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Analysis of Graph Inverses and Algebraic Connectivity in a Systematic Way

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Abstract:
an exhaustive and methodical study of algebraic connectedness and graph inverses. In many contexts, graph theory plays a crucial role, particularly when understanding intricate systems. This paper delves into the fundamental concepts of graph inverses and explains their significance for network analysis and connectivity evaluation. The Laplacian lattice's second-smallest eigenvalue, or algebraic connectivity $\mu N-1$, plays a crucial role in some features like network heartiness, synchronisation security, and diffusion processes. In this study, we focus on the algebraic connectedness in the network-of-networks (NoN), which is the general context of linked networks. A key component of science is the graph hypothesis. Algebraic graph hypothesis refers to the use of algebraic techniques to graph problems. This article concludes a focus on algebraic graph hypothesis and presents and examines a different algebraic and mathematical variety idea to regard as the greatest matching of an undirected graph.

Keywords: Graph Inverses, Algebraic Connectivity, Systematic Way, Network-of-Networks

I.Introduction

In the field of mathematics, it is essential to comprehend the relationships and behaviours between graphs and their inverses. Graphs serve as visual representations of numerical capabilities, and their inverses provide crucial insights into the behaviour of the initial capabilities. Furthermore, analysing the connectivity and architecture of graphs requires a thorough understanding of algebraic connection.

Analysing algebraic connectedness and graph inverses in a methodical manner requires delving into several numerical concepts and techniques. This cycle enables us to uncover patterns, set expectations, and develop a deeper understanding of the numerical patterns that lie beneath the surface.

Let's explore the concept of graph inverses first. In science, planning contributions to yields is switched by using the reverse of a capability. A graph's backwards essentially mirrors it over the line y = x. Gaining an understanding of a graph's qualities and the opposite can provide insights about behaviour, reach, and balance.

A methodical approach to the study of graph inverses is to examine the characteristics of the first capacity and its opposite. Determining whether the first capability is onto (surjective) or balanced (injective) is included in this since these characteristics affect the existence and behavior of the opposing capability. Furthermore, focusing on the area and range of the first skill as well as its opposite might provide light on how they relate to one another.

Conversely, algebraic connectedness represents the percentage of a graph's degree of association. It assesses how much the graph's connectedness is disrupted when a vertex is removed. In the study of networks, transportation systems, and correspondence networks, this concept is very important.

A methodical approach to analysing algebraic connectedness draws on techniques from graph hypothesis and direct variable based mathematics. For instance, the algebraic connectedness of the graph can be determined by processing its eigenvalues and eigenvectors on the nearness lattice. Additionally, focusing on the connection characteristics of the graph, such as its measurement and normal degree, might provide additional insights about its design.

Analysing various numerical concepts, traits, and techniques is a necessary step in deconstructing graph inverses and algebraic connection methodically. Mathematicians and scientists are able to reveal hidden examples, establish expectations, and enhance our understanding of how we might interpret complicated systems by understanding the relationships between graphs and their inverses as well as the connectivity of graphs themselves.

More and more attention has recently been paid to algebraic connectivity in reviews and books, in applications on trees, in applications on challenging problems in graph theory, such as the extending properties of graphs, weighted graphs, outright algebraic connectivity, variety, and isoperimetric number of a graph, in the study of the asymptotic behaviour of algebraic connectivity for irregular graphs, and in applications on combinatorial enhancement problems, such as the specific streaming cycle, the greatest cut, and the mobile sales representative problems.

Additionally, the algebraic connectivity is relevant to the flexibility hypothesis, the correspondence between continuous and discrete science (Cheeger obtained a discrepancy for the endless simple of the algebraic connectivity in reduced Riemannian manifolds), and Mohar's analysis of a transmission capacity type problem using the unearthly boundary. Similarly, describe a method to extend the second-smallest eigenvalue throughout the curved Laplacian frame of graphs inside a given family, which is a problem for improvement. Fiedler vectors, which are eigenvectors that compare to the algebraic connectedness, are also significant. Convinced by, these eigenvectors are quite notable these days.

More recently, there has been interest in the cutoff points of Laplacian spectra and hence in the algebraic connectedness. Finally, the presentation of extreme graphs satisfying certain maximum and trivial invariants seems to have placed a remarkable emphasis on graph hypothesis research at the beginning of this millennium. Therefore, in terms of the algebraic connection, this cannot be different.

II.LITERATURE REVIEW

Cheng Ye et al. (2015) provide a novel approach to the thermodynamic representation of networks using graph polynomials. They describe a system that deconstructs confusing networks' core features using concepts from factual mechanics and graph hypothesis. Graph polynomials, such as the matching and trademark polynomials, can be used to eliminate important information about the thermodynamic behaviour, connectivity, and topology of the network. The authors demonstrate the practicality of their method for capturing basic network features and provide tidbits of information about the relationship between graph polynomials and network characteristics. This work provides new avenues for focusing on the thermodynamic features of complex systems and advances the development of network inquiry methodologies.

D. Zhou et al. (2012) analyse how assortativity affects the resilience of dependent networks. They demonstrate through logical and mathematical analyses how assortative blending, in which hubs typically associate with others of similar degrees, can effectively reduce the robustness of dependent networks against flowing disappointments. The analysis reveals that assortative mixing contributes to dependent systems' susceptibility to targeted attacks, underscoring the need of taking network assortativity into account when assessing resilience to setbacks and disruptions. This analysis sheds light on the erratic relationship between network architecture and robustness, providing valuable insights for designing more robust systems.

G. D'Agostino et al. (2012) examine how diffusion like cycles in networks without size can be powerful and assortative. They explore how diffusive oddities, such as diseases, are disseminated, and how network geography, assortativity, and power interchange with one another. The authors make clear the confusing relationship between assortativity and sans scale networks' adaptability to dissemination processes through theoretical research and mathematical simulations. According to the specific characteristics of the dissemination interaction and the network structure, assortativity can both increase and decrease the power of without scale networks, as demonstrated by their findings. With recommendations for many real characteristics ranging from data scattering to pestilence spreading, this study provides important insights into the components of dispersion in complicated networks.

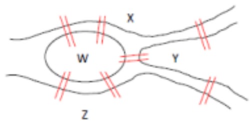
H. Wang et al. (2013) examine how the plague edge is affected by interconnected network topologies. They investigate the implications of different network connectivity designs for the spread of pestilences using logical and mathematical techniques. The review reveals that a major factor in determining the plague edge—the fundamental rate of transmission at which a disease transitions from localised outbreaks to a widespread epidemic—is the topography of interconnected networks. Through consideration of many interrelated network models, the authors provide insights into the factors influencing plague elements in complex systems. This analysis advances our understanding of the mechanisms by which scourges spread via interconnected networks and offers recommendations for methods of mitigating and preventing infections.

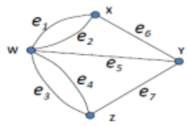
J. Chen et al. (2012) examine the relationship between synchronisation mechanisms on complex networks and Laplacian spectra. They investigate the emergence of synchronisation oddities in networks of coupled dynamical systems by focusing on the phantom aspects of the network Laplacian framework. The authors elucidate the relationship between network structure, Laplacian eigenvalues, and synchronisation factors through theoretical research and mathematical replications. Their analysis reveals that the Laplacian lattice's range holds important information on the synchronisation behaviour of the network, providing tidbits of insight into the conditions that allow synchronisation to occur in intricate networks. This study advances the synchronisation concept and its applications in a variety of domains, including as power systems, neuroscience, and interpersonal organisations.

Jini and Hemalatha(2020) highlight the excellent matching of extension graphs with respect to algebraic and mathematical variety. They study the characteristics of scaffold graphs and provide algorithms for locating optimal matchings in them. The authors explore the relationships between the algebraic and mathematical multiplicities of eigenvalues in span graphs and their recommendations for remarkable matching computations through theoretical analysis and computer experiments. Their analysis advances useful computations for solving matching problems in span graphs, with anticipated applications in several domains, such as computer networks, transportation systems, and human networks. This work advances our understanding of graph matching computations and provides practical solutions for combinatorial streamlining problems in intricate networks.

III.GRAPH THEORY:

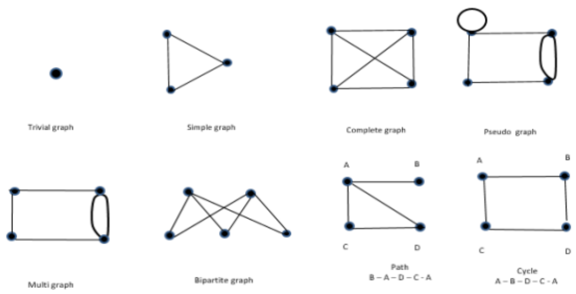
Graph hypothesis, a mathematical analysis of numerical designs, is used to show pairwise relationships between things. A graph is made up of hubs, or vertices, connected by lines, or edges. For example:





The Konigsberg Extension is entirely replaced with a graph in the above graphic., G= {V, E} where vertices V= {w, x, y, z} are the region on either side of the river and edges E = {e1, e2, e3, e4, e5, e6, e7} are the path over the bridge.

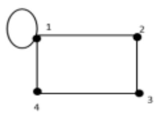
The concept of graphs in Graph Hypothesis includes some basic terminology such as Level of Vertices, Contiguousness Framework, Frequency Grid, Way, Cycle, Walk, and so on. It also includes some basic types of graphs. As an example,



IV.ALGEBRAIC GRAPH THEORY:

There are numerous fascinating algebraic objects related to graphs, and algebraic devices are used for elaborate proofs in algebraic hypotheses. There are currently more books covering various aspects of this topic. The books written by the authors of the algebraic graph hypothesis are very data-rich. Here, we focus on the investigation presented in algebraic graph theory, specifically in the areas of Direct and Digest polynomial algebra.
Before focusing entirely on algebraic graph hypothesis, a few basic terminology and models are clarified.

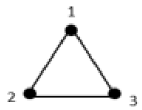
If components $aij = aji$ or the representation of the framework is the real network, then a n x n lattice is expected to be symmetric. According to graph hypothesis, the contiguousness grid of every undirected graph, regardless of self-circles, is a symmetric framework.
The contiguousness grid in graph hypothesis is a square network of a constrained graph whose components are transmitted as 1 in the unlikely event that the graph's vertex sets are close together; otherwise, it is zero. For example



$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Undirected graph with self-loop Symmetric and adjacency matrix

The range of graph is the Eigen upsides of a graph's contiguousness network.
Consider a complete graph, for example, that has three vertices and three edges.



$$A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$$

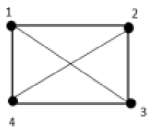
The characteristic polynomial of the adjacency matrix is,

$$\boldsymbol{\lambda^3-3\lambda-2=0} \qquad (\mathbf{1})$$

$\boldsymbol{\lambda=-1,-1,2}$ are the graph's Eigen upsides and represent the graph's range. Known as the graph's degree grid, the corner-to-corner elements of a slanting lattice in a graph hypothesis are its vertices' levels.

In graph theory, the Laplacian grid is represented as an undirected graph with multiple edges and no self-circles. It is written as L = D - A, where D denotes the degree network and An denotes the contiguousness framework.

Consider an undirected graph with four vertices and six edges, for example.



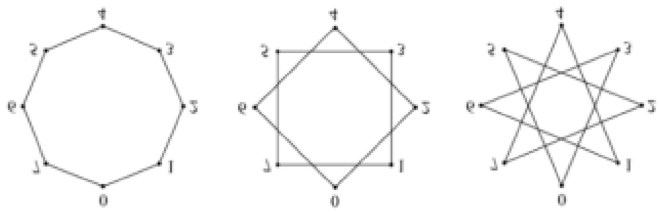
$$A=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix} \qquad D=\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{bmatrix}$$

Adjacency matrix Degree matrix

The matrix Laplacian is provided by,

$$L=D-A=\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{bmatrix}-\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}=\begin{bmatrix}3&-1&-1&-1\\-1&3&-1&-1\\-1&-1&3&-1\\-1&-1&-1&3\end{bmatrix}$$

An undirected cyclic group of graph symmetries that connects any vertex to any other vertex is known as the circulant graph in graph theory.
For instance, from,



Giving an undirected graph in graph hypothesis an algebraic design is known as polynomial math of graphs, and it serves a variety of uses in general polynomial math.

For example,

Take into consideration a graph G with positive-numbered vertices that is addressed by a pair (V, E), where V denotes the vertex arrangement and E is the edge arrangement (V x V subset). At that time, the algebraic construction is provided by,

The structure $(G, +, \rightarrow, \varepsilon)$ introduced in the above satisfies the usual laws,

- ε is the empty set of the graph.
- $(G, +, \varepsilon)$ is an idempotent commutative monoid.
- $(G, \rightarrow, \varepsilon)$ is a monoid.
- $\rightarrow$ distributive over +, (e.g.) $1 \rightarrow (2+3) = (1 \rightarrow 2) + (1 \rightarrow 3)$

A. Study of Graph theory using Linear Algebraic concept:

The main focus of modern math introductions is on straight polynomial algebra, which includes characterising basic objects like lines, planes, and curves.

The main component of algebraic graph hypothesis, which includes the investigation of graph regarding straight variable based math, is the range of contiguousness grid or the Laplacian lattice known as the "ghostly graph hypothesis" in direct variable based math. It is possible to factorise a diagonalizable network into a canonical structure that addresses the lattice to the extent that its range is known for its Eigen values and Eigen vectors.

The study of a graph's characteristics that are similar to its Eigen values, Eigen vectors of frameworks connected to the graph, and trademark polynomial is known as "ghastly graph hypothesis." The Eigen values of a completely symmetric grid are whole integers that are algebraic in nature and symmetrically diagonalizable.

If a limited simple graph is undirected, its contiguousness framework is an undirected (0-1) lattice with zeros on one side of it, and the network is symmetric. The relationship between a graph and the Eigen esteem, Eigen vectors of its nearness framework are investigated in horrifying graph hypothesis.

Simple Polynomial Calculation Applied to graph hypothesis is the best method for analysing traffic flow in courses on networking, electronic circuits or conveyance. It is an application of graph hypothesis to straight variable based maths. In order to handle this problem, various grid jobs that are well-characterized on the nearness framework are considered and a numerical approach to addressing a graph is outlined.

The relationships between graph hypothesis associated with network depictions and the framework properties investigated in straight variable based math are explained by graph hypothesis and direct polynomial math. It identifies certain noteworthy characteristics of exceptional types of graphs, such as full graphs and non-cyclic graphs, and how their network properties reflect their specialisation in graph theory.

Graph structure is represented thermodynamically using graph polynomials, which are based on a characteristic polynomial from the standard The Laplacian lattice illustrates the relationship between the polynomial and the network's Boltzmann segment capabilities. This makes it possible to find the network's average energy and entropy, among other thermodynamic quantities. Furthermore, these thermodynamic parameters can be analysed with respect to fundamental network properties, such as the total number of hubs and hub degree insights for hubs connected by edges. The following thermodynamic representation can be used for real time-varying networks that handle complex systems in both the natural and financial domains.

A recursion equation to generate all the identicalness classes of related Graphs with coefficients provided by the inverses of the sets of their gatherings of automorphisms is known as an algebraic Portrayal of Graphs and Applications to Graph Count. The result is used to the identification of related graphs, each of whose biconnected portions has an equal number of vertices and edges, using an algebraic graph representation. The proof summarises Dziobek's original proof of Cayley's equation and serves the goals of Abel's binomial hypothesis.

The set of all m x n lattices of 0's and 1's with ri1's in column I and sj1's in section j has shown fresh results. The frameworks of zeros and ones with fixed line and segment aggregate vector compose this set. The results can also be determined in the bipartite graph layout with bipartition into m and n vertices with degree grouping R and S independently. It is also typically arranged as a collection of hypergraphs with n edges whose cardinalities are denoted as S and m vertices with degree grouping R.

B. Study of Graph Theory in Abstract Algebra concept:

Gathering hypothesis research is the study of algebraic design. Notable algebraic designs for extracurricular activities and proverbs are rings and fields.

Investigation of Automorphism gatherings and mathematical gathering hypotheses, such as symmetric graphs, vertex-transitive, edge-transitive, and so on, are specifically included in algebraic graph hypothesis in bunch hypothesis. The evenness quality of range, which may be found in its nearness lattice, is also related to the study of gathering hypotheses in algebraic graph hypothesis.

The term "automorphism of a graph" refers to the balance type of a graph that is planned onto itself while maintaining edge-vertex connection. It can be used to both coordinated and undirected graphs.

The study of finitely generated bunches establishing a connection between the algebraic characteristics of these groups and the mathematical properties in which they operate. A key idea of mathematical graph theory is that finitely formed bunches themselves function as mathematical things.

Applications of Compiling combines with algebraic graph theory to handle a graph and an example of a subgroup of an automorphism bunch. If the group acts transitively on the arrangement of length s paths in the graph, it is referred to as s-transitively on the graph. This is shown as a hypothesis for a 1-transitive gathering, where s will be the largest integer such that the subgroup operates in an s-transitive manner in the end.

A hypothetical result about the degree succession, vertex colorings, and vertex freedom number—which are used to infer hypotheses about restricted gatherings—occurs when graph hypothesis is applied to limited bunches.

A numerical plan for determining the boundary between living and non-living systems is provided by the survey and application of hypothesis gathering to subatomic systems science. In this collection of hypotheses, subatomic systems science is used using unique variable-based mathematics.

V.DEFINITIONS

A. Graph Theory Basics

An assortment of center points joined by an enormous number of associations G (N,L) structure a graph G. Suppose there are two networks, G1 = (N1,L1) and G2 = (N2,L2), each having an enormous number of center points (N1, N2) and associations (L1,L2) in confinement. To keep things basic, we will accept that each reliance association in the going with is symmetric, implying that all networks are undirected, for instance.

N = N1 ∪ N2 and L = L1 ∪ L2 ∪ L12 make up the worldwide system that outcomes from the association of the two networks, which is network G with N1 ∪ N2 center points and L1 ∪ L2 "intralinks" notwithstanding unique "interlinks" L12 joining the two networks. Therefore, (N,L) = G def = (N1 ∪ N2,L1 ∪ L2 ∪ L12).

Utilizing the accompanying documentation, we can characterize Ni as the quantity of center points in |Ni |, Li as the quantity of associations in |Li |, N as N1 + N2, and L as L1 + L2. We can likewise characterize An as the continuity structure of the whole system G, whose segments or parts are aij = 1 in the event that center I is associated with center j, and aij = 0 in any case. The N × N cross section lattice An is characterized as the place where the two networks independent (L12 = ∅):

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A_2} \end{bmatrix}. \qquad (\boldsymbol{2})$$

The nearness grid receives non-paltry off-block terms, denoted by Bij, at the point where a connection is offered (L12 6= ∅). Bij is the Ni × Nj interconnection lattice that addresses the interlinks between G1 and G2. Next, the interdependent network B is

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B_{12}} \\ \boldsymbol{B_{12}^T} & \boldsymbol{0} \end{bmatrix}. \qquad (3)$$

and the overall system's adjacency matrix can be expressed as:

$$\boldsymbol{A} + \propto \boldsymbol{B} = \begin{bmatrix} \boldsymbol{A_1} & \propto \boldsymbol{B_{12}} \\ \propto \boldsymbol{B_{12}^T} & \boldsymbol{A_2} \end{bmatrix}. \qquad (4)$$

where α is the communication's coupling strength.Similar to the nearness lattice, the Laplacian framework Q = D − A could be shown; here, D represents the slanting degree grid, and the level of the I-th hub is$\boldsymbol{d_i} \overset{\text{def}}{=} \sum_j \boldsymbol{a_{ij}}$. In the same vein, one may define the diagonal matrices:

$$\begin{cases} (\boldsymbol{D_1})_{ii} \overset{\text{def}}{=} \sum_j (\boldsymbol{B_{12}})_{ij,} \\ (\boldsymbol{D_2})_{ii} \overset{\text{def}}{=} \sum_j (\boldsymbol{B_{21}})_{ij} = \sum_j (\boldsymbol{B_{12}^T})_{ij;} \end{cases}. \qquad (5)$$

Moreover, the overall system G's Laplacian Q is as follows:

:

$$\boldsymbol{Q} = \boldsymbol{Q_A} + \propto \boldsymbol{Q_B} = \begin{bmatrix} \boldsymbol{Q_1} + \propto \boldsymbol{D_1} & -\propto \boldsymbol{B_{12}} \\ -\propto \boldsymbol{B_{12}^T} & \boldsymbol{Q_2} + \propto \boldsymbol{D_2} \end{bmatrix}. \qquad (6)$$

where QB is the Laplacian that only represents the interlinks, and Q1 = Q2 is the Laplacian matrix of A1 = A2:

$$\boldsymbol{Q_B} = \boldsymbol{D} - \boldsymbol{B} = \begin{bmatrix} \boldsymbol{D_1} & -\boldsymbol{B_{12}} \\ -\boldsymbol{B_{12}^T} & \boldsymbol{D_2} \end{bmatrix}. \qquad (7)$$

B. Fiedler Partitioning

Two unmistakable designs of center points {R, S}, where R ∪ S = N, are known as a graph bipartition. Also, we characterize the typical package of G as the bipartition comparing to the two unmistakable center point sets, R = N1 and S = N2, as displayed in Fig. 1. The quantity of center points in R and still up in the air by their separate cardinality, |R| and |S|.

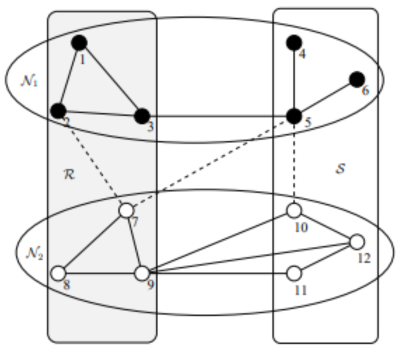


Figure 1: The four primary partition sets that are visible are N1 (set of black nodes), N2 (set of white nodes), R (set of nodes inside the grey rectangle), and S (set of nodes inside the white rectangle). The partition sets and the dashed lines, which serve as interlinks, were selected at random for the sake of example and do not correspond to the relevant Fiedler partition.

Since Q is a non-negative really symmetric grid, it has N true eigenvalues; we are looking for a non-decreasingly $0 = \mu N < \mu N-1 \leq \bullet \bullet \bullet \leq \mu 1$. The name of the unique bipartition generated from the eigenvector xN−1, which is associated with the primary non-zero eigenvalue μN−1, honours Fiedler, who derived the majority of its properties. We shall essentially compose (μ, x) in order to work on the documentation of the eigenpair (μN−1, xN−1), since this paper only deals with the Fiedler eigenvector. Fiedler parcelling divides the hubs of N into two bunches so that two hubs, I and j, share a position with a related group if xixj>0. For example, the Fiedler eigenvector x's comparing parts have the same sign. For instance, if the coupling strength α in (3) is zero, the bipartition that results from Fiedler splitting is the same as the two regular bunches, R = G1 and S = G2.

To assess how the Fiedler parcel affects dependent networks, we will concentrate on the three following measurements of extraterrestrial allocation.:

• Fiedler cut-size,$\stackrel{\text{def}}{=} \frac{\boldsymbol{l(\mathcal{R},\mathcal{S})}}{\boldsymbol{L_1+L_2}}$, where l(R, S) = l(S, R) comes close to the number of connections with one end hub in R and an additional end hub in S. Regardless of the directionality of the connection, the Fiedler cut-size takes into account the insignificant portion of connections having one end in R and another end in S over the initial number of connections.

• The angle between the normalised Fielder vector x and the versor is known as the interdependence angle.

$\boldsymbol{x^{(0)} = \frac{1}{\sqrt{N}}(1,\ldots,1,-1,\ldots,-1)}$, which we thoroughly explain in Reference section A. When the Fiedler vector is aligned with x (0), for instance, at the point where the Fiedler parcel fits the regular segment, the dependence point is limited.

• entropy H of the squared Fiedler vector components $\boldsymbol{H \stackrel{\text{def}}{=} -\sum_{i=1}^{N} xi^2 \log_{xi} 2}$. The entropy demonstrates how homogeneous the qualities of x are concerning the help extent or vector limitation, as well as Shannon's information speculation metric. There is less dispersal among the characteristics in x the higher the entropy.

In the unlikely event that the Laplacian framework Q has flaws, some parcel quality assessments may be imprecise. More specifically, any direct blend with the second and third littlest eigenvalues of QA + αQB equals μN−1 = μN−2. $\boldsymbol{x' = ax_{N-1} + bx_{N-2}}$ is besides an eigenvector of Q with an eigenvalue of μN−1, in this way the Fiedler vector isn't clear cut. Nonetheless, we will disregard these occurrences, which every now and again just emerge in graphs with deterministic plans, (for example, the cycle graph).

VI.ANALYTICAL RESULTS

This section introduces two open scientific approaches to figure μ for the associated graph arrangement shown in the previous section. The mean-field hypothesis and the irritation hypothesis, which separately lengthen Area 3.1 and Segment 3.2, are the foundations of the two approaches. Both of the hypotheses that have been put forth agree with each other for a small number of additional links, which validates our research.

A. Exact results for mean-field theory

a) Diagonal interlinking

To begin, we should consider the instance of two indistinguishable networks associated by L12 related interlinks, or what we allude to as "slanting interlinking". One association can be added, and there can be however many joins as N. It was concluded that via cautiously adding few interlinks, this system would have the best conceivable effect. Moving toward such a system mean-field involves zeroing in on a graph of two indistinguishable networks collaborating through N1 weighted relationship between each associating center. The weight of every link, denoted by α = L12 N1, increases with the number of hubs linked to their neighbour for comparison inside that particular network. Ultimately, B12 = I, to the degree that the border α modifies the synchronisation dependency:

$$\boldsymbol{Q_A} + \propto \boldsymbol{Q_B} = \begin{bmatrix} \boldsymbol{Q_1} + \propto \boldsymbol{I} & -\propto \boldsymbol{I} \\ -\propto \boldsymbol{I} & \boldsymbol{Q_1} + \propto \boldsymbol{I} \end{bmatrix}. \qquad (\boldsymbol{8})$$

In terms of physical science, α refers to the coupling constant of the network's cooperation. This system will also be referred to as the mean-field model of the corner-to-corner interlinking approach, in keeping with the rest of the work. Whatever its origin, this system has some intriguing characteristics that are worthwhile investigating.

Let ξN1, ξN1−1, ..., ξ1 is the configuration of eigenvectors for the Laplacian of the single network A1, and their corresponding eigenvalues are ωN1, ωN1−1,..., ω1. Since the bother QB drives with QA, the corresponding graph's eigenvectors remain unchanged. Since all of the (unperturbed) eigenvalues are degenerate two by two, a group of eigenvectors can be characterised using the single network's eigenvalues as a guide:

$$\begin{cases} \boldsymbol{x_{2i}} = \begin{bmatrix} \xi_i \\ \xi_i \end{bmatrix}. \\ \boldsymbol{x_{2i+1}} = \begin{bmatrix} \xi_i \\ -\xi_i \end{bmatrix}. \end{cases} \qquad (\boldsymbol{9})$$

The eigenvalues of the unperturbed system, μ2i = μ2i+1 = ωi, are equivalent to those of the outright non-coordinating system (e.g., α = 0). Thusly, the rising succession of eigenvalues for the non-natural system is ωN1 = 0, 0, ωN1−1, ωN1−1..., ω1, ω1. A trivial demonstration demonstrates that the odd eigenvalues increase directly by a comparable total, whereas the even eigenvalues remain unchanged at the moment where cooperation is activated (e.g., α 6= 0), anticipating that A1 = A2. 2α,

$$\begin{cases} \boldsymbol{\mu_{2i}} = \boldsymbol{\omega_i}, \\ \boldsymbol{\mu_{2i}} + \boldsymbol{1} = \boldsymbol{\omega_i} + \boldsymbol{2} \propto. \end{cases} \qquad (\boldsymbol{10})$$

The eigenvector placement is unchanged for α close to zero: μN = ωN1 = 0, μN−1 = 2α, μN−2 = ωN1−1, μN−3 = ωN1−1 + 2α,..., ω1, ω1 + 2α. Regardless, the second and third least significant eigenvalues of the dependent network (μN−1 and μN−2) trade when α > ωN1−1 2. As a result, the leading non-zero eigenvalue increases directly with 2α until it reaches a level equal to the value of the disconnected networks ωN1−1. Concerning condition (1), the natural system can synchronize with a practically identical immediacy to the single restricted network when α is more eminent than the breaking point αI = ωN1−1 2. Thusly, the canny system takes more time to arrive at balance than a solitary network when the system coupling channel is quicker than the legitimate time. The crucial worth αI for the specific model relates to a basic worth of associations lI that should be remembered for request to accomplish the single network's speed:

$$\boldsymbol{l_I} = \propto_{\boldsymbol{I}} \boldsymbol{N_1} = \frac{\boldsymbol{\omega_{N-1}} \cdot \boldsymbol{N_1}}{\boldsymbol{2}}. \qquad (\boldsymbol{11})$$

b) Random interlinking

We have introduced a second method, which we have dubbed the random interlinking system, as an algebraically solvable variant of the constrained inclining interlinking. It is related to the mean field guess of two indistinguishable networks cooperating through L12 irregular linkages.

With every unitary portion, the mean-field technique generates a relationship lattice: B12 = J, where J is the entire network; the weight of each interlink is α = L12 N2 1, and

$$\boldsymbol{Q} = \boldsymbol{Q_A} + \propto \boldsymbol{Q_B} = \begin{bmatrix} \boldsymbol{Q_1} + \propto \boldsymbol{N_1 I} & -\propto \boldsymbol{J} \\ -\propto \boldsymbol{J} & \boldsymbol{Q_2} + \propto \boldsymbol{N_1 I} \end{bmatrix}. \quad (\mathbf{12})$$

The QB grid drives with QA, just like in the previous instance, thus an eigenvector arrangement that is usual can be chosen. All other eigenvalues experience some increment for a non-paltry α, except for the invalid eigenvalue μN, which is always present. All eigenvalues μi for I that is less than N − 1 increment for a respectable sum αN1, but μN−1 increments by double that amount,

$$\begin{cases} \boldsymbol{\mu_N} = \mathbf{0}, \\ \boldsymbol{\mu_{N-1}} = \mathbf{2} \propto \boldsymbol{N_1}, \\ \boldsymbol{\mu_i} = \boldsymbol{\omega_i} + \propto \boldsymbol{N_1}, \boldsymbol{for}\ \boldsymbol{i} \leq \boldsymbol{N} - \mathbf{1}. \end{cases} \quad (\mathbf{13})$$

The varying development rates indicate that the second and third eigenvectors (μN−1 and μN−2) trade beyond a fundamental worth αJ. It is possible to obtain the limit αJ by using the intersection condition μN−1 = μN−2,

$$\propto \boldsymbol{J} = \frac{\boldsymbol{\omega_{N-1}}}{\boldsymbol{N_1}}. \quad (\mathbf{14})$$

We can estimate the minimum number of connections for the arbitrary interlinking technique as follows, given that α = L12 N2 1.

$$\boldsymbol{l_J} = \propto \boldsymbol{J N_1^2} = \boldsymbol{\omega N} - \mathbf{1}.\boldsymbol{N_1}. \quad (\mathbf{15})$$

It is crucial that general interlink techniques and the basic number of interlinks in relation to the inclining mean-field hypothesis differ by roughly a factor of two.

c) Physical interpretation

On the off chance that we characterize network power as a system's capacity to play out its capability in case of harm or assaults, it merits exploring what happens when two networks, A1 and A2, which were at first connected together by B's corner-to-corner interlinking, are exposed to purposeful assaults or interlink setback. Our basic, express model exhibits that the reaction of the hard and fast interfacing system A + αB happens at a similar rate as the single part network A1 when these two completely related networks rely upon little interlink issues. Eventually, worldwide system synchronizability doesn't diminish when the operability of the control channel through α is fairly decreased. Notwithstanding, in the event that the association gadgets' usefulness falls apart beneath the major worth αI, the synchronization connection dials back

This proposes that, from the outlook of the mean-field technique, the system might lose a small level of interlinks while keeping up with its synchronization time.

According to the viewpoint of estimation, the limit α can be perceived as a coupling consistent. On the off chance that the Fiedler eigenvalue μN−1 is related to the interior energy of a thermodynamic system, then, at that point, its most unmistakable subordinate shows a progress from zero to limited esteem. One way or the other, this subordinate remains dedicated to frame request 1 development in a concise way. Be that as it may, on the off chance that the Fiedler eigenvalue is utilized to survey synchronizability and is viewed as a thermodynamically anticipated amount, for example, the free enthalpy, then, at that point, its Legendre change looks at to the inner energy and shows a brokenness at α = αI. From this vantage point, the noticed startling change could be deciphered as the progression of the main solicitation stage. In spite of this charming equality, it is critical to recollect that the Fiedler eigenvalue and related Legendre change are not enormous qualities and, thusly, can't be viewed as thermodynamically attainable. Whether or not this unforeseen shift is the result of some secret authentic vestige stays unanswered.

One could look at the topological characteristics of the eigenvectors in order to understand the near concept of the stage change. The Fiedler segment's cut connections are limited to being outside the first networks (interlinks, for example) below the basic worth αI, while all cut connections lie inside the initially detached networks above the basic worth (intralinks, for example). This means that the intralinks in αB overwhelm the synchronisation below αI, and the complete system, A + αB, is synchronised beyond αI.

B. Approximating μ using perturbation theory

We carried out a perturbative analysis of the Laplacian spectra in order to obtain approval for our mean-field study. The theory of irritation provides fragments of information on dispersion processes by methodically representing μN−1 in cases when the diffusive coupling is asymptotically small or large, across the range of G1 and G2.

Tracking down the foundation of the related quadratic design in the unitary circle (x T x = 1) with the requirement u T x = the issue lies with 0.

$$\boldsymbol{\mu} = \boldsymbol{\mu_{N-1}} = \underset{\boldsymbol{x} \neq \boldsymbol{0}, \boldsymbol{u^T x} = \boldsymbol{0}}{\boldsymbol{inf}} \frac{\boldsymbol{x^T Q x}}{\boldsymbol{x^T x}}; \quad (\mathbf{16})$$

The lattice Q in our case is the sum of a grid QA that connects only hubs within a similar net and a "bother" αQB that relates only hubs within different networks (QA + αQB). Therefore, we must locate the base that satisfies the otherworldly requirements:

$$\begin{cases} (\boldsymbol{Q_A} + \propto \boldsymbol{Q_B} - \boldsymbol{\mu I})\boldsymbol{x} = \boldsymbol{0}, \\ \boldsymbol{x^T x} = \boldsymbol{1}, \\ \boldsymbol{u^T x} = \boldsymbol{0}. \end{cases} \quad (\mathbf{17})$$

By using Taylor expansion, μ and x can be expressed as follows when the solution is analytical in α:

$$\boldsymbol{\mu} = \sum_{\boldsymbol{k=0}}^{\infty} \boldsymbol{\mu^{(k)}} \propto^{\boldsymbol{k}} \quad (\mathbf{18})$$

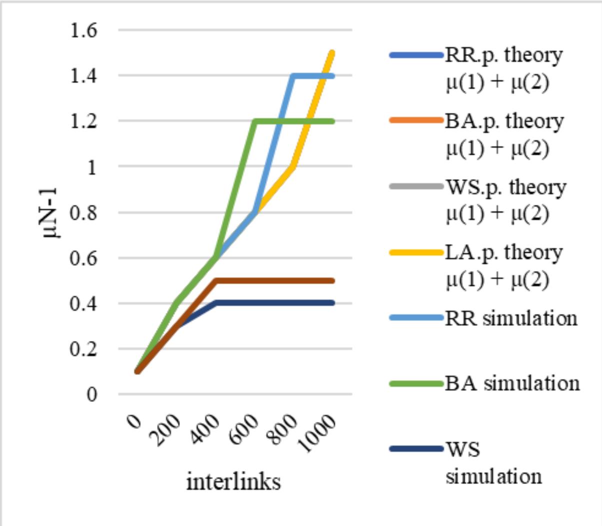
$$\boldsymbol{x} = \sum_{\boldsymbol{k=0}}^{\infty} \boldsymbol{x^{(k)}} \propto^{\boldsymbol{k}} \quad (\mathbf{19})$$

A ranking of conditions is obtained by substituting the development in the eigenvalue condition. Taking on the final choice up until the next request results in the accompanying approximations,

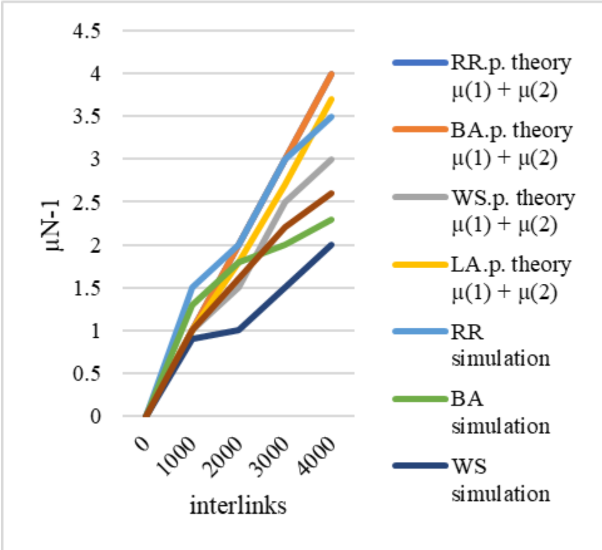
$$\boldsymbol{\mu^{(1)}} = \left(\boldsymbol{x^{(0)}}\right)^{\boldsymbol{T}} \boldsymbol{Q_{Bx}}^{(\mathbf{0})} \quad (\mathbf{20})$$

$$\boldsymbol{\mu^{(2)}} = \left(\boldsymbol{x^{(0)}}\right)^{\boldsymbol{T}} \boldsymbol{Q_B}\left(\boldsymbol{x^{(1)}}\right) = -\left(\boldsymbol{x^{(1)}}\right)^{\boldsymbol{T}} \boldsymbol{Q_A}\left(\boldsymbol{x^{(1)}}\right) \leq \boldsymbol{0}. \quad (\mathbf{21})$$

Appendix A outlines the steps taken to arrive at the solution. Anticipatingly, the first order μ (1) relies solely on the zero-order eigenvector x (0); conversely, the second order μ (2) is negative, so augmenting the algebraic connectivity estimate. Fig. 2 shows the previous perturbation estimates along with a numerical simulation.



(a) Diagonal strategy



(b) Random Strategy

Figure 2: Algebraic connectedness µN−1 of four graph models with N = 1000 nodes, as approximated by dashed lines and simulated over several networks (solid lines).

VII.CONCLUSION

This paper provides an overview of the algebraic hypothesis study in the graph hypothesis through the definitions and applications of the first three components of the algebraic hypothesis. Similarly, a few exploratory articles were considered to conduct further analysis in the subject of algebraic graph theory in light of algebraic hypotheses. The study that was presented looks at components of linked networks that are driven by dissemination, focusing in particular on synchronisation applications and the algebraic connectedness of these networks. Reviewing the algebraic connection reveals a stage change phenomenon that occurs when the number of interlinks between initially isolated networks increases. Based on the network topology, this transition point is determined to follow a straight line with the algebraic connectedness of the separate networks, which increases with assortativity. Science establishes the fundamental advancement boundaries for different interlinking systems by means of mean field approximations and leading reenactments. Furthermore, the review suggests that the homogeneity of the networks' spectra can be examined to predict the advancement. The investigation is continuing, looking into heterogeneous dependent networks and considering extensions, such as additional graph models and interlinking techniques.

REFERENCES

[1] Angela Mestre, 2013, "An Algebraic Representation of Graphs and Applications to Graph Enumeration", Hindawi Publishing Corporation, International Journal of Combinatorics, Volume 2013, Article ID 347613.

[2] Cheng Ye, Cesar H.Comin, Thomas K.DM.Peron, Filipi N. Silva, Francisco A. Rodrigues, Luciano da F.Costa, Andrea Torsello, Edwin R.Hancock, (2015) "Thermodynamic Characterization of Networks using graph Polynomials", Physical Review 92, 032810.

[3] D. Zhou, E. Stanley, G. DAgostino, A. Scala, Assortativity decreases the robustness of interdependent networks, Physical Review E 86 (6) (2012) 066103.

[4] G. D'Agostino, A. Scala, V. Zlati´c, G. Caldarelli, Robustness and assortativity for diffusion-like processes in scale-free networks, Europhysics Letters 97 (6) (2012) 68006.

[5] H. Wang, Q. Li, G. D'Agostino, S. Havlin, H. E. Stanley, P. Van Mieghem, Effect of the interconnected network structure on the epidemic threshold, Phys. Rev. E 88 (2013) 022801.

[6] J. Chen, J. Lu, C. Zhan, G. Chen, Laplacian Spectra and Synchronization Processes on Complex Networks, Springer Optimization and Its Applications, 2012, Ch. 4, pp. 81–113.

[7] Jini J, Hemalatha S, (2020) "Perfect Matching of Bridge Graph based on Geometric and Algebraic Multiplicity", Journal of Xidian University", ISSN NO: 1001-2400, Volume 14, ISSUE 11.
[8] Li, Yiyong& Yang, Qingzhi& Yang, Yuning. (2015). A new definition of geometric multiplicity of eigenvalues of Tensors and some results based on it. Frontiers of Mathematics in China. 10. 10.1007/s11464-015-0412-z.
[9] Ogiwara K., Fukami T., Takahashi N., Maximizing algebraic connectivity in the space of graphs with fixed number of vertices and edges, IEEE Trans. Control Netw. Syst., 2017, 4(2), 359–368.10.1109/TCNS.2015.2503561Search in Google Scholar
[10] Raja singh, Indra & Manuel, Paul &Arockiaraj, Micheal&Rajan, Bharati. (2013). "Embedding of circulant networks" Journal of Combinatorial Optimization. 26. 10.1007/s10878-011-9443-x
[11] S. G´omez, A. D´ıaz-Guilera, J. G´omez-Garde˜nes, C. J. P´erez-Vicente, Y. Moreno, A. Arenas, Diffusion dynamics on multiplex networks, Phys. Rev. Lett. 110 (2013) 028701. doi:10.1103/PhysRevLett.110.028701.
[12] Wei P., Spiers G., Sun D., Algebraic connectivity maximization for air transportation networks, IEEE Trans. Intell. Transp. Syst., 2014, 15(2), 685–698.
[13] Y. Shang, Synchronization in networks of coupled harmonic oscillators with stochastic perturbation and time delays, Mathematics and its Applications: Annals of the Academy of Romanian Scientists 4 (1) (2012) 44.
[14] Yunyun Yang and Gang Xie, (2016) "Maximum Matchings of a Digraph Based on the Largest Geometric Multiplicity", Hindawi Publishing Corporation, Vol 2016, Article ID 4702387.
[15] Z. Z. Yuan, C. Zhao, Z. R. Di, W.-X. Wang, and Y.-C. Lai, (2013) "Exact controllability of complex Networks," Nature Communications, vol. 4, article 2447.