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Bounded Closed Interval-Valued Decagonal Fuzzy Number and its Application

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Abstract:
In this paper, we introduced the notation of Algebraic operations of bounded closed interval-valued decagonal fuzzy number. To overcome the uncertainty here we used fuzzy numbers. Many researchers have focused their research with Triangular and Trapezoidal fuzzy numbers, in some cases it is not fit to the problem when the vagueness arises in ten different opinions so our motto is to develop the theorems for bounded closed interval using Algebraic operations such as Addition, Subtraction, Multiplication and Division via Decagonal Fuzzy Number.

Keywords: Fuzzy sets, Fuzzy numbers, Decagonal Fuzzy numbers, Bounded Closed interval

1. Introduction

The fuzzy set theory was first introduced by Zadeh which helps to deal uncertainty in the problem [1]. Gorzalczany [2] and Turksen [3] introduced the notation of fuzzy set theory. Dubois and Prade have defined fuzzy numbers as a fuzzy subset of the real line [9]. A fuzzy number is a multi-valued quantity whose value is precise, rather than a single-valued quantity. Most of the researchers have used triangular and trapezoidal fuzzy numbers to handle imprecision in real life situations [10, 11, 12, 15-18]. Hexagonal, heptagonal, nonagonal, decagonal fuzzy numbers have also been introduced to tackle the vagueness [5, 13, 14, 19, 20]. Wang and Li [4] in 1998, defined interval-valued fuzzy numbers and gave their extended operations. Karthik et.al [6] developed a fuzzy decision-making system using triangular fuzzy numbers to study the impact of pesticides on human health. Selvaraj et.al proposed linear and nonlinear hexagonal fuzzy number with symmetry and asymmetry for solving transportation problem, and their respective alpha cuts also have been derived [8]. Karthik et.al introduced heptagonal fuzzy number for both symmetrical and asymmetrical structures, derived alpha cuts to solve assignment problem. Felix et.al [5] has introduced a new operation on a decagonal fuzzy number (DFN) under certain linguistic environment is presented.

This paper consists of four sections. The first section describes the introduction and basic definitions of fuzzy. We introducing the definition for Bounded closed interval-valued Decagonal fuzzy numbers in the second section. Section three seeks about the theorem for Bounded closed interval-valued Decagonal fuzzy numbers. Finally, the last section is the conclusion based on theorems part.

2. Preliminaries

This section describes the preliminary definitions of fuzzy sets,fuzzy numbers and Decagonal fuzzy numbers.

2.1 Fuzzy Set

A fuzzy set $\tilde{A}$ is a subset of a universe of discourse *X*, which is characterized by a membership function $\mu_{\tilde{A}}(\theta)$ representing a mapping $\mu_{\tilde{A}}: X \to [0,1]$. The function value of $\mu_{\tilde{A}}(\theta)$ is called the membership value, which represents the degree of truth that θ is an element of the fuzzy set $\tilde{A}$.

2.2 Fuzzy numbers

A fuzzy set $\tilde{A}$ defined on the set of real numbers R is said to be a fuzzy number and its membership function $\tilde{A}: R \to [0,1]$ has the following characteristics,

(i) $\tilde{A}$ is convex.
$\mu_{\tilde{A}}(\lambda\theta_1 + (1-\lambda)\theta_2) \geq \min(\mu_{\tilde{A}}(\theta_1), \mu_{\tilde{A}}(\theta_2)), \forall \theta \in [\theta_1, \theta_2], \lambda \in [0,1]$.

(ii) $\tilde{A}$ is normal if $\max \mu_{\tilde{A}}(\theta) = 1$.

(iii) $\tilde{A}$ is piecewise continuous.

2.3 Decagonal Fuzzy Number

A Decagonal fuzzy number $\tilde{D}$ =(a,b,c,d,e,f,g,h,i,j) and the membership function is defined as

$$\mu_{\tilde{D}}(\theta)=\begin{cases} \frac{1}{4}\frac{(\theta-a)}{(b-a)} & a \leq \theta \leq b \\ \frac{1}{4}+\frac{1}{4}\frac{(\theta-b)}{(c-b)} & b \leq \theta \leq c \\ \frac{1}{2}+\frac{1}{4}\frac{(\theta-c)}{(d-c)} & c \leq \theta \leq d \\ \frac{3}{4}+\frac{1}{4}\frac{(\theta-d)}{(e-d)} & d \leq \theta \leq e \\ 1 & e \leq \theta \leq f \\ 1-\frac{1}{4}\frac{(\theta-f)}{(g-f)} & f \leq \theta \leq g \\ \frac{3}{4}-\frac{1}{4}\frac{(\theta-g)}{(h-g)} & g \leq \theta \leq h \\ \frac{1}{2}-\frac{1}{4}\frac{(\theta-h)}{(i-h)} & h \leq \theta \leq i \\ \frac{1}{4}\frac{(j-\theta)}{(j-i)} & i \leq \theta \leq j \\ 0 & \text{otherwise} \end{cases}$$

3. Bounded closed interval-valued Decagonal Fuzzy number

The definitions of Bounded closed interval-valued Decagonal Fuzzy number are given below

Definition 3.1

$\boldsymbol{A} = \left[\tilde{\tilde{\boldsymbol{A}}}^{\boldsymbol{L}}, \tilde{\tilde{\boldsymbol{A}}}^{\boldsymbol{U}}\right] \boldsymbol{or} \left[\tilde{\tilde{\boldsymbol{A}}}^{\boldsymbol{1}}, \tilde{\tilde{\boldsymbol{A}}}^{\boldsymbol{2}}\right] \in [\boldsymbol{I}]^{\boldsymbol{R}}$,If A is a interval-valued Decagonal fuzzy set of R is called a bounded closed interval-valued decagonal fuzzy number (BCIDFN). If $\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U \in Bc[R]$. The interval-valued decagonal fuzzy number $\tilde{\tilde{A}}$ has two elements. Lower fuzzy number $\tilde{\tilde{A}}^L$ and upper fuzzy number $\tilde{\tilde{A}}^U$.

The interval valued decagonal fuzzy numbers $\tilde{\tilde{A}}$ can be represented as

$$\tilde{\tilde{A}} = \begin{bmatrix} (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \end{bmatrix}$$

Where $(\alpha_1^{1.1} \le \alpha_2^{1.2} \le \alpha_3^{1.3} \le \alpha_4^{1.4} \le \alpha_5^{1.5} \le \alpha_6^{1.6} \le \alpha_7^{1.7} \le \alpha_8^{1.8} \le \alpha_9^{1.9} \le \alpha_{10}^{1.0})$

$$(\alpha_1^{2.1} \le \alpha_2^{2.2} \le \alpha_3^{2.3} \le \alpha_4^{2.4} \le \alpha_5^{2.5} \le \alpha_6^{2.6} \le \alpha_7^{2.7} \le \alpha_8^{2.8} \le \alpha_9^{2.9} \le \alpha_{10}^{2.10})$$

$$0 < \hat{w}\tilde{\tilde{A}}^L \le \hat{w}\tilde{\tilde{A}}^U \le 1 \quad \text{dem} \ \tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U \ \text{de}$$

$$\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U \in B \subset [R]$$

The set of all BCIDFNS on R is denoted by $B \subset [R]$.

Definition 3.2

Let

$$\tilde{\tilde{\boldsymbol{A}}} = \begin{bmatrix} (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \end{bmatrix} \in BC[R]$$

If $\tilde{\tilde{A}}$ is called a positive bounded closed interval- valued decagonal fuzzy number (BCIDFNS) if $A^U(x) = 0$ each $x \le 0$. All positive BCIDFNS is denoted by $B \subset [R^+]$

If $\tilde{\tilde{A}}$ is called negative bounded closed interval- valued decagonal fuzzy number (BCIDFNS) if $A^L(x) = 0$ each $x \ge 0$. All negative BCIDFNS is denoted by $B \subset [R^-]$.

Definition 3.3 (Algebraic operations on BCIDFNS)

Let $\circledast \in \{\oplus, \ominus, \otimes, \oslash\}$ be the binary operation on R for $\widetilde{\widetilde{A^1}}, \widetilde{\widetilde{B^1}} \in B \subset [R]$

$\widetilde{\widetilde{A^1}} \circledast \widetilde{\widetilde{B^1}}$ is defined as follows,

where

$$\widetilde{\widetilde{A^1}} = \begin{bmatrix} (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \end{bmatrix}$$

$$\widetilde{\widetilde{B^1}} = \begin{bmatrix} (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L, \\ (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \end{bmatrix}$$

$$\left(\widetilde{\widetilde{A^1}} \circledast \widetilde{\widetilde{B^1}}\right)(z) = \underset{Z = X \circledast Y}{V} \left(\widetilde{\widetilde{A^1}}\,(X) \wedge \widetilde{\widetilde{B^1}}\,(y)\right), z \in R$$

where $\widetilde{\widetilde{\widetilde{A^1}}}\,(X) \wedge \widetilde{\widetilde{\widetilde{B^1}}}\,(y) = \left[\widetilde{\widetilde{\widetilde{A^1}}}(x), \widetilde{\widetilde{\widetilde{A^2}}}(x)\right] \wedge \left[\widetilde{\widetilde{\widetilde{B^1}}}(x), \widetilde{\widetilde{\widetilde{B^2}}}(x)\right]$

Theorem 1

Let $\widetilde{\widetilde{\widetilde{\mathrm{A}^1}}}, \widetilde{\widetilde{\widetilde{\mathrm{B}^1}}} \in B \ \in C[R]$ then

$(i)\widetilde{\widetilde{\widetilde{\mathrm{A}^1}}} \ \circledast \ \widetilde{\widetilde{\widetilde{\mathrm{B}^1}}} \in B\ C[R]\ for\ \circledast \in \{\oplus, \ominus, \otimes,\}$

$(ii)\widetilde{\widetilde{\widetilde{\mathrm{A}^1}}} \ \emptyset\ \widetilde{\widetilde{\widetilde{\mathrm{B}^1}}} \in B\ C\ [R]\ for\ B \in B\ C\ [R^+]\ or\ B \in B\ C\ [R^-]$ Where

$$\widetilde{\widetilde{\widetilde{\mathrm{A}^1}}} = \left[(a_1^L, a_2^L, a_3^L, a_4^L, a_5^L, a_6^L, a_7^L, a_8^L, a_9^L, a_{10}^L)(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})\right]$$

and

$$\widetilde{\widetilde{\widetilde{\mathrm{B}^1}}} = \left[\left((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}\right)(b_1^U, b_2^U, b_3^U, b_4^U, b_5^U, b_6^U, b_7^U, b, b_9^U, b_{10}^U)\right]$$

Proof: (i) For $\circledast \in \{\oplus, \ominus, \otimes,\}\ and\ each\ Z\ \in R$

$$\left(\widetilde{\widetilde{\widetilde{\mathrm{A}^1}}} \ \circledast\ \widetilde{\widetilde{\widetilde{\mathrm{B}^1}}}\right)(Z)$$

$$= \bigvee_{Z = X \circledast Y} \begin{matrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})\,(x) \\ \wedge\left((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}\right)(\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})(y) \end{matrix}$$

$$= \begin{matrix} \bigvee_{Z = X \circledast Y} \left(((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})(x) \wedge ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})(y)\right), \\ \bigvee_{Z = X \circledast Y} ((\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})(x) \wedge (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})(y)) \end{matrix}$$

$$= \begin{pmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})(z), \\ ((\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})(\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})(z)) \end{pmatrix}$$

On the other hand

$$\left(\widetilde{\widetilde{\widetilde{\mathrm{A}^1}}} \circledast \widetilde{\widetilde{\widetilde{\mathrm{B}^1}}}\right)(Z)$$

$$= \begin{pmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \circledast ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})(z) \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \circledast (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})(z) \end{pmatrix}$$

Hence

$$\left(\widetilde{\widetilde{\widetilde{\mathrm{A}^1}}} \circledast \widetilde{\widetilde{\widetilde{\mathrm{B}^1}}}\right)^L = \left(((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \circledast (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})\right)$$

$$\left(\widetilde{\widetilde{\widetilde{\mathrm{A}^2}}} \circledast \widetilde{\widetilde{\widetilde{\mathrm{B}^2}}}\right)^U = \left((\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \circledast \left(\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}\right)\right)$$

That $A^L \circledast \tilde{\tilde{B}}^L, A^U \circledast \tilde{\tilde{B}}^U \in B\ C(R)\ \ this\ shows\ A \circledast \tilde{\tilde{B}} \in B\ C\ [R]$

$(ii)\left(\widetilde{\widetilde{\widetilde{\mathrm{A}^1}}} \div \widetilde{\widetilde{\widetilde{\mathrm{B}^1}}}\right) \in B\ C\ [R]\ for\ B \in B\ C[R^+]\ or\ B \in B\ C[R^-]$

$\left(\widetilde{\widetilde{\mathrm{A}^1}} \ominus \widetilde{\widetilde{\mathrm{B}^1}}\right)(Z)$

$= \underset{Z = X \ominus Y}{V} \Big(((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})(x)$
$\wedge ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})(\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})(y)\Big)$

$= \underset{Z = X \ominus Y}{V} \begin{matrix}[((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})(x) \wedge ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})(y)], \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})(x) \wedge (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})(y)]\end{matrix}$

$= \underset{Z = X \ominus Y}{V} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})(x)$
$\wedge ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})(y)],$

$\underset{Z = X \ominus Y}{V} [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})(x)$
$\wedge (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})(y)]$

$= \begin{bmatrix}((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})(z), \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})(z)\end{bmatrix}$

On the other hand

$$\left(\widetilde{\widetilde{\mathrm{A}^1}} \ominus \widetilde{\widetilde{\mathrm{B}^1}}\right)(z) = \left[\left(\widetilde{\widetilde{\mathrm{A}^1}} \ominus \widetilde{\widetilde{\mathrm{B}^1}}\right)^L(z), \left(\widetilde{\widetilde{\mathrm{A}^2}} \ominus \widetilde{\widetilde{\mathrm{B}^2}}\right)^U(z)\right]$$

$$\left(\widetilde{\widetilde{\mathrm{A}^1}} \ominus \widetilde{\widetilde{\mathrm{B}^1}}\right)^L = \left[\left((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}\right)\right) \ominus \left((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}\right)(z)\right]$$

$$\left(\widetilde{\widetilde{\mathrm{A}^2}} \ominus \widetilde{\widetilde{\mathrm{B}^2}}\right)^U = \left[(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus \left(\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}\right)(z)\right]$$

It follows that

$$\widetilde{\widetilde{\mathrm{A}^1}}^L \ominus \widetilde{\widetilde{\mathrm{B}^1}}^L, \widetilde{\widetilde{\mathrm{A}^2}}^U \ominus \widetilde{\widetilde{\mathrm{B}^2}}^U \in B\ C\ (R)\ this\ shows\ \mathrm{A}^1 \ominus \mathrm{B}^1 \in B\ C\ [R]$$

Theorem 2

Let $A^1, B^1, \Gamma^1 \in B\ C\ [R]$ then

A^1
$= \left[(\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})\right]$

B^1
$= \left[(\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})\left(\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}\right)\right]$

Γ^1
$= \left[\left(\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}\right), \left(\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}\right)\right]$

Proof: ${A^1}^L \wedge {B^1}^L, {A^2}^U \wedge {B^2}^U$

$(i) A^1 + B^1 = [(A^1 + B^1)^L, (A^2 + B^2)^U] = \left[{A^1}^L + {B^1}^L, {A^2}^U + {B^2}^U\right]$

$$= \left[{B^1}^L + {A^1}^L, {B^2}^U + {A^2}^U\right]$$

$$= [(B^1 + A^1)^L, (B^2 + A^2)^U]$$

$$= B^1 + A^1$$

$$A + B = A + B = [(\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) + (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}))]^L$$

$$[(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) + (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U$$

$$[(\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})) +_1 (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}))]^L$$

$$[(\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) + (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}))]^U$$

$$= B + A$$

(ii) $A^1 \otimes B^1 = B^1 \otimes A^1$

$$A^1 \otimes B^1 = [(A^1 \otimes B^1)^L, (A^2 \otimes B^2)^U] = \left[{A^1}^L \otimes {B^1}^L, {A^2}^U \otimes {B^2}^U\right]$$

$$= \left[{B^1}^L \otimes {A^1}^L, {B^2}^U \otimes {A^2}^U\right]$$

$$= [(B^1 \otimes A^1)^L, (B^2 \otimes A^2)^U]$$

$$= B^1 \otimes A^1$$

$$A^1 \otimes B^1 = [(\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})]^L$$

$$[(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \otimes (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U$$

$$= [(\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \otimes_{\square}, (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})]^L$$

$$[(\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \otimes_{\square} (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})]^U$$

$$= B^1 \otimes A^1$$

(iii) $(A^1 \oplus B^1) \oplus \Gamma^1 = A^1 \oplus (B^1 \oplus \Gamma^1)$

$$(A^1 \oplus B^1) \oplus \Gamma^1 = \left[(A^1 \oplus B^1)^L \oplus {\Gamma^1}^L, (A^2 \oplus B^2)^U \oplus {\Gamma^2}^U\right]$$

$$= \begin{bmatrix} [(\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^{\square} \oplus (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}))]^L \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L, \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^{\square} \oplus (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^{\square} \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^{\square} \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \left[{A^1}^L \oplus (B^1 \oplus \Gamma^1)^L, {A^2}^U \oplus (B^2 \oplus \Gamma^2)^U\right]$$

$$(A^1 \otimes B^1) \otimes \Gamma^1 = A^1 \otimes (B^1 \otimes \Gamma^1)$$

$$(A^1 \otimes B^1) \otimes \Gamma^1 = \left[(A^1 \otimes B^1)^L \otimes {\Gamma^1}^L, (A^2 \otimes B^2)^U \otimes {\Gamma^2}^U\right]$$

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$$= \begin{bmatrix} [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^{\square}\otimes(\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})]^L\otimes(\gamma_1^{1.1},\gamma_2^{1.2},\gamma_3^{1.3},\gamma_4^{1.4},\gamma_5^{1.5},\gamma_6^{1.6},\gamma_7^{1.7},\gamma_8^{1.8},\gamma_9^{1.9},\gamma_{10}^{1.10}))^L, \\ [(\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^{\square}\otimes(\beta_1^{2.1},\beta_2^{2.2},\beta_3^{2.3},\beta_4^{2.4},\beta_5^{2.5},\beta_6^{2.6},\beta_7^{2.7},\beta_8^{2.8},\beta_9^{2.9},\beta_{10}^{2.10})]^U\otimes(\gamma_1^{2.1},\gamma_2^{2.2},\gamma_3^{2.3},\gamma_4^{2.4},\gamma_5^{2.5},\gamma_6^{2.6},\gamma_7^{2.7},\gamma_8^{2.8},\gamma_9^{2.9},\gamma_{10}^{2.10})^U \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L\otimes(\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0}))^L\otimes(\gamma_1^{1.1},\gamma_2^{1.2},\gamma_3^{1.3},\gamma_4^{1.4},\gamma_5^{1.5},\gamma_6^{1.6},\gamma_7^{1.7},\gamma_8^{1.8},\gamma_9^{1.9},\gamma_{10}^{1.10})^L, \\ ((\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U\otimes(\beta_1^{2.1},\beta_2^{2.2},\beta_3^{2.3},\beta_4^{2.4},\beta_5^{2.5},\beta_6^{2.6},\beta_7^{2.7},\beta_8^{2.8},\beta_9^{2.9},\beta_{10}^{2.10})^U\otimes(\gamma_1^{2.1},\gamma_2^{2.2},\gamma_3^{2.3},\gamma_4^{2.4},\gamma_5^{2.5},\gamma_6^{2.6},\gamma_7^{2.7},\gamma_8^{2.8},\gamma_9^{2.9},\gamma_{10}^{2.10}))^U \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L\otimes(\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^{\square}\otimes(\gamma_1^{1.1},\gamma_2^{1.2},\gamma_3^{1.3},\gamma_4^{1.4},\gamma_5^{1.5},\gamma_6^{1.6},\gamma_7^{1.7},\gamma_8^{1.8},\gamma_9^{1.9},\gamma_{10}^{1.10})]^L, \\ (\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U\otimes(\beta_1^{2.1},\beta_2^{2.2},\beta_3^{2.3},\beta_4^{2.4},\beta_5^{2.5},\beta_6^{2.6},\beta_7^{2.7},\beta_8^{2.8},\beta_9^{2.9},\beta_{10}^{2.10})^{\square}\otimes(\gamma_1^{2.1},\gamma_2^{2.2},\gamma_3^{2.3},\gamma_4^{2.4},\gamma_5^{2.5},\gamma_6^{2.6},\gamma_7^{2.7},\gamma_8^{2.8},\gamma_9^{2.9},\gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \left[{A^1}^L\otimes(B^1\otimes\Gamma^1)^L, {A^2}^U\otimes(B^2\otimes\Gamma^2)^U\right]$$

Theorem 3

If $A^1, B^1, \Gamma^1 \in B\ C[R^+]$, then

$$A^1 = [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L,(\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U]$$

$$B^1 = [(((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0}))^L,(((\beta_1^{2.1},\beta_2^{2.2},\beta_3^{2.3},\beta_4^{2.4},\beta_5^{2.5},\beta_6^{2.6},\beta_7^{2.7},\beta_8^{2.8},\beta_9^{2.9},\beta_{10}^{2.10}))^U]$$

$$\Gamma^1 = \left((\gamma_1^{1.1},\gamma_2^{1.2},\gamma_3^{1.3},\gamma_4^{1.4},\gamma_5^{1.5},\gamma_6^{1.6},\gamma_7^{1.7},\gamma_8^{1.8},\gamma_9^{1.9},\gamma_{10}^{1.10})\right)^L,\left((\gamma_1^{2.1},\gamma_2^{2.2},\gamma_3^{2.3},\gamma_4^{2.4},\gamma_5^{2.5},\gamma_6^{2.6},\gamma_7^{2.7},\gamma_8^{2.8},\gamma_9^{2.9},\gamma_{10}^{2.10})\right)^U$$

$$A^1\otimes(B^1\oplus\Gamma^1) = (A^1\otimes B^1)\oplus(A^1\otimes C^1)$$

$$A^1\otimes(B^1\oplus\Gamma^1) = ({A^1}^L\otimes(B^1\oplus C^1)^L, {A^2}^U\otimes(B^2\oplus C^2)^U)$$

$$= \begin{bmatrix} (\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L\otimes[(\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})\oplus(\gamma_1^{1.1},\gamma_2^{1.2},\gamma_3^{1.3},\gamma_4^{1.4},\gamma_5^{1.5},\gamma_6^{1.6},\gamma_7^{1.7},\gamma_8^{1.8},\gamma_9^{1.9},\gamma_{10}^{1.10},\gamma_5^1,\gamma_6^1,\gamma_7^1,\gamma_8^1,\gamma_9^1,\gamma_{10}^1)]^L, \\ (\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U\otimes[((\beta_1^{2.1},\beta_2^{2.2},\beta_3^{2.3},\beta_4^{2.4},\beta_5^{2.5},\beta_6^{2.6},\beta_7^{2.7},\beta_8^{2.8},\beta_9^{2.9},\beta_{10}^{2.10})\oplus(\gamma_1^{2.1},\gamma_2^{2.2},\gamma_3^{2.3},\gamma_4^{2.4},\gamma_5^{2.5},\gamma_6^{2.6},\gamma_7^{2.7},\gamma_8^{2.8},\gamma_9^{2.9},\gamma_{10}^{2.10},\gamma_8^2,\gamma_9^2,\gamma_{10}^2)]^U \end{bmatrix}$$

$$A^1\otimes(B^1\oplus\Gamma^1) = \left({A^1}^L\otimes{B^1}^L\right)\left({A^2}^U\otimes{B^2}^U\right)\oplus\left({A^1}^L\otimes{A^1}^L\right)\left({A^2}^U\otimes{\Gamma^2}^U\right)$$

$$A^1\otimes(B^1\oplus\Gamma^1) = [(A^1\otimes B^1)\oplus(A^1\otimes\Gamma^1)]$$

Theorem 4

If $A^1, B^1 \in B\ C[R], then\ A^1 \vee B^1, A^1 \wedge B^1 \in B\ C[R]$

where

$$A^1 = [((\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L,(\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U]$$

$$B^1 = [((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L,((\beta_1^{2.1},\beta_2^{2.2},\beta_3^{2.3},\beta_4^{2.4},\beta_5^{2.5},\beta_6^{2.6},\beta_7^{2.7},\beta_8^{2.8},\beta_9^{2.9},\beta_{10}^{2.10})^U]$$

Proof: only prove $A^1 \vee B^1 \in BC[R]\ for\ Z \in [R]$

$$(i)(A^1 \vee B^1)(Z) = \underset{Z = X \vee Y}{V}(A^1(x) \wedge B^1(y))$$

$$(A^1 \vee B^1)(Z) = \underset{Z = X \vee Y}{V} \begin{matrix} [((\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L(x),(\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U(x)] \wedge \\ [((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L(y),((\beta_1^{2.1},\beta_2^{2.2},\beta_3^{2.3},\beta_4^{2.4},\beta_5^{2.5},\beta_6^{2.6},\beta_7^{2.7},\beta_8^{2.8},\beta_9^{2.9},\beta_{10}^{2.10})^U(y)] \end{matrix}$$

$$(A^1 \vee B^1)(Z) = \underset{Z = X \vee Y}{V} \begin{matrix} [((\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L(x) \wedge (\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U(x)] \\ [((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L(y) \wedge ((\beta_1^{2.1},\beta_2^{2.2},\beta_3^{2.3},\beta_4^{2.4},\beta_5^{2.5},\beta_6^{2.6},\beta_7^{2.7},\beta_8^{2.8},\beta_9^{2.9},\beta_{10}^{2.10})^U(y)] \end{matrix}$$

$$(A^1 \vee B^1)(Z) = \underset{Z = X \vee Y}{V}[((\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L(x) \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L(y))],$$

$$\underset{Z = X \vee Y}{V}[((\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^U(x) \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^U(y)]$$

$$= \begin{pmatrix} [((\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L(x) \vee ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L(y))](z), \\ [((\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^U(x) \vee ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^U(y))](z) \end{pmatrix}$$

Note that

$$(A^1 \vee B^1)(Z) =$$

$$= \begin{pmatrix} [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0}) \vee ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})]^L(z) \\ [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0}) \vee ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})]^U(z) \end{pmatrix}$$

$$(A^1 \vee B^1)^L = \Big[(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L \vee \big((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0}\big)^L\Big]$$

$$(A^1 \vee B^1)^U = \Big[(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^U \vee \big((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0}\big)^U\Big]$$

Since ${A^1}^L, {B^1}^L, {A^2}^U, {B^2}^U \in BC(R)$ *it follows that*

$${A^1}^L \vee {B^1}^L, {A^2}^U \vee {B^2}^U \in BC\ (R) \ \textit{and so}\ (A^1 \vee B^1)^L, (A^1 \vee B^1)^U \in BC(R)$$

Hence $A^1 \vee B^1 \in BC[R]$

$$(ii)(A^1 \wedge B^1)(Z) = \bigvee_{Z = X \wedge Y} (A^1(x) \wedge B^1(y))$$

$$(A^1 \wedge B^1)(Z) = \bigvee_{Z = X \wedge Y} \begin{matrix} [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L(x),(\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U(x)] \wedge \\ [((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L(y),((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^U(y)] \end{matrix}$$

$$(A^1 \wedge B^1)(Z) = \bigvee_{Z = X \wedge Y} \begin{matrix} [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L(x) \wedge (\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U(x)] \\ [((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L(y) \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^U(y)] \end{matrix}$$

$$(A^1 \wedge B^1)(Z) = \bigvee_{Z = X \wedge Y} [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L(x) \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L(y))],$$

$$\bigvee_{Z = X \wedge Y} [(\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U(x) \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^U(y)]$$

$$= \begin{pmatrix} [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^L](z), \\ [(\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10})^U \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})^U](z) \end{pmatrix}$$

Note that

$$(A^1 \wedge B^1)(Z) = \begin{pmatrix} [(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0}) \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})]^L(z) \\ [(\alpha_1^{2.1},\alpha_2^{2.2},\alpha_3^{2.3},\alpha_4^{2.4},\alpha_5^{2.5},\alpha_6^{2.6},\alpha_7^{2.7},\alpha_8^{2.8},\alpha_9^{2.9},\alpha_{10}^{2.10}) \wedge ((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0})]^{L^U}(z) \end{pmatrix}$$

$$(A^1 \wedge B^1)^L = \Big[(\alpha_1^{1.1},\alpha_2^{1.2},\alpha_3^{1.3},\alpha_4^{1.4},\alpha_5^{1.5},\alpha_6^{1.6},\alpha_7^{1.7},\alpha_8^{1.8},\alpha_9^{1.9},\alpha_{10}^{1.0})^L \wedge \big((\beta_1^{1.1},\beta_2^{1.2},\beta_3^{1.3},\beta_4^{1.4},\beta_5^{1.5},\beta_6^{1.6},\beta_7^{1.7},\beta_8^{1.8},\beta_9^{1.9},\beta_{10}^{1.0}\big)^L\Big]$$

$$(A^1 \wedge B^1)^U = \Big[(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \wedge \big((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}\big)^U\Big]$$

Since ${A^1}^L, {B^1}^L, {A^2}^U, {B^2}^U \in BC\ (R)\ \textit{it follows that}$

${A^1}^L \wedge {B^1}^L, {A^2}^U \wedge {B^2}^U \in BC\ (R)\ \textit{and so}\ (A^1 \wedge B^1)^L, (A^2 \wedge B^2)^U \in BC(R)$,

Hence $A^{1,2} \wedge B^{1,2} \in BC[R]$

Theorem 5

If $A^1, B^1, \Gamma^1 \in BC[R]$

$$A^1 = [(\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L, (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U]$$

$$B^1 = [(((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}))^L, (((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}))^U]$$

$$\Gamma^1 = ((\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}))^L, ((\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}))^U$$

Then $(i)\ A^1 \oplus (B^1 \vee \Gamma^1) = (A^1 \oplus B^1) \vee (A^1 \oplus \Gamma^1)$

Proof:

$$A^1 \oplus (B^1 \vee \Gamma^1) = \Big[\big(A^1 \oplus (B^1 \vee \Gamma^1)\big)^L, \big(A^1 \oplus (B^1 \vee \Gamma^1)\big)^U\Big]$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus \big[\big((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \vee \big(\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})\big]^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus \big[\big((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \vee \big(\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})\big]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus \Big[\big((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}\big)^L \vee \big(\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}\big)^L\Big] \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus \Big[\big((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}\big)^U \vee \big(\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}\big)^U\Big] \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \vee \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \vee \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})]^L \vee \\ [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U \vee \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})] \vee \\ (((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}))]^L \vee, \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})] \vee \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \vee \\ (((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}))]^L \vee \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \vee \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus \big((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}\big)^L \\ \big((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}\big)^U \vee ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus \\ \big(\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}\big)^L \big(\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}\big)^U \end{bmatrix}$$

$= (A^1 - B^1) \wedge (A^1 - \Gamma^1)$

$(ii) A^1 \oplus (B^1 \wedge \Gamma^1) = (A^1 \oplus B^1) \wedge (A^1 \oplus \Gamma^1)$

Proof:

$$A^1 \oplus (B^1 \wedge \Gamma^1) = \left[\left(A^1 \oplus (B^1 \wedge \Gamma^1)\right)^L, \left(A^1 \oplus (B^1 \wedge \Gamma^1)\right)^U\right]$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus [((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \wedge (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus \left[((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \wedge (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}, \gamma_{10}^{1})^L\right] \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus \left[((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U\right] \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \wedge \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L \end{bmatrix},$$

$$\begin{bmatrix} (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})]^L \wedge \\ [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L \end{bmatrix},$$

$$\begin{bmatrix} (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U \wedge \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \wedge \\ (((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}))]^L \wedge \end{bmatrix},$$

$$\begin{bmatrix} (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$= [[((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})$
$\wedge ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]$

$= [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})$
$\wedge ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oplus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L,$

$[(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})$
$\wedge (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oplus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U$

$= (A^1 \oplus B^1) \wedge (A^1 \oplus \Gamma^1)$

$(iii)\ A^1 \ominus (B^1 \vee \Gamma^1) = (A^1 \ominus B^1) \wedge (A^1 \ominus \Gamma^1)$

Proof:

$$A^1 \ominus (B^1 \vee \Gamma^1) = [[A^1 \ominus (B^1 \vee \Gamma^1)]^L, [A^1 \ominus (B^1 \vee \Gamma^1)]^U]$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \vee (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}, \gamma_9^1, \gamma_{10}^1)]^L \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \vee (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}, \gamma_{10}^2)]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \ominus [((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \vee (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10} \gamma_8^1, \gamma_9^1, \gamma_{10}^1)]^L \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) U \ominus [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \vee (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}, \gamma_9^2, \gamma_{10}^2)]^U] \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \ominus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L] \vee \\ [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \ominus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L], \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U] \vee \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix}$$

$$\begin{bmatrix} \left[\left((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})\right)^L\right] \vee \\ [(((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus \gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L], \\ \left[\left((\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})\right)^U\right] \vee \\ \left((\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})\right)^U \end{bmatrix}$$

$$= \begin{bmatrix} \left[\left((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})\right) \wedge\right] \\ \left(((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})\right)^L, \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})] \wedge \\ \left[(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus \left(\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}\right)\right]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus \\ ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L (\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus \\ (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix}$$

$= (\mathrm{A}^1 - \mathrm{B}^1) \wedge (\mathrm{A}^1 - \Gamma^1)$

$(iv)\ \mathrm{A}^1 - (\mathrm{B}^1 \wedge \Gamma^1) = (\mathrm{A}^1 \ominus \mathrm{B}^1) \vee (\mathrm{A}^1 \ominus \Gamma^1)$

Proof:

$\mathrm{A}^1 - (\mathrm{B}^1 \wedge \Gamma^1) = [(\mathrm{A}^1 \ominus (\mathrm{B}^1 \wedge \Gamma^1)^L, (\mathrm{A}^1 \ominus (\mathrm{B}^1 \wedge \Gamma^1)^U]$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus [(\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \wedge (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ \alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \ominus \left[(\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \wedge (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}, \gamma_{10}^{1})^L\right], \\ \left[\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus \left[((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U\right]\right] \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \ominus (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \vee \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \ominus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L \end{bmatrix}, \\ \begin{bmatrix} \alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \vee \\ \alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})]^L \vee \\ [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L \end{bmatrix}, \\ \begin{bmatrix} [\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U \vee \\ [\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus (\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \wedge \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10}) \end{pmatrix}^L, \\ \begin{pmatrix} \alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge \\ \alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}) \end{pmatrix}^U \end{bmatrix}$$

$$= [(\mathrm{A}^1 \ominus \mathrm{B}^1) \wedge (\mathrm{A}^1 \ominus \mathrm{C}^1)]$$

Theorem 6

If $\mathrm{A}, \mathrm{B}, \mathrm{C} \in [\mathrm{R}^+]$

$where\ \mathrm{A}^1 = [(\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})]$

$\mathrm{B}^1 = [(b_1^L, b_2^L, b_3^L, b_4^L, b_5^L, b_6^L, b_7^L, b_8^L, b_9^L, b_{10}^L)(b_1^U, b_2^U, b_3^U, b_4^U, b_5^U, b_6^U, b_7^U, b_8^U, b_9^U, b_{10}^U)]$

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$$C = [(c_1^L, c_2^L, c_3^L, c_4^L, c_5^L, c_6^L, c_7^L, c_8^L, c_9^L, c_{10}^L)(c_1^U, c_2^U, c_3^U, c_4^U, c_5^U, c_6^U, c_7^U, c_8^U, c_9^U, c_{10}^U)]$$

$then \quad (i) A^1 \otimes (B^1 \vee C) = (A^1 \otimes B^1) \vee (A^1 \otimes C)$

$(ii) A^1 \otimes (B^1 \wedge C) = (A^1 \otimes B^1) \wedge (A^1 \otimes C)$

$(iii) A^1 \ominus (B^1 \vee C) = (A^1 \ominus B^1) \wedge (A^1 \ominus C)$

$(iv) A^1 \ominus (B^1 \wedge C) = (A^1 \ominus B^1) \vee (A^1 \ominus C)$

(i) $A^1 \otimes (B^1 \vee C) = (A^1 \otimes B^1) \vee (A^1 \otimes C)$

$Proof$:

$$A^1 \otimes (B^1 \vee C) = [[A^1 \otimes (B^1 \vee C)]^L, [A^1 \otimes (B^1 \vee C)]^U]$$

$$= \left[\left[{A^1}^L \otimes (B^1 \vee C)^L, \right] [A^U \otimes (B^1 \vee C)^U]\right]$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \otimes [(\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \vee (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \vee (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \otimes [((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \vee (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \vee (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \otimes ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \vee \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \otimes (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L], \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \vee \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U] \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})]^L \vee \\ [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \otimes ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U \vee \\ [\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \otimes (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \vee \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \otimes ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \vee \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \otimes (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes \\ ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U] \vee \\ [(\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes \\ (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U] \end{bmatrix}$$

$$= (A \otimes B) \vee (A \otimes C)$$

(ii) $A \otimes (B \wedge C) = (A \otimes B) \wedge (A \otimes C)$

$Proof$:

$$A \otimes (B \wedge C) = [[A \otimes (B \wedge C)]^L, [A \otimes (B \wedge C)]^L]$$

$$= [[A^L \otimes (B \wedge C)]^L, [A^U \otimes (B \wedge C)]^U]$$

$$= \begin{bmatrix} (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}{}^L \otimes [((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \wedge (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \otimes [((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \wedge (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \otimes ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \wedge \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \otimes (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L], \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U] \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})]^L \wedge \\ [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10})]^L, \\ [(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \otimes ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U \wedge \\ [(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \otimes (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \wedge \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \otimes (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \otimes ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \otimes (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes \\ ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U] \wedge \\ [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \otimes \\ (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U] \end{bmatrix}$$

$= (A \otimes B) \wedge (A \otimes C)$

$(iii)\ A \oslash (B \vee C) = (A \oslash B) \wedge (A \oslash C)$

Proof: $A \oslash (B \vee C) = [[A \oslash (B \vee C)]^L, [A \oslash (B \vee C)]^U]$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oslash [((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \vee (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oslash [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \vee (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oslash ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \vee (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oslash ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \vee (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oslash ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \wedge \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \oslash (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L], \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oslash ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oslash (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U] \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oslash ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \wedge \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oslash (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oslash ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oslash (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0^L} (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oslash \\ ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \oslash \\ (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})^L (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U \end{bmatrix}$$

$= (A \oslash B) \wedge (A \oslash C)$

$(iv) A \oslash (B \wedge C) = (A \oslash B) \vee (A \oslash C)$

Proof:

$A \oslash (B \wedge C) = [A \oslash (B \wedge C)^L, A \oslash (B \wedge C)^L]$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \oslash \left((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0} \wedge (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})\right)^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \oslash \left(((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10}\right)^U \end{bmatrix}$$

$$= \begin{bmatrix} ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \ominus [((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \wedge (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus [((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0})^L \ominus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})^L \wedge \gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10 L}], \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10})^U \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})^U \wedge (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})^U] \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0})]^L \vee \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L, \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10})]^U \vee \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= \begin{bmatrix} [((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus ((\beta_1^{1.1}, \beta_2^{1.2}, \beta_3^{1.3}, \beta_4^{1.4}, \beta_5^{1.5}, \beta_6^{1.6}, \beta_7^{1.7}, \beta_8^{1.8}, \beta_9^{1.9}, \beta_{10}^{1.0}) \vee \\ ((\alpha_1^{1.1}, \alpha_2^{1.2}, \alpha_3^{1.3}, \alpha_4^{1.4}, \alpha_5^{1.5}, \alpha_6^{1.6}, \alpha_7^{1.7}, \alpha_8^{1.8}, \alpha_9^{1.9}, \alpha_{10}^{1.0}) \ominus (\gamma_1^{1.1}, \gamma_2^{1.2}, \gamma_3^{1.3}, \gamma_4^{1.4}, \gamma_5^{1.5}, \gamma_6^{1.6}, \gamma_7^{1.7}, \gamma_8^{1.8}, \gamma_9^{1.9}, \gamma_{10}^{1.10})]^L \\ [(\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus ((\beta_1^{2.1}, \beta_2^{2.2}, \beta_3^{2.3}, \beta_4^{2.4}, \beta_5^{2.5}, \beta_6^{2.6}, \beta_7^{2.7}, \beta_8^{2.8}, \beta_9^{2.9}, \beta_{10}^{2.10}) \vee \\ (\alpha_1^{2.1}, \alpha_2^{2.2}, \alpha_3^{2.3}, \alpha_4^{2.4}, \alpha_5^{2.5}, \alpha_6^{2.6}, \alpha_7^{2.7}, \alpha_8^{2.8}, \alpha_9^{2.9}, \alpha_{10}^{2.10}) \ominus (\gamma_1^{2.1}, \gamma_2^{2.2}, \gamma_3^{2.3}, \gamma_4^{2.4}, \gamma_5^{2.5}, \gamma_6^{2.6}, \gamma_7^{2.7}, \gamma_8^{2.8}, \gamma_9^{2.9}, \gamma_{10}^{2.10})]^U \end{bmatrix}$$

$$= (A \ominus B) \vee (A \ominus C)$$

4. Conclusion

In this paper we successfully developed decagonal fuzzy number and derived some important theorem for bounded closed interval-valued decagonal fuzzy numbers. Further, the alpha cuts also have derived for the decagonal fuzzy number, and some important theorems have also been proved using alpha cuts. Application of this theorem is used in many fields such as medical, control system, automatic system, engineering, science and technology, etc. for handling the vagueness in the decision making problem.

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