{\delta})) \in \lrcorner(\widehat{Z}_{(t,s)})$. Now, $\widehat{\mu}_{\widehat{Z}}(\hbar) \cdot e^{i2\pi\widehat{\beta}_{\widehat{Z}}(\hbar)} = \widehat{\mu}_{\widehat{\Upsilon}}(\lrcorner(\hbar)) \cdot e^{i2\pi\widehat{\beta}_{\widehat{\Upsilon}}(\lrcorner(\hbar))}$.

$e^{i2\pi\widehat{\beta}_\Upsilon(\triangleleft(\hbar))} \succeq t, \widehat{\mu}_Z(\overline{\delta}) \cdot e^{i2\pi\widehat{\beta}_Z(\overline{\delta})} = \widehat{\mu}_\Upsilon(\triangleleft(\overline{\delta})) \cdot e^{i2\pi\widehat{\beta}_\Upsilon(\triangleleft(\overline{\delta}))} \succeq t$. Thus, $\widehat{\mu}_Z((\hbar \vee_1 \overline{\delta})) \cdot e^{i2\pi\widehat{\beta}_Z((\hbar \vee_1 \overline{\delta}))} \succeq \min\{\widehat{\mu}_Z(\hbar) \cdot e^{i2\pi\widehat{\beta}_Z(\hbar)}, \widehat{\mu}_Z(\overline{\delta}) \cdot e^{i2\pi\widehat{\beta}_Z(\overline{\delta})}\} \succeq t$. Now, $\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)} = \widehat{\nu}_\Upsilon(\triangleleft(\hbar)) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon(\triangleleft(\hbar))} \preceq s, \widehat{\nu}_Z(\overline{\delta}) \cdot e^{i2\pi\widehat{\gamma}_Z(\overline{\delta})} = \widehat{\nu}_\Upsilon(\triangleleft(\overline{\delta})) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon(\triangleleft(\overline{\delta}))} \preceq s$. Thus, $\widehat{\nu}_Z((\hbar \vee_1 \overline{\delta})) \cdot e^{i2\pi\widehat{\gamma}_Z((\hbar \vee_1 \overline{\delta}))} = \widehat{\nu}_\Upsilon((\triangleleft(\hbar) \wedge_1 \triangleleft(\overline{\delta}))) \cdot e^{i2\pi\widehat{\gamma}_\Upsilon((\triangleleft(\hbar) \wedge_1 \triangleleft(\overline{\delta})))} \preceq \max\{\widehat{\nu}_Z(\hbar) \cdot e^{i2\pi\widehat{\gamma}_Z(\hbar)}, \widehat{\nu}_Z(\overline{\delta}) \cdot e^{i2\pi\widehat{\gamma}_Z(\overline{\delta})}\} \preceq s$, for all $\hbar, \overline{\delta} \in \mathcal{B}_1$. Similarly other operations, $\widehat{Z}_{(t,s)}$ is a subbisemiring of ComCIFSBS Z of \mathcal{B}_1 .

The mapping $\triangleleft: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be any homomorphism.

We have $\triangleleft((\hbar \vee_1 \overline{\delta})) = \triangleleft(\hbar) \wedge_1 \triangleleft(\overline{\delta}), \triangleleft((\hbar \vee_2 \overline{\delta})) = \triangleleft(\hbar) \wedge_2 \triangleleft(\overline{\delta})$ and $\triangleleft((\hbar \vee_3 \overline{\delta})) = \triangleleft(\hbar) \wedge_3 \triangleleft(\overline{\delta})$ for all $\hbar, \overline{\delta} \in \mathcal{B}_1$. Let $\Upsilon = \triangleleft(Z), \Upsilon$ is a ComCIFSBS of \mathcal{B}_2 . By Theorem 3.15, Z is a ComCIFSBS of \mathcal{B}_1 . Let $\triangleleft(Z_{(t,s)})$ be a subbisemiring of Υ . Suppose that $\triangleleft(\hbar), \triangleleft(\overline{\delta}) \in \triangleleft(Z_{(t,s)})$. Now, $\triangleleft((\hbar \vee_1 \overline{\delta})), \triangleleft((\hbar \vee_2 \overline{\delta}))$ and $\triangleleft((\hbar \vee_3 \overline{\delta})) \in \triangleleft(Z_{(t,s)})$. Now, $\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)} = \mu_\Upsilon(\triangleleft(\hbar)) \cdot e^{i2\pi\beta_\Upsilon(\triangleleft(\hbar))} \succeq t, \mu_Z(\overline{\delta}) \cdot e^{i2\pi\beta_Z(\overline{\delta})} = \mu_\Upsilon(\triangleleft(\overline{\delta})) \cdot e^{i2\pi\beta_\Upsilon(\triangleleft(\overline{\delta}))} \succeq t$. Thus, $\mu_Z((\hbar \vee_1 \overline{\delta})) \cdot e^{i2\pi\beta_Z((\hbar \vee_1 \overline{\delta}))} \succeq \min\{\mu_Z(\hbar) \cdot e^{i2\pi\beta_Z(\hbar)}, \mu_Z(\overline{\delta}) \cdot e^{i2\pi\beta_Z(\overline{\delta})}\} \succeq t$. Now, $\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)} = \nu_\Upsilon(\triangleleft(\hbar)) \cdot e^{i2\pi\gamma_\Upsilon(\triangleleft(\hbar))} \preceq s, \nu_Z(\overline{\delta}) \cdot e^{i2\pi\gamma_Z(\overline{\delta})} = \nu_\Upsilon(\triangleleft(\overline{\delta})) \cdot e^{i2\pi\gamma_\Upsilon(\triangleleft(\overline{\delta}))} \preceq s$. Thus, $\nu_Z((\hbar \vee_1 \overline{\delta})) \cdot e^{i2\pi\gamma_Z((\hbar \vee_1 \overline{\delta}))} = \nu_\Upsilon((\triangleleft(\hbar) \wedge_1 \triangleleft(\overline{\delta}))) \cdot e^{i2\pi\gamma_\Upsilon((\triangleleft(\hbar) \wedge_1 \triangleleft(\overline{\delta})))} \preceq \max\{\nu_Z(\hbar) \cdot e^{i2\pi\gamma_Z(\hbar)}, \nu_Z(\overline{\delta}) \cdot e^{i2\pi\gamma_Z(\overline{\delta})}\} \preceq s$, for all $\hbar, \overline{\delta} \in \mathcal{B}_1$. Similarly other operations, $Z_{(t,s)}$ is a subbisemiring of ComCIFSBS Z of \mathcal{B}_1 .

References

- [1] L. A. Zadeh, Fuzzy sets, Information and Control, 8, (1965), 338-353.
- [2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1), (1986) 87-96.
- [3] R. R. Yager, Pythagorean membership grades in multi criteria decision-making, IEEE Trans. Fuzzy Systems, 22, (2014), 958-965.
- [4] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani and T. Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision making problems, Journal of Intelligent and Fuzzy Systems, 36, (2019), 2829-284.
- [5] B.C. Cuong and V. Kreinovich, Picture fuzzy sets a new concept for computational intelligence problems, in Proceedings of 2013 Third World Congress on Information and Communication Technologies (WICT 2013), IEEE, (2013), 1-6.
- [6] F. Smarandache, A unifying field in logics Neutrosophy Neutrosophic Probability, Set and Logic, Rehoboth American Research Press (1999).
- [7] Daniel Ramot, Ron Milo, Menahem Friedman, and Abraham Kandel, Complex fuzzy set, IEEE Transactions on Fuzzy System, 10(2), 2002.
- [8] S.J Golan, Semirings and their Applications, Kluwer Academic Publishers, London, 1999.
- [9] Faward Hussian, Raja Muhammad Hashism, Ajab Khan, Muhammad Naeem, Generalization of bisemirings, International Journal of Computer Science and Information Security, 14(9), (2016), 275-289.

- [10] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. Intelligent Technologies Bangkok, Thailand, (2000) 307-312.
- [11] Javed Ahsan, John N. Mordeson, and Muhammad Shabir, Fuzzy Semirings with Applications to Automata Theory, Springer Heidelberg New York Dordrecht, London, 2012.
- [12] M.K Sen, S. Ghosh An introduction to bisemirings, Southeast Asian Bulletin of Mathematics, 28(3), (2001), 547-559.
- [13] Palanikumar M, Arulmozhi K, On intuitionistic fuzzy normal subbisemirings of bisemirings, Nonlinear studies, 28(3), 2021, 717-721.
- [14] Palanikumar M, Selvi G, Ganeshsree Selvachandran and Tan S.L, New approach to bisemiring theory via the bipolar-valued neutrosophic normal sets, Neutrosophic Sets and Systems, 55, 427-450, 2023.
- [15] SG Quek, H Garg, G Selvachandran, M Palanikumar, K Arulmozhi, VIKOR and TOPSIS framework with a truthful-distance measure for the (t, s)-regulated interval-valued neutrosophic soft set, Soft Computing, 1–27, 2023.
- [16] M Palanikumar, K Arulmozhi, A Iampan, Multi criteria group decision making based on VIKOR and TOPSIS methods for Fermatean fuzzy soft with aggregation operators, ICIC Express Letters 16 (10), (2022), 1129–1138.
- [17] M Palanikumar, K Arulmozhi, MCGDM based on TOPSIS and VIKOR using Pythagorean neutrosophic soft with aggregation operators, Neutrosophic Sets and Systems, (2022), 538–555.
- [18] M Palanikumar, S Broumi, Square root (l_1, l_2) phantine neutrosophic normal interval-valued sets and their aggregated operators in application to multiple attribute decision making, International Journal of Neutrosophic Science, 4, (2022).
- [19] M Palanikumar, K Arulmozhi, Novel possibility Pythagorean interval valued fuzzy soft set method for a decision making, TWMS J. App. and Eng. Math., 13(1), (2023), 327–340.
- [20] M Palanikumar, K Arulmozhi, Novel possibility Pythagorean interval valued fuzzy soft set method for a decision making, TWMS J. App. and Eng. Math., 13(1), (2023), 327–340.
- [21] M Palanikumar, N Kausar, H Garg, A Iampan, S Kadry, M Sharaf, Medical robotic engineering selection based on square root neutrosophic normal interval-valued sets and their aggregated operators, AIMS Mathematics, 8(8), (2023), 17402–17432.
- [22] Raed Hatamleh, Ayman Hazaymeh, Finding Minimal Units In Several Two-Fold Fuzzy Finite Neutrosophic Rings, Neutrosophic Sets and Systems, 70, (2024), 1-16.
- [23] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Mowafaq Omar Al-Qadri, Abdallah Al-Husban, On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For $2^n \times 3$, Neutrosophic Sets and Systems, 68, (2024), 8-25.

- [24] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban, An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, *Neutrosophic Sets and Systems*, 67, (2024), 169-178.
- [25] Raed Hatamleh , Ayman Hazaymeh, On Some Topological Spaces Based On Symbolic n -Plithogenic Intervals, *International Journal of Neutrosophic Science*, 25(1), (2025), 23-37.
- [26] Raed Hatamleh , Ayman Hazaymeh, On The Topological Spaces of Neutrosophic Real Intervals, *International Journal of Neutrosophic Science*, 25(1), (2025), 130-136.
- [27] Raed Hatamleh , Ayman Hazaymeh, The Properties of Two-Fold Algebra Based on the n -standard Fuzzy Number Theoretical System, *International Journal of Neutrosophic Science*, 25(1), (2025), 172-178.
- [28] Dima Alrwashdeh , Talat Alkhouli , Ahmed Soiman Rashed Alhawiti , Ali Allouf , Hussein Edduweh , Abdallah Al-Husban, On Two Novel Generalized Versions of Diffie-Hellman Key Exchange Algorithm Based on Neutrosophic and Split-Complex Integers and their Complexity Analysis, *International Journal of Neutrosophic Science*, 25(2), (2025) 01-10.
- [29] Isra Al-Shbeil , Ibraheem Abu Falahah , Talat Alkhouli , Ahmed Soiman Rashed Alhawiti , Jenan Shtayat , Abdallah Al-Husban, On the Two-Fold Maximal Units in Some Two-Fold Finite Neutrosophic Rings Modulo Integers for $2 \leq 5$, *International Journal of Neutrosophic Science*, 25(2), (2025), 155-164.
- [30] Ahmad A. Abubaker , Raed Hatamleh , Khaled Matarneh , Abdallah Al-Husban, On the Irreversible k -Threshold Conversion Number for Some Graph Products and Neutrosophic Graphs, *International Journal of Neutrosophic Science*, 25(2), (2025), 197-205.
- [31] Hazaymeh, Ayman. Time Effective Fuzzy Soft Set and Its Some Applications with and Without a Neutrosophic. *International Journal of Neutrosophic Science*, 2024, 129-149.
- [32] Sivakumar, C. , Omar, Mowafaq. , shihadeh, Abdallah. , Atallah, Ahmed. , Al-Husban, Abdallah. , Maragatha, P. , Rajesh, N. , Palanikumar, M.. q -rung square root interval-valued neutrosophic sets with respect to aggregated operators using multiple attribute decision making. *International Journal of Neutrosophic Science*, 2024, 154-174.