Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 3 (2025)

New approach towards complex bipolar intuitionistic fuzzy subbisemiring applied to homomorphism

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Received: 14-03-2024 Revised: 27-08-2024 Accepted: 19-09-2024.

Abstract

We establish the concept of complex bipolar intuitionistic fuzzy subbisemiring (CBIFSBS). We analyze the homomorphic features and important properties of CBIFSBS. We construct the CBIFSBS level sets for bisemirings. Suppose that L is a subset of $\mathscr{B}$. Then $R = (\mu_L^{\mathscr{T}^-} \cdot e^{\delta\chi_L^{\mathscr{T}^-}}, \mu_L^{\mathscr{F}^-} \cdot e^{\delta\chi_L^{\mathscr{F}^-}}, \mu_L^{\mathscr{T}^+} \cdot e^{\delta\chi_L^{\mathscr{T}^+}}, \mu_L^{\mathscr{F}^+} \cdot e^{\delta\chi_L^{\mathscr{F}^+}})$ is a CBIFSBS of $\mathscr{B}$ if and only if $\mu^{(t,s)}$ is a SBS of $\mathscr{B}$ for all $t, s \in [-1, 0] \times [0, 1]$. It is shown that there are homomorphic pre-images and homomorphic images for every CBIFSBS. We illustrate the results based on numerical examples.

1 Introduction

Zadeh[1] developed fuzzy set (FS) theory, which succeeds when dealing with ambiguity and uncertainty. If an element in an FS has a single value inside the interval, it is considered a member. The degree of non-membership does not necessarily equal one minus the degree of membership, however, because resistance might exist in real-world situations. As FS theory advances, a growing number of hybrid fuzzy models appear. Numerous uncertain theories, including FS,[1] intuitionistic FS (IFS),[2] Pythagorean FS (PFS),[3] and spherical FS (SFS),[4] have been developed as a result of the uncertainties. An FS is made up of MG sets, or sets with grades ranging from 0 to 1. IFS is classified as MG in spite of Atanassov's[2] assertion that non-membership grades (NMG) may only have a value of 1. Throughout the entirety of a decision-making process, the total of MGs and NMGs may occasionally reach 1. The generalized MG and NMG logic, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs, was developed by Yager[3] using PFS logic. These theories are unable to represent the neutral state since it is neither positive nor negative. Cuong[5] discussed the photo FS with coworkers. Three grading points were utilized by FS: positive, neutral, and negative. These grades added together could not total more than 1. In a few of scenarios, it also performs better than PFS and IFS. It is an independent generalization of three models that handles the truth, indeterminacy, and falsity of FS and IFS. The concept of bipolar fuzzy sets was introduced in Lee.[6] In typical fuzzy sets, the membership degree range is $[0, 1]$. Bipolar fuzzy sets (FSs) are those whose membership degree range is extended from the interval $[0, 1]$ to the interval $[-1, 1]$. A bipolar fuzzy set has elements whose membership degree is $[-1, 0]$, which indicates that the implicit contrary property is partially satisfied, and elements whose membership degree is $(0, 1]$, which indicates that the property is not relevant to any extent possible.

The neutrosophic set (NS) was developed by Smarandache[7] to deal with contradictory and ambiguous data. This logic establishes the degree to which a notion is true, ambiguous, or wrong. In,[8] Ramot et al. present the idea of the complex fuzzy set (CFS). The values of the membership functions in CFS transactions might vary greatly. The unit circle of the complex plane has been extended to $[0, 1]$, but the unit circle of a fuzzy membership function stays fixed. The membership function $\mu_X(x)$ of the CFS X extends to the unit circle in the complex plane instead of only to $[0, 1]$. Thus, $\mu_X(x)$ is a complex-valued function that yields a grade of membership of the kind $\eta_X(x) \cdot e^{i\tau_X(x)}$, where $i = \sqrt{-1}$, for each element x in the discourse universe. The value of $\mu_X(x)$ is defined by the two real-valued variables, $\eta_X(x)$ and $\tau_X(x)$, where $\eta_X(x) \in [0, 1]$. Semiring logic and its uses were introduced by Golan.[9] Hussian et al.[10] discussed about the idea and application of bisemirings. Ahsan et al. researched fuzzy semirings.[11] The notion of bisemirings was first presented by

Sen et al.[12] An intuitionistic fuzzy normal subbisemiring of bisemiring was introduced by Palanikumar et al. (2019).[13] Bisemiring was first proposed by Palanikumar et al.[14] employing bipolar valued neutrosophic normal sets. We will look at specific components of the SBS and CBIFSBS concepts and make some assumptions. The article is divided into the following three sections. In Section 1 is called an introduced. Section 2 contains basic information. Section 3 has a list of CBIFSBS concepts. Section 4 deals that homomoprhism based on CBIFSBS concepts.

2 Preliminaries

Definition 2.1. [12] An algebraic structure $(\mathscr{B}, \sqcap, \ominus, \odot)$ is a bisemiring, if $(\mathscr{B}, \sqcap, \ominus)$ and $(\mathscr{B}, \ominus, \odot)$ are semirings, ie.,$(\mathscr{B}, \sqcap)$, $(\mathscr{B}, \ominus)$ and $(\mathscr{B}, \odot)$ are semigroups and

1. $a \ominus (b \sqcap c) = (a \ominus b) \sqcap (a \ominus c)$,
2. $(b \sqcap c) \ominus a = (b \ominus a) \sqcap (c \ominus a)$,
3. $a \odot (b \ominus c) = (a \odot b) \ominus (a \odot c)$,
4. $(b \ominus c) \odot a = (b \odot a) \ominus (c \odot a), \ \forall \ a, b, c \in \mathscr{B}$.

Definition 2.2. A bipolar fuzzy set μ in a universe $\mathcal{U}$ is an object having the form $\mu = \left\{\left\langle x, A_\mu^+(x), A_\mu^-(x)\right\rangle : x \in \mathcal{U}\right\}$, where $A_\mu^+ : \mathcal{U} \to [0,1]$ and $A_\mu^- : \mathcal{U} \to [-1,0]$.
Here $A_\mu^+(x)$ represents the degree of satisfaction of the element x to the property of $A_\mu^-(x)$ represents the degree of satisfaction of x to some implicit counter property of μ. For simplicity the symbol $\left\langle A_\mu^+, A_\mu^-\right\rangle$ is used for the bipolar fuzzy set $\mu = \left\{\left\langle x, A_\mu^+(x), A_\mu^-(x)\right\rangle : x \in \mathcal{U}\right\}$.

Definition 2.3. For two bipolar fuzzy subsets $\mu = (\mu^+, \mu^-)$ and $\Psi = (\Psi^+, \Psi^-)$, the product of μ and Ψ is denoted by $\mu \circ \Psi$ and is defined as

$$(\mu^+ \circ \Psi^+)(x) = \begin{cases} \sup\limits_{(s,t)\in A_x} \left\{\mu^+(s) \wedge \Psi^+(t)\right\} \ if \ A_x \neq 0 \\ 0 \quad if \qquad\qquad A_x = 0 \end{cases}$$

$$(\mu^- \circ \Psi^-)(x) = \begin{cases} \inf\limits_{(s,t)\in A_x} \left\{\Psi^-(s) \vee \Psi^-(t)\right\} \ if \ A_x \neq 0 \\ -1 \quad if \qquad\qquad A_x = 0 \end{cases}$$

Definition 2.4. [7] A NS μ in the universe $\mathcal{U}$ is $\mu = \{x, A_\mu^T(x), A_\mu^I(x), A_\mu^F(x) | x \in \mathcal{U}\}$,where $A_\mu^T(x)$,$A_\mu^I(x)$ $A_\mu^F(x)$ represents the TD,ID and FD of v respectively. Consider the mapping $A_\mu^T : \mathcal{U} \to [0,1], A_\mu^I : \mathcal{U} \to [0,1], A_\mu^F : \mathcal{U} \to [0,1]$ and $0 \leq A_\mu^T(x) + A_\mu^I(x) + A_\mu^F(x) \leq 3$.

Definition 2.5. [7] Let $\mu_1 = \langle \chi_{\mu_1}^T, \chi_{\mu_1}^I, \chi_{\mu_1}^F \rangle$,$\mu_2 = \langle \chi_{\mu_2}^T, \chi_{\mu_2}^I, \chi_{\mu_2}^F \rangle$ and $\mu_3 = \langle \chi_{\mu_3}^T, \chi_{\mu_3}^I, \chi_{\mu_3}^F \rangle$ be the three neutrosophic numbers over $\mathcal{U}$. Then

1. $\mu_2 \ominus \mu_3 = \Big\langle \max(\chi_{\mu_2}^T, \chi_{\mu_3}^T), \min(\chi_{\mu_2}^I, \chi_{\mu_3}^I), \min(\chi_{\mu_2}^F, \chi_{\mu_3}^F) \Big\rangle$,
2. $\mu_2 \uplus \mu_3 = \Big\langle \min(\chi_{\mu_2}^T, \chi_{\mu_3}^T), \max(\chi_{\mu_2}^I, \chi_{\mu_3}^I), \max(\chi_{\mu_2}^F, \chi_{\mu_3}^F) \Big\rangle$,
3. $\mu_2 \geq \mu_3$ iff $\chi_{\mu_2}^T \geq \chi_{\mu_3}^T$ and $\chi_{\mu_2}^I \leq \chi_{\mu_3}^I$ and $\chi_{\mu_2}^F \leq \chi_{\mu_3}^F$,
4. $\mu_2 = \mu_3$ iff $\chi_{\mu_2}^T = \chi_{\mu_3}^T$ and $\chi_{\mu_2}^I = \chi_{\mu_3}^I$ and $\chi_{\mu_2}^F = \chi_{\mu_3}^F$.

Definition 2.6. [7] For any NS $\mu = \{x, A_\mu^T(x), A_\mu^I(x), A_\mu^F(x)\}$ of $\mathcal{U}$. Then (τ, β)-cut is defined as $\{x \in U | A_\mu^T(x) \geq \tau, A_\mu^I(x) \geq \tau, A_\mu^F(x) \leq \beta\}$.

3 Complex bipolar intuitionistic fuzzy subbisemiring

Here $\mathscr{B}$ denotes bisemiring unless other stated, μ stands for real part and χ stands for imaginary part and $z = i2\pi$.

Definition 3.1. For any complex bipolar intuitionistic fuzzy set(CBIFS) L in universal set U,
$L = \{l, \mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)}, \mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-(l)}}, \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)}, \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}} : l \in U\}$, where $\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-(l)}}, \mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-(l)}} : U \to [-1, 0] \times [0, 1]$ and $\mu_L^T(l) \cdot e^{\delta\chi_L^{T(l)}}, \mu_L^F(l) \cdot e^{\delta\chi_L^{F(l)}} : U \to [-1, 0] \times [0, 1]$ represents the truth degree,indeterminacy degree and false degree respectively.
For simplicity the symbol $\mu_L^{\mathscr{T}^-}, \mu_L^{\mathscr{F}^-}$ is CBIFS $L = \{l, \mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-(l)}}$
$\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-(l)}}, \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+(l)}}, \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}} : l \in U\}$.

Definition 3.2. Let $L = \Big\{l, \mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-(l)}}, \mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-(l)}}, \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+(l)}}, \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}}\Big\}$ and $M = \Big\{l, \mu_M^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_M^{\mathscr{T}^-(l)}}, \mu_M^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_M^{\mathscr{F}^-(l)}}, \mu_M^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_M^{\mathscr{T}^+(l)}}, \mu_M^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_M^{\mathscr{F}^+(l)}}\Big\}$ be two CBIFSs of U.
Then we define the intersection and union operation is defined as
(i) $L \cap M = \Big\{\Big(l, \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-(l)}}, \mu_M^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_M^{\mathscr{T}^-(l)}}\}, \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-(l)}}, \mu_M^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_M^{\mathscr{F}^-(l)}}\},$
$\min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+(l)}}, \mu_M^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_M^{\mathscr{T}^+(l)}}\}, \max\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}}, \mu_M^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_M^{\mathscr{F}^+(l)}}\}\Big)\Big| l \in U\Big\}$.
(ii) $L \cup M = \Big\{\Big(l, \min\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-(l)}}, \mu_M^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_M^{\mathscr{T}^-(l)}}\}, \max\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-(l)}}, \mu_M^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_M^{\mathscr{F}^-(l)}}\},$
$\max\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+(l)}}, \mu_M^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_M^{\mathscr{T}^+(l)}}\}, \max\{\mu_L^{I^+}(l) \cdot e^{\delta\chi_L^{I^+(l)}}$
$\min\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}}, \mu_M^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_M^{\mathscr{F}^+(l)}}\}\Big)\Big| l \in U\Big\}$.

Definition 3.3. For any CBIFS $L = \Big\{l, \mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-(l)}}, \mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-(l)}}, \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+(l)}}, \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}}\Big\}$ of a universal set U. Then (t, s)-cut is defined as $\Big\{l \in U | \mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-(l)}} \leq t, \mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-(l)}} \geq s, \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+(l)}} \geq t, \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}} \leq s\Big\}$.

Definition 3.4. The Cartesian product of L and M is defined as
$L \times M = \Big\{\mu_{L\times M}^{\mathscr{T}^-}((l, m)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-((l,m))}}, \mu_{L\times M}^{\mathscr{F}^-}(l, m) \cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-((l,m))}},$
$\mu_{L\times M}^{\mathscr{T}^+}((l, m)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+((l,m))}}, \mu_{L\times M}^{\mathscr{F}^+}(l, m) \cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^+((l,m))}} \mid for\ all\ l, m \in S\Big\}$,
where L and M be the CBIFS of U

$$\left\{\begin{array}{l} \mu_{L\times M}^{\mathscr{T}^-}((l, m)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-((l,m))}} = \max\Big\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-(l)}}, \mu_M^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_M^{\mathscr{T}^-(m)}}\Big\} \\ \mu_{L\times M}^{\mathscr{F}^-}((l, m)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-((l,m))}} = \min\Big\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}}, \mu_M^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_M^{\mathscr{F}^-(m)}}\Big\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \mu_{L\times M}^{\mathscr{T}^+}((l, m)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+((l,m))}} = \min\Big\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+(l)}}, \mu_M^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_M^{\mathscr{T}^+(m)}}\Big\} \\ \mu_{L\times M}^{\mathscr{F}^+}((l, m)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^+((l,m))}} = \max\Big\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+(l)}}, \mu_M^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_M^{\mathscr{F}^+(m)}}\Big\} \end{array}\right\}$$

Definition 3.5. For any CBIFS L of $\mathscr{B}$ is said to be a Q-complex bipolar intuitionistic fuzzy SBS (CBIFSBS) of $\mathscr{B}$ if

$$\left\{\begin{array}{l} \mu_L^{\mathscr{T}^-}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l\ddagger_1 m))} \leq \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l)}, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((m)}\} \\ \mu_L^{\mathscr{T}^-}((l \ddagger_2 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l\ddagger_2 m))} \leq \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l)}, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((m)}\} \\ \mu_L^{\mathscr{T}^-}((l \ddagger_3 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l\ddagger_3 m))} \leq \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l)}, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((m)}\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \mu_L^{\mathscr{F}^-}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\ddagger_1 m))} \geq \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l)}, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((m)}\} \\ \mu_L^{\mathscr{F}^-}((l \ddagger_2 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\ddagger_2 m))} \geq \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l)}, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((m)}\} \\ \mu_L^{\mathscr{F}^-}((l \ddagger_3 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\ddagger_3 m))} \geq \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l)}, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((m)}\} \end{array}\right\}$$

$$\begin{Bmatrix} \mu_L^{\mathscr{T}^+}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\ddagger_1 m))} \geq \min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l)}, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((m)}\} \\ \mu_L^{\mathscr{T}^+}((l \ddagger_2 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\ddagger_2 m))} \geq \min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l)}, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((m)}\} \\ \mu_L^{\mathscr{T}^+}((l \ddagger_3 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\ddagger_3 m))} \geq \min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l)}, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((m)}\} \end{Bmatrix}$$

$$\begin{Bmatrix} \mu_L^{\mathscr{F}^+}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\ddagger_1 m))} \leq \max\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l)}, \mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((m)}\} \\ \mu_L^{\mathscr{F}^+}((l \ddagger_2 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\ddagger_2 m))} \leq \max\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l)}, \mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((m)}\} \\ \mu_L^{\mathscr{F}^+}((l \ddagger_3 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\ddagger_3 m))} \leq \max\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l)}, \mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((m)}\} \end{Bmatrix}$$

for all $l, m \in \mathscr{B}$.

Example 3.6. Consider the bisemiring $\mathscr{B} = \{a, b, c, d\}$ with the Cayley table:

$\ddagger_1$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

$\ddagger_2$	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	c	d	c	d
d	d	d	d	d

$\ddagger_3$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	d	d	d	d
d	d	d	d	d

	$(\flat) = a$	$(\flat) = b$	$(\flat) = c$	$(\flat) = d$
$(\mu_L^{\mathscr{T}^-}, \chi_L^{\mathscr{T}^-})(\flat)$	$-0.8e^{i2\pi(-0.65)}$	$-0.75e^{i2\pi(-0.60)}$	$-0.65e^{i2\pi(-0.50)}$	$-0.7e^{i2\pi(0-.55)}$
$(\mu_L^{\mathscr{F}^-}, \chi_L^{\mathscr{F}^-})(\flat)$	$-0.7e^{i2\pi(-0.55)}$	$-0.8e^{i2\pi(-0.65)}$	$-0.9e^{i2\pi(-0.75)}$	$-0.85e^{i2\pi(-0.7)}$

	$(\flat) = a$	$(\flat) = b$	$(\flat) = c$	$(\flat) = d$
$(\mu_L^{\mathscr{T}^+}, \chi_L^{\mathscr{T}^+})(\flat)$	$0.7e^{i2\pi(0.6)}$	$0.6e^{i2\pi(0.5)}$	$0.4e^{i2\pi(0.3)}$	$0.5e^{i2\pi(0.4)}$
$(\mu_L^{\mathscr{F}^+}, \chi_L^{\mathscr{F}^+})(\flat)$	$0.6e^{i2\pi(0.5)}$	$0.7e^{i2\pi(0.6)}$	$0.9e^{i2\pi(0.8)}$	$0.8e^{i2\pi(0.7)}$

Hence, L is a CBIFSBS of $\mathscr{B}$.

Theorem 3.7. *The intersection of a every CBIFSBSs is again a CBIFSBS of $\mathscr{B}$.*

Proof. Let $\{\tau_i : i \in I\}$ be the family of CBIFSBSs of $\mathscr{B}$ and $L = \Cap_{i \in I} \tau_i$. Let $l, m \in \mathscr{B}$.
Now,

$$\begin{aligned} \mu_L^{\mathscr{T}^-}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l\ddagger_1 m))} &= \sup_{i \in I} \ \mu_{\tau_i}^{\mathscr{T}^-}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l\ddagger_1 m))} \\ &\leq \sup_{i \in I} \ \max\{\mu_{\tau_i}^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{T}^-}(l)}, \mu_{\tau_i}^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{T}^-}(m)}\} \\ &= \max\left\{\sup_{i \in I} \ \mu_{\tau_i}^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{T}^-}(l)}, \sup_{i \in I} \ \mu_{\tau_i}^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{T}^-}(m)}\right\} \\ &= \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)}, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m)}\} \end{aligned}$$

Similarly,
$\mu_L^{\mathscr{T}^-}((l \ddagger_2 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l\ddagger_2 m))} \leq \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)}, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m)}\}$,
$\mu_L^{\mathscr{T}^-}((l \ddagger_3 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l\ddagger_3 m))} \leq \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)}, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m)}\}$.

Now,

$$\begin{aligned} \mu_L^{\mathscr{F}^-}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\ddagger_1 m))} &= \inf_{i \in I} \ \mu_{\tau_i}^{\mathscr{F}^-}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\ddagger_1 m))} \\ &\geq \inf_{i \in I} \ \min\{\mu_{\tau_i}^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{F}^-}(l)}, \mu_{\tau_i}^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{F}^-}(m)}\} \\ &= \min\left\{\inf_{i \in I} \ \mu_{\tau_i}^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{F}^-}(l)}, \inf_{i \in I} \ \mu_{\tau_i}^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{F}^-}(m)}\right\} \\ &= \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)}, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m)}\} \end{aligned}$$

Similarly,
$\mu_L^{\mathscr{F}^-}((l \ddagger_2 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\ddagger_2 m))} \geq \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)}, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m)}\}$ and
$\mu_L^{\mathscr{F}^-}((l \ddagger_3 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\ddagger_3 m))} \geq \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)}, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m)}\}$.
Let $\{\tau_i : i \in I\}$ be the family of CBIFSBSs of $\mathscr{B}$ and $L = \Cap_{i\in I}\tau_i$. Let $l, m \in \mathscr{B}$.

Now,

$$\begin{aligned}
\mu_L^{\mathscr{T}^+}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\ddagger_1 m))} &= \inf_{i\in I}\ \mu_{\tau_i}^{\mathscr{T}^+}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\ddagger_1 m))} \\
&\geq \inf_{i\in I}\ \min\{\mu_{\tau_i}^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{T}^+}(l)}, \mu_{\tau_i}^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{T}^+}(m)}\} \\
&= \min\left\{\inf_{i\in I}\ \mu_{\tau_i}^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{T}^+}(l)}, \inf_{i\in I}\ \mu_{\tau_i}^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{T}^+}(m)}\right\} \\
&= \min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)}, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m)}\}
\end{aligned}$$

Similarly,
$\mu_L^{\mathscr{T}^+}((l \ddagger_2 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\ddagger_2 m))} \geq \min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)}, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m)}\}$,
$\mu_L^{\mathscr{T}^+}((l \ddagger_3 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\ddagger_3 m))} \geq \min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)}, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m)}\}$.
Now,

$$\begin{aligned}
\mu_L^{\mathscr{F}^+}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\ddagger_1 m))} &= \sup_{i\in I}\ \mu_{\tau_i}^{\mathscr{F}^+}((l \ddagger_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\ddagger_1 m))} \\
&\leq \sup_{i\in I}\ \max\{\mu_{\tau_i}^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{F}^+}(l)}, \mu_{\tau_i}^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{F}^+}(m)}\} \\
&= \max\left\{\sup_{i\in I}\ \mu_{\tau_i}^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{F}^+}(l)}, \sup_{i\in I}\ \mu_{\tau_i}^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_{\tau_i}^{\mathscr{F}^+}(m)}\right\} \\
&= \max\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l)}, \mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m)}\}
\end{aligned}$$

Similarly,
$\mu_L^{\mathscr{F}^+}((l \ddagger_2 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\ddagger_2 m))} \leq \max\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l)}, \mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m)}\}$ and
$\mu_L^{\mathscr{F}^+}((l \ddagger_3 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\ddagger_3 m))} \leq \max\{\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l)}, \mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m)}\}$.
Thus,L is a CBIFSBS of $\mathscr{B}$.

Theorem 3.8. *If L and M be the CBIFSBSs of $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively, then $L \times M$is a CBIFSBS of $\mathscr{B}_1 \times \mathscr{B}_2$.*

Proof. Let $l_1, l_2 \in \mathscr{B}_1$ and $m_1, m_2 \in \mathscr{B}_2$. Then (l_1, m_1) and (l_2, m_2) are in $\mathscr{B}_1 \times \mathscr{B}_2$. Now

$$\begin{aligned}
& \mu_{L\times M}^{\mathscr{T}^-}[((l_1, m_1) \ddagger_1 (l_2, m_2))] \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}[((l_1,m_1)\ddagger_1(l_2,m_2))]} \\
= \quad & \mu_{L\times M}^{\mathscr{T}^-}((l_1 \ddagger_1 l_2, m_1 \ddagger_1 m_2)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}((l_1\ddagger_1 l_2, m_1\ddagger_1 m_2))} \\
= \quad & \max\{\mu_L^{\mathscr{T}^-}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_1\ddagger_1 l_2))}, \mu_M^{\mathscr{T}^-}((m_1 \ddagger_1 m_2)) \cdot e^{\delta\chi_M^{\mathscr{T}^-}((m_1\ddagger_1 m_2))}\} \\
\leq \quad & \max\{\max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l_1}), \mu_L^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l_2)}\}, \\
& \max\{\mu_M^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_1)}, \mu_M^{\mathscr{T}^-}(m_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_2)}\}\} \\
= \quad & \max\{\max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l_1}), \mu_M^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_1})\}, \\
& \max\{\mu_L^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l_2)}, \mu_M^{\mathscr{T}^-}(m_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_2)}\}\} \\
= \quad & \max\{\mu_{L\times M}^{\mathscr{T}^-}((l_1, m_1)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}((l_1,m_1))}, \mu_{L\times M}^{\mathscr{T}^-}((l_2, m_2)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}((l_2,m_2))}\}
\end{aligned}$$

Also $\mu_{L\times M}^{\mathscr{T}^-}[((l_1, m_1) \ddagger_2 (l_2, m_2))] \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}[((l_1,m_1)\ddagger_2(l_2,m_2))]}$
$\leq \max\{\mu_{L\times M}^{\mathscr{T}^-}((l_1, m_1)) \cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}((l_1,m_1))}, \mu_{L\times M}^{\mathscr{T}^-}((l_2, m_2))e^{\delta\chi_{L\times M}^{\mathscr{T}^-}((l_2,m_2))}\}$

and $\mu_{L\times M}^{\mathscr{T}^-}[((l_1,m_1)\ddagger_3(l_2,m_2))]\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}[((l_1,m_1)\ddagger_3(l_2,m_2))]}$
$\leq \max\{\mu_{L\times M}^{\mathscr{T}^-}((l_1,m_1))\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}((l_1,m_1))},\mu_{L\times M}^{\mathscr{T}^-}((l_2,m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^-}((l_2,m_2))}\}$.

Now,

$$\begin{aligned}
& \mu_{L\times M}^{\mathscr{F}^-}[((l_1,m_1)\ddagger_1(l_2,m_2))]\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}[((l_1,m_1)\ddagger_1(l_2,m_2))]} \\
= \; & \mu_{L\times M}^{\mathscr{F}^-}((l_1\ddagger_1 l_2,m_1\ddagger_1 m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}((l_1\ddagger_1 l_2,m_1\ddagger_1 m_2))} \\
= \; & \min\{\mu_L^{\mathscr{F}^-}((l_1\ddagger_1 l_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}((l_1\ddagger_1 l_2))},\mu_M^{\mathscr{F}^-}((m_1\ddagger_1 m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}((m_1\ddagger_1 m_2))}\} \\
\geq \; & \min\{\min\{\mu_L^{\mathscr{F}^-}(l_1)\cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)},\mu_L^{\mathscr{F}^-}(l_2)\cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_2)}\}, \\
& \min\{\mu_M^{\mathscr{F}^-}(m_1)\cdot e^{\delta\chi_M^{\mathscr{F}^-}(m_1)},\mu_M^{\mathscr{F}^-}(m_2)\cdot e^{\delta\chi_M^{\mathscr{F}^-}(m_2)}\}\} \\
= \; & \min\{\min\{\mu_L^{\mathscr{F}^-}(l_1)\cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)},\mu_M^{\mathscr{F}^-}(m_1)\cdot e^{\delta\chi_M^{\mathscr{F}^-}(m_1)}\}, \\
& \min\{\mu_L^{\mathscr{F}^-}(l_2)\cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_2)},\mu_M^{\mathscr{F}^-}(m_2)\cdot e^{\delta\chi_M^{\mathscr{F}^-}(m_2)}\}\} \\
= \; & \min\{\mu_{L\times M}^{\mathscr{F}^-}((l_1,m_1))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}((l_1,m_1))},\mu_{L\times M}^{\mathscr{F}^-}((l_2,m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}((l_2,m_2))}\}
\end{aligned}$$

Also $\mu_{L\times M}^{\mathscr{F}^-}[((l_1,m_1)\ddagger_2(l_2,m_2))]\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}[((l_1,m_1)\ddagger_2(l_2,m_2))]}$
$\geq \min\{\mu_{L\times M}^{\mathscr{F}^-}((l_1,m_1))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}((l_1,m_1))},\mu_{L\times M}^{\mathscr{F}^-}((l_2,m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}}((l_2,m_2))\}$,
$\mu_{L\times M}^{\mathscr{F}^-}[((l_1,m_1)\ddagger_3(l_2,m_2))]\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}[((l_1,m_1)\ddagger_3(l_2,m_2))]}$
$\geq \min\{\mu_{L\times M}^{\mathscr{F}^-}((l_1,m_1))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}((l_1,m_1))},\mu_{L\times M}^{\mathscr{F}^-}((l_2,m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{F}^-}((l_2,m_2))}\}$.

Let $l_1,l_2\in\mathscr{B}_1$ and $m_1,m_2\in\mathscr{B}_2$. Then (l_1,m_1) and (l_2,m_2) are in $\mathscr{B}_1\times\mathscr{B}_2$. Now

$$\begin{aligned}
& \mu_{L\times M}^{\mathscr{T}^+}[((l_1,m_1)\ddagger_1(l_2,m_2))]\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}[((l_1,m_1)\ddagger_1(l_2,m_2))]} \\
= \; & \mu_{L\times M}^{\mathscr{T}^+}((l_1\ddagger_1 l_2,m_1\ddagger_1 m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}((l_1\ddagger_1 l_2,m_1\ddagger_1 m_2))} \\
= \; & \min\{\mu_L^{\mathscr{T}^+}((l_1\ddagger_1 l_2))\cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_1 l_2))},\mu_M^{\mathscr{T}^+}((m_1\ddagger_1 m_2))\cdot e^{\delta\chi_M^{\mathscr{T}^+}((m_1\ddagger_1 m_2))}\} \\
\geq \; & \min\{\min\{\mu_L^{\mathscr{T}^+}(l_1)\cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1}),\mu_L^{\mathscr{T}^+}(l_2)\cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_2)}\}, \\
& \min\{\mu_M^{\mathscr{T}^+}(m_1)\cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)},\mu_M^{\mathscr{T}^+}(m_2)\cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_2)}\}\} \\
= \; & \min\{\min\{\mu_L^{\mathscr{T}^+}(l_1)\cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1}),\mu_M^{\mathscr{T}^+}(m_1)\cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1})\}, \\
& \min\{\mu_L^{\mathscr{T}^+}(l_2)\cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_2)},\mu_M^{\mathscr{T}^+}(m_2)\cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_2)}\}\} \\
= \; & \min\{\mu_{L\times M}^{\mathscr{T}^+}((l_1,m_1))\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}((l_1,m_1))},\mu_{L\times M}^{\mathscr{T}^+}((l_2,m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}((l_2,m_2))}\}
\end{aligned}$$

Also $\mu_{L\times M}^{\mathscr{T}^+}[((l_1,m_1)\ddagger_2(l_2,m_2))]\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}[((l_1,m_1)\ddagger_2(l_2,m_2))]}$
$\geq \min\{\mu_{L\times M}^{\mathscr{T}^+}((l_1,m_1))\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}((l_1,m_1))},\mu_{L\times M}^{\mathscr{T}^+}((l_2,m_2))e^{\delta\chi_{L\times M}^{\mathscr{T}^+}((l_2,m_2))}\}$

and $\mu_{L\times M}^{\mathscr{T}^+}[((l_1,m_1)\ddagger_3(l_2,m_2))]\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}[((l_1,m_1)\ddagger_3(l_2,m_2))]}$
$\geq \min\{\mu_{L\times M}^{\mathscr{T}^+}((l_1,m_1))\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}((l_1,m_1))},\mu_{L\times M}^{\mathscr{T}^+}((l_2,m_2))\cdot e^{\delta\chi_{L\times M}^{\mathscr{T}^+}((l_2,m_2))}\}$.

Now,

$$\begin{aligned}
& \mu^{\mathscr{F}^+}_{L\times M}[((l_1,m_1)\ddagger_1(l_2,m_2))]\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}[((l_1,m_1)\ddagger_1(l_2,m_2))]} \\
= \;& \mu^{\mathscr{F}^+}_{L\times M}((l_1\ddagger_1 l_2, m_1\ddagger_1 m_2))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}((l_1\ddagger_1 l_2,m_1\ddagger_1 m_2))} \\
= \;& \max\{\mu^{\mathscr{F}^+}_{L}((l_1\ddagger_1 l_2))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}((l_1\ddagger_1 l_2))}, \mu^{\mathscr{F}^+}_{M}((m_1\ddagger_1 m_2))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}((m_1\ddagger_1 m_2))}\} \\
\leq \;& \max\{\max\{\mu^{\mathscr{F}^+}_{L}(l_1)\cdot e^{\delta\chi^{\mathscr{F}^+}_{L}(l_1)}, \mu^{\mathscr{F}^+}_{L}(l_2)\cdot e^{\delta\chi^{\mathscr{F}^+}_{L}(l_2)}\}, \\
& \max\{\mu^{\mathscr{F}^+}_{M}(m_1)\cdot e^{\delta\chi^{\mathscr{F}^+}_{M}(m_1)}, \mu^{\mathscr{F}^+}_{M}(m_2)\cdot e^{\delta\chi^{\mathscr{F}^+}_{M}(m_2)}\}\} \\
= \;& \max\{\max\{\mu^{\mathscr{F}^+}_{L}(l_1)\cdot e^{\delta\chi^{\mathscr{F}^+}_{L}(l_1)}, \mu^{\mathscr{F}^+}_{M}(m_1)\cdot e^{\delta\chi^{\mathscr{F}^+}_{M}(m_1)}\}, \\
& \max\{\mu^{\mathscr{F}^+}_{L}(l_2)\cdot e^{\delta\chi^{\mathscr{F}^+}_{L}(l_2)}, \mu^{\mathscr{F}^+}_{M}(m_2)\cdot e^{\delta\chi^{\mathscr{F}^+}_{M}(m_2)}\}\} \\
= \;& \max\{\mu^{\mathscr{F}^+}_{L\times M}((l_1,m_1))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}((l_1,m_1))}, \mu^{\mathscr{F}^+}_{L\times M}((l_2,m_2))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}((l_2,m_2))}\}
\end{aligned}$$

Also $\mu^{\mathscr{F}^+}_{L\times M}[((l_1,m_1)\ddagger_2(l_2,m_2))]\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}[((l_1,m_1)\ddagger_2(l_2,m_2))]}$
$\leq \max\{\mu^{\mathscr{F}^+}_{L\times M}((l_1,m_1))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}((l_1,m_1))}$,
$\mu^{\mathscr{F}^+}_{L\times M}((l_2,m_2))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}}((l_2,m_2))\}$,
$\mu^{\mathscr{F}^+}_{L\times M}[((l_1,m_1)\ddagger_3(l_2,m_2))]\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}[((l_1,m_1)\ddagger_3(l_2,m_2))]}$
$\leq \max\{\mu^{\mathscr{F}^+}_{L\times M}((l_1,m_1))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}((l_1,m_1))}, \mu^{\mathscr{F}^+}_{L\times M}((l_2,m_2))\cdot e^{\delta\chi^{\mathscr{F}^+}_{L\times M}((l_2,m_2))}\}$.
Thus,$L\times M$ is a CBIFSBS of $\mathscr{B}$.

Corollary 3.9. *If $L_1, L_2, ..., L_-$ be the finite collection of CBIFSBSs of $\mathscr{B}_1, \mathscr{B}_2, ..., \mathscr{B}_n$ respectively. Then $L_1\times L_2\times ...\times L_n$ is a CBIFSBS of $\mathscr{B}_1\times\mathscr{B}_2\times ...\times\mathscr{B}_n$.*

Definition 3.10. Let $L\subseteq\mathscr{B}$,the strongest CBN relation on $\mathscr{B}$ is

$$\left\{\begin{array}{l}
\mu^{\mathscr{T}^-}_{\tau}((l,m))\cdot e^{\delta\chi^{\mathscr{T}^-}_{\tau}((l,m))} = min\{\mu^{\mathscr{T}^-}_{L}(l)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(l)}, \mu^{\mathscr{T}^-}_{L}(m)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(m)}\} \\
\mu^{\mathscr{F}^-}_{\tau}((l,m))\cdot e^{\delta\chi^{\mathscr{F}^-}_{\tau}((l,m))} = max\{\mu^{\mathscr{F}^-}_{L}(l)\cdot e^{\delta\chi^{\mathscr{F}^-}_{L}(l)}, \mu^{\mathscr{F}^-}_{L}(m)\cdot e^{\delta\chi^{\mathscr{F}^-}_{L}(m)}\}
\end{array}\right\}$$

$$\left\{\begin{array}{l}
\mu^{\mathscr{T}^+}_{\tau}((l,m))\cdot e^{\delta\chi^{\mathscr{T}^+}_{\tau}((l,m))} = \max\{\mu^{\mathscr{T}^+}_{L}(l)\cdot e^{\delta\chi^{\mathscr{T}^+}_{L}(l)}, \mu^{\mathscr{T}^+}_{L}(m)\cdot e^{\delta\chi^{\mathscr{T}^+}_{L}(m)}\} \\
\mu^{\mathscr{F}^+}_{\tau}((l,m))\cdot e^{\delta\chi^{\mathscr{F}^+}_{\tau}((l,m))} = \min\{\mu^{\mathscr{F}^+}_{L}(l)\cdot e^{\delta\chi^{\mathscr{F}^+}_{L}(l)}, \mu^{\mathscr{F}^+}_{L}(m)\cdot e^{\delta\chi^{\mathscr{F}^+}_{L}(m)}\}
\end{array}\right\}$$

Theorem 3.11. *Let L be a CBIFSBS of $\mathscr{B}$ and τ be a strongest complex bipolar intuitionistic fuzzy relation of $\mathscr{B}$. Then L is a CBIFSBS of $\mathscr{B}\times\mathscr{B}$ if and only if τ is a CBIFSBS of $\mathscr{B}\times\mathscr{B}$.*

Proof. Suppose L is a CBIFSBS of $\mathscr{B}\times\mathscr{B}$ and τ be the strongest complex bipolar intuitionistic fuzzy relation of $\mathscr{B}$.
For any $l=(l_1,l_2), m=(m_1,m_2)\in\mathscr{B}\times\mathscr{B}$. Now,

$$\begin{aligned}
& \mu^{\mathscr{T}^-}_{\tau}((l\ddagger_1 m))\cdot e^{\delta\chi^{\mathscr{T}^-}_{\tau}((l\ddagger_1 m))} \\
= \;& \mu^{\mathscr{T}^-}_{\tau}[((((l_1,l_2))\ddagger_1((m_1,m_2))]\cdot e^{\delta\chi^{\mathscr{T}^-}_{\tau}[((((l_1,l_2))\ddagger_1((m_1,m_2))]} \\
= \;& \mu^{\mathscr{T}^-}_{\tau}(l_1\ddagger_1 m_1, l_2\ddagger_1 m_2)\cdot e^{\delta\chi^{\mathscr{T}^-}_{\tau}(l_1\ddagger_1 m_1,l_2\ddagger_1 m_2)} \\
= \;& \max\{\mu^{\mathscr{T}^-}_{L}((l_1\ddagger_1 m_1))\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}((l_1\ddagger_1 m_1))}, \mu^{\mathscr{T}^-}_{L}((l_2\ddagger_1 m_2))\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}((l_2\ddagger_1 m_2))}\} \\
\leq \;& \max\{\max\{\mu^{\mathscr{T}^-}_{L}(l_1)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(l_1)}, \mu^{\mathscr{T}^-}_{L}(m_1)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(m_1)}\}, \\
& \max\{\mu^{\mathscr{T}^-}_{L}(l_2)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(l_2)}, \mu^{\mathscr{T}^-}_{L}(m_2)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(m_2)}\}\} \\
= \;& \max\{\max\{\mu^{\mathscr{T}^-}_{L}(l_1)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(l_1)}, \mu^{\mathscr{T}^-}_{L}(l_2)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(l_2)}\}, \\
& \max\{\mu^{\mathscr{T}^-}_{L}(m_1)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(m_1)}, \mu^{\mathscr{T}^-}_{L}(m_2)\cdot e^{\delta\chi^{\mathscr{T}^-}_{L}(m_2)}\}\} \\
= \;& \max\{\mu^{\mathscr{T}^-}_{\tau}((l_1,l_2))\cdot e^{\delta\chi^{\mathscr{T}^-}_{\tau}((l_1,l_2))}, \mu^{\mathscr{T}^-}_{\tau}((m_1,m_2))\cdot e^{\delta\chi^{\mathscr{T}^-}_{\tau}((m_1,m_2))}\} \\
= \;& \max\{\mu^{\mathscr{T}^-}_{\tau}(l)\cdot e^{\delta\chi^{\mathscr{T}^-}_{\tau}(l)}, \mu^{\mathscr{T}^-}_{\tau}(m)\cdot e^{\delta\chi^{\mathscr{T}^-}_{\tau}(m)}\}
\end{aligned}$$

Also $\mu_{\tau}^{\mathscr{T}^-}((l \ddagger_2 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}((l\ddagger_2 m))} \leq \max\{\mu_{\tau}^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}(l)}, \mu_{\tau}^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}(m)}\}$,
$\mu_{\tau}^{\mathscr{T}^-}((l \ddagger_3 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}((l\ddagger_3 m))} \leq \max\{\mu_{\tau}^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}(l)}, \mu_{\tau}^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}(m)}\}$.
Similarly,$\mu_{\tau}^{\mathscr{F}^-}((l \ddagger_1 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^-}((l\ddagger_1 m))} \geq \min\{\mu_{\tau}^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^-}(l)\cdot}, \mu_{\tau}^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^-}(m)}\}$,
$\mu_{\tau}^{\mathscr{F}^-}((l \ddagger_2 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^-}((l\ddagger_2 m))} \geq \min\{\mu_{\tau}^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^-}(l)}, \mu_{\tau}^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^-}(m)}\}$ and
$\mu_{\tau}^{\mathscr{F}^-}((l \ddagger_3 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^-}((l\ddagger_3 m))} \geq \min\{\mu_{\tau}^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^-}(l)}, \mu_{\tau}^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_{\tau}^{I^-}(m}\}$.

For any $l = (l_1, l_2), m = (m_1, m_2) \in \mathscr{B} \times \mathscr{B}$. Now,

$$\begin{aligned}
& \mu_{\tau}^{\mathscr{T}^+}((l \ddagger_1 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}((l\ddagger_1 m))} \\
= \quad & \mu_{\tau}^{\mathscr{T}^+}[((((l_1, l_2)) \ddagger_1 ((m_1, m_2))] \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}[((((l_1,l_2))\ddagger_1((m_1,m_2))]} \\
= \quad & \mu_{\tau}^{\mathscr{T}^+}(l_1 \ddagger_1 m_1, l_2 \ddagger_1 m_2) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}(l_1\ddagger_1 m_1, l_2\ddagger_1 m_2)} \\
= \quad & \min\{\mu_{L}^{\mathscr{T}^+}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}((l_1\ddagger_1 m_1))}, \mu_{L}^{\mathscr{T}^+}((l_2 \ddagger_1 m_2)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}((l_2\ddagger_1 m_2))}\} \\
\geq \quad & \min\{\min\{\mu_{L}^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}(l_1)}, \mu_{L}^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}(m_1)}\}, \\
& \min\{\mu_{L}^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}(l_2)}, \mu_{L}^{\mathscr{T}^+}(m_2) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}(m_2)}\}\} \\
= \quad & \min\{\min\{\mu_{L}^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}(l_1)}, \mu_{L}^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}(l_2)}\}, \\
& \min\{\mu_{L}^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}(m_1)}, \mu_{L}^{\mathscr{T}^+}(m_2) \cdot e^{\delta\chi_{L}^{\mathscr{T}^+}(m_2)}\}\} \\
= \quad & \min\{\mu_{\tau}^{\mathscr{T}^+}((l_1, l_2)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}((l_1,l_2))}, \mu_{\tau}^{\mathscr{T}^+}((m_1, m_2)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}((m_1,m_2))}\} \\
= \quad & \min\{\mu_{\tau}^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}(l)}, \mu_{\tau}^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}(m)}\}
\end{aligned}$$

Also $\mu_{\tau}^{\mathscr{T}^+}((l \ddagger_2 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}((l\ddagger_2 m))} \geq \min\{\mu_{\tau}^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}(l)}, \mu_{\tau}^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}(m)}\}$,
$\mu_{\tau}^{\mathscr{T}^+}((l \ddagger_3 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}((l\ddagger_3 m))} \geq \min\{\mu_{\tau}^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}(l)}, \mu_{\tau}^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^+}(m)}\}$.

Similarly, $\mu_{\tau}^{\mathscr{F}^+}((l \ddagger_1 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}((l\ddagger_1 m))} \leq \max\{\mu_{\tau}^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}(l)\cdot}, \mu_{\tau}^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}(m)}\}$,
$\mu_{\tau}^{\mathscr{F}^+}((l \ddagger_2 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}((l\ddagger_2 m))} \leq \max\{\mu_{\tau}^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}(l)}, \mu_{\tau}^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}(m)}\}$ and
$\mu_{\tau}^{\mathscr{F}^+}((l \ddagger_3 m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}((l\ddagger_3 m))} \leq \max\{\mu_{\tau}^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}(l)}, \mu_{\tau}^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^+}(m}\}$.
Therefore, τ is a CBIFSBS of $\mathscr{B} \times \mathscr{B}$.
Conversely,suppose that τ is a CBIFSBS of $\mathscr{B} \times \mathscr{B}$.
Let $l = ((l_1, l_2)), m = ((m_1, m_2)) \in \mathscr{B} \times \mathscr{B}$. Now,

$$\begin{aligned}
& \max\{\mu_{L}^{\mathscr{T}^-}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}((l_1\ddagger_1 m_1))}, \mu_{L}^{\mathscr{T}^-}((l_2 \ddagger_1 m_2)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}((l_2\ddagger_1 m_2))}\} \\
& = \mu_{\tau}^{\mathscr{T}^-}(l_1 \ddagger_1 m_1, l_2 \ddagger_1 m_2) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}(l_1\ddagger_1 m_1, l_2\ddagger_1 m_2)} \\
& = \mu_{\tau}^{\mathscr{T}^-}[((l_1, l_2)) \ddagger_1 ((m_1, m_2))] \cdot e^{\delta\chi_{v}^{\mathscr{T}^+}[((l_1,l_2))\ddagger_1((m_1,m_2))]} \\
& = \mu_{\tau}^{\mathscr{T}^-}(l \ddagger_1 m) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}(l\ddagger_1 m)} \\
& \leq \max\{\mu_{\tau}^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}(l)}, \mu_{\tau}^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}(m)}\} \\
& = \max\{\mu_{\tau}^{\mathscr{T}^-}((l_1, l_2))\} \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}((l_1,l_2))\}}, \mu_{\tau}^{\mathscr{T}^-}((m_1, m_2)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^-}((m_1,m_2))}\} \\
& = \max\{\max\{\mu_{L}^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(l_1)}, \mu_{L}^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(l_2)}\}, \\
& \max\{\mu_{L}^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(m_1)}, \mu_{L}^{\mathscr{T}^-}(m_2) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(m_2)}\}\}
\end{aligned}$$

If $\mu_{L}^{\mathscr{T}^-}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}((l_1\ddagger_1 m_1))} \geq \mu_{L}^{\mathscr{T}^-}((l_2 \ddagger_1 m_2)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}((l_2\ddagger_1 m_2))}$, then $\mu_{L}^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(l_1)} \geq \mu_{L}^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(l_2)}$ and $\mu_{L}^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(m_1)} \geq \mu_{L}^{\mathscr{T}^-}(m_2) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(m_2)}$. We get $\mu_{L}^{\mathscr{T}^-}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}((l_1\ddagger_1 m_1))} \leq \max\{\mu_{L}^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(l_1)}, \mu_{L}^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(m_1)}\}$ for all $l_1, m_1 \in \mathscr{B}$,and
$\max\{\mu_{L}^{\mathscr{T}^-}((l_1 \ddagger_2 m_1)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}((l_1\ddagger_2 m_1))}, \mu_{L}^{\mathscr{T}^-}((l_2 \ddagger_2 m_2)) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}((l_2\ddagger_2 m_2))}\} \leq \max\{\max\{\mu_{L}^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_{L}^{\mathscr{T}^-}(l_1)}, \mu_{L}^{\mathscr{T}^-}(l_2) \cdot$

$e^{\delta\chi_L^{\mathscr{T}^-}(l_2)}\}, \max\{\mu_L^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_1)}, \mu_L^{\mathscr{T}^-}(m_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_2)}\}\}$

If $\mu_L^{\mathscr{T}^-}((l_1\ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_1\ddagger_2 m_1))} \geq \mu_L^{\mathscr{T}^-}((l_2\ddagger_2 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_2\ddagger_2 m_2))}$,then $\mu_L^{\mathscr{T}^-}((l_1\ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_1\ddagger_2 m_1))} \leq \max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l_1)}, \mu_L^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_1)}\}$.

$\max\{\mu_L^{\mathscr{T}^-}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_1\ddagger_3 m_1))}, \mu_L^{\mathscr{T}^-}((l_2\ddagger_3 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_2\ddagger_3 m_2))}\} \leq \max\{\max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l_1)}, \mu_L^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l_2)}\}, \max\{\mu_L^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_1)}, \mu_L^{\mathscr{T}^-}(m_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_2)}\}\}$.

If $\mu_L^{\mathscr{T}^-}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_1\ddagger_3 m_1))} \geq \mu_L^{\mathscr{T}^-}((l_2\ddagger_3 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_2\ddagger_3 m_2))}$,then $\mu_L^{\mathscr{T}^-}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l_1\ddagger_3 m_1))} \leq \max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l_1)}, \mu_L^{\mathscr{T}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m_1)}\}$.

Similarly to prove that

$\min\{\mu_L^{\mathscr{F}^-}((l_1\ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_1 m_1))}, \mu_L^{\mathscr{F}^-}((l_2\ddagger_1 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_2\ddagger_1 m_2))}\} \geq \min\{\min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)}, \mu_L^{\mathscr{F}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_2)}\}, \min\{\mu_L^{\mathscr{F}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_1)}, \mu_L^{\mathscr{F}^-}(m_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_2)}\}\}$

If $\mu_L^{\mathscr{F}^-}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_1 m_1))} \leq \mu_L^{\mathscr{F}^-}((l_2 \ddagger_1 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_2\ddagger_1 m_2))}$,then $\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)} \leq \mu_L^{\mathscr{F}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_2)}$ and $\mu_L^{\mathscr{F}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_1)} \leq \mu_L^{\mathscr{F}^-}(m_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_2)}$.

We get $\mu_L^{\mathscr{F}^-}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_1 m_1))} \geq \min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)}, \mu_L^{\mathscr{F}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_1)}\}$.

$\min\{\mu_L^{\mathscr{F}^-}((l_1 \ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_2 m_1))}, \mu_L^{\mathscr{F}^-}((l_2 \ddagger_2 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_2\ddagger_2 m_2))}\}$
$\geq \min\{\min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)}, \mu_L^{\mathscr{F}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_2)}\}, \min\{\mu_L^{\mathscr{F}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_1)}, \mu_L^{\mathscr{F}^-}(m_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}\}\}$

If $\mu_L^{\mathscr{F}^-}((l_1\ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_2 m_1))} \leq \mu_L^{\mathscr{F}^-}((l_2\ddagger_2 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_2\ddagger_2 m_2))}$,then $\mu_L^{\mathscr{F}^-}((l_1\ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_2 m_1))} \geq \min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)}, \mu_L^{\mathscr{F}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_1)}\}$.

$\min\{\mu_L^{\mathscr{F}^-}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_3 m_1))}, \mu_L^{\mathscr{F}^-}((l_2\ddagger_3 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_2\ddagger_3 m_2))}\} \geq \min\{\min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)}, \mu_L^{\mathscr{F}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_2)}\}, \min\{\mu_L^{\mathscr{F}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_1)}, \mu_L^{\mathscr{F}^-}(m_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_2)}\}\}$

If $\mu_L^{\mathscr{F}^-}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_3 m_1))} \leq \mu_L^{\mathscr{F}^-}((l_2\ddagger_3 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_2\ddagger_3 m_2))}$,then $\mu_L^{\mathscr{F}^-}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l_1\ddagger_3 m_1))} \geq \min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l_1)}, \mu_L^{\mathscr{F}^-}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m_1)}\}$.

Let $l = ((l_1, l_2)), m = ((m_1, m_2)) \in \mathscr{B} \times \mathscr{B}$. Now,

$$\begin{aligned}
&\min\{\mu_L^{\mathscr{T}^+}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_1 m_1))}, \mu_L^{\mathscr{T}^+}((l_2 \ddagger_1 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_2\ddagger_1 m_2))}\} \\
&= \mu_\tau^{\mathscr{T}^+}(l_1 \ddagger_1 m_1, l_2 \ddagger_1 m_2) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}(l_1\ddagger_1 m_1, l_2\ddagger_1 m_2)} \\
&= \mu_\tau^{\mathscr{T}^+}[((l_1, l_2)) \ddagger_1 ((m_1, m_2))] \cdot e^{\delta\chi_v^{\mathscr{T}^+}[((l_1,l_2))\ddagger_1((m_1,m_2))]} \\
&= \mu_\tau^{\mathscr{T}^+}(l \ddagger_1 m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}(l\ddagger_1 m)} \\
&\geq \min\{\mu_\tau^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}(l)}, \mu_\tau^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}(m)}\} \\
&= \min\{\mu_\tau^{\mathscr{T}^+}((l_1, l_2))\} \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}((l_1,l_2))\}}, \mu_\tau^{\mathscr{T}^+}((m_1, m_2)) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}((m_1,m_2))}\} \\
&= \min\{\min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1)}, \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_2)}\}, \\
&\min\{\mu_L^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)}, \mu_L^{\mathscr{T}^+}(m_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_2)}\}\}
\end{aligned}$$

If $\mu_L^{\mathscr{T}^+}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_1 m_1))} \leq \mu_L^{\mathscr{T}^+}((l_2 \ddagger_1 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_2\ddagger_1 m_2))}$, then $\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1)} \leq \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_2)}$ and $\mu_L^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)} \leq \mu_L^{\mathscr{T}^+}(m_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_2)}$. We get $\mu_L^{\mathscr{T}^+}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_1 m_1))} \geq \min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1)}, \mu_L^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)}\}$ for all $l_1, m_1 \in \mathscr{B}$,and

$\min\{\mu_L^{\mathscr{T}^+}((l_1\ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_2 m_1))}, \mu_L^{\mathscr{T}^+}((l_2\ddagger_2 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_2\ddagger_2 m_2))}\} \geq \min\{\min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1)}, \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_2)}\}, \min\{\mu_L^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)}, \mu_L^{\mathscr{T}^+}(m_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_2)}\}\}$

If $\mu_L^{\mathscr{T}^+}((l_1\ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_2 m_1))} \leq \mu_L^{\mathscr{T}^+}((l_2\ddagger_2 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_2\ddagger_2 m_2))}$,then $\mu_L^{\mathscr{T}^+}((l_1\ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_2 m_1))} \geq \min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1)}, \mu_L^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)}\}$.

$\min\{\mu_L^{\mathscr{T}^+}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_3 m_1))}, \mu_L^{\mathscr{T}^+}((l_2\ddagger_3 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_2\ddagger_3 m_2))}\} \geq \min\{\min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1)}, \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_2)}\}, \min\{\mu_L^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)}, \mu_L^{\mathscr{T}^+}(m_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_2)}\}\}$

If $\mu_L^{\mathscr{T}^+}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_3 m_1))} \leq \mu_L^{\mathscr{T}^+}((l_2\ddagger_3 m_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_2\ddagger_3 m_2))}$,then $\mu_L^{\mathscr{T}^+}((l_1\ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_3 m_1))} \geq \min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l_1)}, \mu_L^{\mathscr{T}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)}\}$.

Similarly to prove that
$\max\{\mu_L^{\mathscr{F}^+}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_1\ddagger_1 m_1))}, \mu_L^{\mathscr{F}^+}((l_2 \ddagger_1 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_2\ddagger_1 m_2))}\}$
$\leq \max\{\max\{\mu_L^{\mathscr{F}^+}(l_1)\cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_1)}, \mu_L^{\mathscr{F}^+}(l_2)\cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_2)}\}, \max\{\mu_L^{\mathscr{F}^+}(m_1)\cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_1)}, \mu_L^{\mathscr{F}^+}(m_2)\cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_2)}\}\}$
If $\mu_L^{\mathscr{F}^+}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_1\ddagger_1 m_1))} \geq \mu_L^{\mathscr{F}^+}((l_2 \ddagger_1 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_2\ddagger_1 m_2))}$, then
$\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_1)} \geq \mu_L^{\mathscr{F}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_2)}$ and $\mu_L^{\mathscr{F}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_1)} \geq \mu_L^{\mathscr{F}^+}(m_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_2)}$.
We get $\mu_L^{\mathscr{F}^+}((l_1 \ddagger_1 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_1\ddagger_1 m_1))} \leq \max\{\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_1)}, \mu_L^{\mathscr{F}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_1)}\}$.
$\max\{\mu_L^{\mathscr{F}^+}((l_1 \ddagger_2 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_1\ddagger_2 m_1))}, \mu_L^{\mathscr{F}^+}((l_2 \ddagger_2 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_2\ddagger_2 m_2))}\}$
$\leq \max\{\max\{\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_1)}, \mu_L^{\mathscr{F}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_2)}\}, \max\{\mu_L^{\mathscr{F}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_1)}, \mu_L^{\mathscr{F}^+}(m_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}\}\}$
If $\mu_L^{\mathscr{F}^+}((l_1\ddagger_2 m_1))\cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_1\ddagger_2 m_1))} \geq \mu_L^{\mathscr{F}^+}((l_2\ddagger_2 m_2))\cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_2\ddagger_2 m_2))}$,then $\mu_L^{\mathscr{F}^+}((l_1\ddagger_2 m_1))\cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_1\ddagger_2 m_1))} \leq \max\{\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_1)}, \mu_L^{\mathscr{F}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_1)}\}$.
$\max\{\mu_L^{\mathscr{F}^+}((l_1 \ddagger_3 m_1)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_1\ddagger_3 m_1))}, \mu_L^{\mathscr{F}^+}((l_2 \ddagger_3 m_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l_2\ddagger_3 m_2))}\}$
$\leq \max\{\max\{\mu_L^{\mathscr{F}^+}(l_1)\cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_1)}, \mu_L^{\mathscr{F}^+}(l_2)\cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_2)}\}, \max\{\mu_L^{\mathscr{F}^+}(m_1)\cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_1)}, \mu_L^{\mathscr{F}^+}(m_2)\cdot e^{\delta\chi_L^{\mathscr{F}^+}(m_2)}\}\}$
If $\mu_L^{\mathscr{F}^+}((l_1\ddagger_3 m_1))\cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_3 m_1))} \geq \mu_L^{\mathscr{F}^+}((l_2\ddagger_3 m_2))\cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_2\ddagger_3 m_2))}$,then $\mu_L^{\mathscr{F}^+}((l_1\ddagger_3 m_1))\cdot e^{\delta\chi_L^{\mathscr{T}^+}((l_1\ddagger_3 m_1))} \leq \max\{\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l_1)}, \mu_L^{\mathscr{F}^+}(m_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m_1)}\}$.
Therefore, L is a CBIFSBS of $\mathscr{B}$.

Theorem 3.12. *Suppose that L is a subset of $\mathscr{B}$. Then $R = (\mu_L^{\mathscr{T}^-} \cdot e^{\delta\chi_L^{\mathscr{T}^-}}, \mu_L^{\mathscr{F}^-} \cdot e^{\delta\chi_L^{\mathscr{F}^-}}, \mu_L^{\mathscr{T}^+} \cdot e^{\delta\chi_L^{\mathscr{T}^+}}, \mu_L^{\mathscr{F}^+} \cdot e^{\delta\chi_L^{\mathscr{F}^+}})$ is a CBIFSBS of $\mathscr{B}$ if and only if $\mu^{(t,s)}$ is a SBS of $\mathscr{B}$ for all $t, s \in [-1, 0] \times [0, 1]$.*

Proof. Assume that μ is a CBIFSBS of $\mathscr{B}$. For each $t, s \in [-1, 0] \times [0, 1]$ and $l_1, l_2 \in \mu^{(t,s)}$. Now,$\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_1) \leq t, \mu_L^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_2) \leq t$ and $\mu_L^{I^-}(l_1) \cdot e^{\delta\chi_L^{I^-}}(l_1) \leq t, \mu_L^{I^-}(l_2) \cdot e^{\delta\chi_L^{I^-}}(l_2) \leq t$ and $\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_1) \geq s, \mu_L^{\mathscr{F}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_2) \geq s$. Now,$\mu_L^{\mathscr{T}^-}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}((l_1 \ddagger_1 l_2)) \leq \max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_1), \mu_L^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_2)\} \leq t$ and
$\mu_L^{\mathscr{F}^-}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}((l_1 \ddagger_1 l_2)) \geq \min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_1), \mu_L^{\mathscr{F}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_2)\} \geq s$. This implies that $l_1 \ddagger_1 l_2 \in \mu^{(t,s)}$.
Similarly,$l_1 \ddagger_2 l_2 \in \mu^{(t,s)}$ and $l_1 \ddagger_3 l_2 \in \mu^{(t,s)}$. Hence,$\mu^{(t,s)}$ is a SBS of $\mathscr{B}$,for all $t, s \in [-1, 0] \times [0, 1]$.

For each $t, s \in [0, 1]$ and $l_1, l_2 \in \mu^{(t,s)}$. Now,$\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_1) \geq t, \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_2) \geq t$ and $\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_1) \leq s, \mu_L^{\mathscr{F}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_2) \leq s$.
Now, $\mu_L^{\mathscr{T}^+}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}((l_1 \ddagger_1 l_2)) \geq \min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_1), \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_2)\} \geq t$ and $\mu_L^{\mathscr{F}^+}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}((l_1 \ddagger_1 l_2)) \leq \max\{\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_1), \mu_L^{\mathscr{F}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_2)\} \leq s$.
This implies that $l_1 \ddagger_1 l_2 \in \mu^{(t,s)}$. Similarly,$l_1 \ddagger_2 l_2 \in \mu^{(t,s)}$ and $l_1 \ddagger_3 l_2 \in \mu^{(t,s)}$.
Hence,$\mu^{(t,s)}$ is a SBS of $\mathscr{B}$,for all $t, s \in [-1, 0] \times [0, 1]$.

Conversely, assume that $\mu^{(t,s)}$ is a SBS of $\mathscr{B}$ and $t, s \in [-1, 0] \times [0, 1]$. Suppose if there exist $l_1, l_2 \in \mathscr{B}$ such that $\mu_L^{\mathscr{T}^-}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}((l_1 \ddagger_1 l_2)) > \max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_1), \mu_L^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_2)\}$

and $\mu_L^{\mathscr{F}^-}((l_1\ddagger_1 l_2))\cdot e^{\delta\chi_L^{\mathscr{F}^-}}((l_1\ddagger_1 l_2)) < \min\{\mu_L^{\mathscr{F}^-}(l_1)\cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_1), \mu_L^{\mathscr{F}^-}(l_2)\cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_2)\}$. For $t, s \in [-1, 0]\times [0, 1]$ such that $\mu_L^{\mathscr{T}^-}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}((l_1 \ddagger_1 l_2)) > t \geq \max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_1), \mu_L^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_2)\}$ and
$\mu_L^{\mathscr{F}^-}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}((l_1 \ddagger_1 l_2)) < s \leq \min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_1), \mu_L^{\mathscr{F}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_2)\}$. Thus, $l_1, l_2 \in \mu^{(t,s)}$,but $l_1 \ddagger_1 l_2 \notin \mu^{(t,s)}$. This contradicts, $\mu^{(t,s)}$ is a SBS of $\mathscr{B}$.
Therefore $\mu_L^{\mathscr{T}^-}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}((l_1 \ddagger_1 l_2)) \leq \max\{\mu_L^{\mathscr{T}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_1), \mu_L^{\mathscr{T}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l_2)\}$ and $\mu_L^{\mathscr{F}^-}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}((l_1 \ddagger_1 l_2)) \geq \min\{\mu_L^{\mathscr{F}^-}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_1), \mu_L^{\mathscr{F}^-}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l_2)\}$. Similarly,$\ddagger_2$ and $\ddagger_3$ cases.
Hence $\mu = (\mu_L^{\mathscr{T}^-} \cdot e^{\delta\chi_L^{\mathscr{T}^-}}, \mu_L^{\mathscr{F}^-} \cdot e^{\delta\chi_L^{\mathscr{F}^-}})$ is a CBIFSBS of $\mathscr{B}$.
Let us assume that $\mu^{(t,s)}$ is a SBS of $\mathscr{B}$ and $t, s \in [-1, 0] \times [0, 1]$. Suppose if there exist $l_1, l_2 \in \mathscr{B}$ such that $\mu_L^{\mathscr{T}^+}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}((l_1 \ddagger_1 l_2)) > \min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_1), \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_2)\}$ and $\mu_L^{\mathscr{F}^+}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}((l_1 \ddagger_1 l_2)) < \max\{\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_1), \mu_L^{\mathscr{F}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_2)\}$. For $t, s \in [-1, 0] \times [0, 1]$ such that

$\mu_L^{\mathscr{T}^+}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}((l_1 \ddagger_1 l_2)) > t \leq \min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_1), \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_2)\}$ and $\mu_L^{\mathscr{F}^+}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}((l_1 \ddagger_1 l_2)) < s \geq \max\{\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_1), \mu_L^{\mathscr{F}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_2)\}$.
Thus, $l_1, l_2 \in \mu^{(t,s)}$,but $l_1 \ddagger_1 l_2 \notin \mu^{(t,s)}$. This contradicts,$\mu^{(t,s)}$ is a SBS of $\mathscr{B}$.
Therefore $\mu_L^{\mathscr{T}^+}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}((l_1 \ddagger_1 l_2)) \geq \min\{\mu_L^{\mathscr{T}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_1), \mu_L^{\mathscr{T}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l_2)\}$ and $\mu_L^{\mathscr{F}^+}((l_1 \ddagger_1 l_2)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}((l_1 \ddagger_1 l_2)) \leq \max\{\mu_L^{\mathscr{F}^+}(l_1) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_1), \mu_L^{\mathscr{F}^+}(l_2) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l_2)\}$. Similarly, $\ddagger_2$ and $\ddagger_3$ cases. Hence $\mu = (\mu_L^{\mathscr{T}^+} \cdot e^{\delta\chi_L^{\mathscr{T}^+}}, \mu_L^{\mathscr{F}^+} \cdot e^{\delta\chi_L^{\mathscr{F}^+}})$ is a CBIFSBS of $\mathscr{B}$.

4 Homomorphism

Definition 4.1. Let $(\mathscr{B}_1, \Diamond_1, \Diamond_2, \Diamond_3)$ and $(\mathscr{B}_2, \divideontimes_1, \divideontimes_2, \divideontimes_3)$ be any two bisemirings. The mapping $\mho : \mathscr{B}_1 \to \mathscr{B}_2$ and L be any CBIFSBS in $\mathscr{B}_1$, τ be any CBIFSBS in $\mho(\mathscr{B}_1) = \mathscr{B}_2$. If $\mu_L \cdot e^{\delta\chi_L} = [\mu_L^{\mathscr{T}^-} \cdot e^{\delta\chi_L^{\mathscr{T}^-}}, \mu_L^{\mathscr{F}^-} \cdot e^{\delta\chi_L^{\mathscr{F}^-}}, \mu_L \cdot e^{\delta\chi_L}, \mu_L^{\mathscr{T}^+} \cdot e^{\delta\chi_L^{\mathscr{T}^+}}, \mu_L^{\mathscr{F}^+} \cdot e^{\delta\chi_L^{\mathscr{F}^+}}$ is a CBIFS in $\mathscr{B}_1$, then μ_τ is a CBIFS in $\mathscr{B}_2$,defined by

$$\mu_\tau^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(m) = \begin{cases} \inf \mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}}(l) & \text{if } l \in \mho^{-1}m \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_\tau^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(m) = \begin{cases} \sup \mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}}(l) & \text{if } l \in \mho^{-1}m \\ -1 & \text{otherwise} \end{cases}$$

$$\mu_\tau^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(m) = \begin{cases} \sup \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}}(l) & \text{if } l \in \mho^{-1}m \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_\tau^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(m) = \begin{cases} \inf \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}}(l) & \text{if } l \in \mho^{-1}m \\ 1 & \text{otherwise} \end{cases}$$

for all $l \in \mathscr{B}_1$ and $m \in \mathscr{B}_2$ is represents the image of R_L under $\mho$.
Similarly,If $\mu_\tau \cdot e^{\delta\chi_\tau} = [\mu_\tau^{\mathscr{T}^-} \cdot e^{\delta\chi_L^{\mathscr{T}^-}}, \mu_\tau^{I^-} \cdot e^{\delta\chi_\tau^{I^-}}, \mu_\tau^{\mathscr{F}^-} \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}}]$, $\mu_\tau \cdot e^{\delta\chi_\tau}, \mu_\tau^{\mathscr{T}^+} \cdot e^{\delta\chi_L^{\mathscr{T}^+}}, \mu_\tau^{\mathscr{F}^+} \cdot e^{\delta\chi_\tau^{\mathscr{F}^+}}$ is a CBIFS in $\mathscr{B}_2$,then CBIFS $\mu_L = \mho \circ \mu_\tau$ in $\mathscr{B}_1$ ie,the CBIFS defined by $\mu_L(l) \cdot e^{\delta\chi_L(l)}, \mu_\tau(\mho(l)) \cdot e^{\delta\chi_L(\mho(l))}$, $\mu_L = \mho \circ \mu_\tau$ in $\mathscr{B}_1$ [ie, the CBIFS defined by $\mu_L(l) \cdot e^{\delta\chi_L(l)} = \mu_\tau(\mho(l)) \cdot e^{\delta\chi_L(\mho(l))}$ is represents the pre-image of μ_τ under $\mho$.

Theorem 4.2. *The homomorphic image of every CBIFSBS is a CBIFSBS.*

Proof. The mapping $\mho : \mathscr{B}_1 \to \mathscr{B}_2$ be any homomorphism.
Now, $\mho((l \Diamond_1 m)) = \mho(l) \divideontimes_1 \mho(m), \mho((l \Diamond_2 m)) = \mho(l) \divideontimes_2 \mho(m)$ and $\mho((l \Diamond_3 m)) = \mho(l) \divideontimes_3 \mho(m)$ for all $l, m \in \mathscr{B}_1$. Let $\tau = \mho(L)$,L is any CBIFSBS of $\mathscr{B}_1$. Let $\mho(l), \mho(m) \in \mathscr{B}_2$. Let $l \in u\mho^{-1}(\mho(l))$ and $m \in \mho^{-1}(\mho(m))$ be such that $\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)} = \inf\limits_{l \in \mho^{-1}(\mho(l))} \mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)}$ and $\mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m)} = \inf\limits_{l \in \mho^{-1}(\mho(m))} \mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)}$.
Now, $\mu_\tau^{\mathscr{T}^-}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((\mho(l) \divideontimes_1 \mho(m)))}$

$$\begin{aligned}
&= \inf_{(l') \in \mho^{-1}(\mho(l) \divideontimes_1 \mho(m))} \mu_L^{\mathscr{T}^-}(l') \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l')} \\
&= \inf_{(l') \in \mho^{-1}(\mho((l \Diamond_1 m))} \mu_L^{\mathscr{T}^-}(l') \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l')} \\
&= \mu_L^{\mathscr{T}^-}((l \Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l \Diamond_1 m))} \\
&\leq \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)}, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m)}\} \\
&= \max\{\mu_\tau^{\mathscr{T}^-} \mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-} \mho(l)}, \mu_\tau^{\mathscr{T}^-} \mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-} \mho(m)}\}.
\end{aligned}$$

Thus,$\mu_\tau^{\mathscr{T}^-}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}((\mho(l)\divideontimes_1\mho(m)))} \leq \max\{\mu_\tau^{\mathscr{T}^-}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}\mho(l)}, \mu_\tau^{\mathscr{T}^-}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}\mho(m)}\}$.
Similarly,$\mu_\tau^{\mathscr{T}^-}((\mho(l) \divideontimes_2 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}((\mho(l)\divideontimes_2\mho(m)))} \leq \max\{\mu_\tau^{\mathscr{T}^-}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}\mho(l)}, \mu_\tau^{\mathscr{T}^-}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}\mho(m)}\}$
and
$\mu_\tau^{\mathscr{T}^-}((\mho(l) \divideontimes_3 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}((\mho(l)\divideontimes_3\mho(m)))} \leq \max\{\mu_\tau^{\mathscr{T}^-}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}\mho(l)}, \mu_\tau^{\mathscr{T}^-}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}\mho(m)}\}$.

Let $\mho(l), \mho(m) \in \mathscr{B}_2$. Let $l \in \mho^{-1}(\mho(l))$ and $m \in \mho^{-1}(\mho(m))$ be such that $\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)} = \sup\limits_{l \in \mho^{-1}(\mho(l))} \mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)}$ and $\mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m)} = \sup\limits_{l \in \mho^{-1}(\mho(m))} \mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)}$. Now, $\mu_\tau^{\mathscr{F}^-}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}((\mho(l)\divideontimes_1\mho(m)))}$

$$
\begin{aligned}
&= \sup_{(l') \in \mho^{-1}(\mho(l)\divideontimes_1\mho(m))} \mu_L^{\mathscr{F}^-}(l') \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l')} \\
&= \sup_{(l') \in \mho^{-1}(\mho((l\Diamond_1 m))} \mu_L^{\mathscr{F}^-}(l') \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l')} \\
&= \mu_L^{\mathscr{F}^-}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\Diamond_1 m))} \\
&\geq \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)}, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m)}\} \\
&= \min\{\mu_\tau^{\mathscr{F}^-}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}\mho(l)}, \mu_\tau^{\mathscr{F}^-}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}\mho(m)}\}.
\end{aligned}
$$

Thus, $\mu_\tau^{\mathscr{F}^-}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}((\mho(l)\divideontimes_1\mho(m)))} \geq \min\{\mu_\tau^{\mathscr{F}^-}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}\mho(l)}, \mu_\tau^{\mathscr{F}^-}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}\mho(m)}\}$.
Similarly,
$\mu_\tau^{\mathscr{F}^-}((\mho(l) \divideontimes_2 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}((\mho(l)\divideontimes_2\mho(m)))} \geq \min\{\mu_\tau^{\mathscr{F}^-}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}\mho(l)}, \mu_\tau^{\mathscr{F}^-}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}\mho(m)}\}$ and
$\mu_\tau^{\mathscr{F}^-}((\mho(l) \divideontimes_3 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}((\mho(l)\divideontimes_1\mho(m)))} \geq \min\{\mu_\tau^{\mathscr{F}^-}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}\mho(m)}, \mu_\tau^{\mathscr{F}^-}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}\mho(m)}\}$.
The mapping $\mho : \mathscr{B}_1 \to \mathscr{B}_2$ be any homomorphism. Now,
$\mho((l\Diamond_1 m)) = \mho(l) \divideontimes_1 \mho(m), \mho((l\Diamond_2 m)) = \mho(l) \divideontimes_2 \mho(m)$ and
$\mho((l\Diamond_3 m)) = \mho(l) \divideontimes_3 \mho(m)$ for all $l, m \in \mathscr{B}_1$. Let $\tau = \mho(L)$,L is any CBIFSBS of $\mathscr{B}_1$. Let $\mho(l), \mho(m) \in \mathscr{B}_2$. Let $l \in \mho^{-1}(\mho(l))$ and $m \in \mho^{-1}(\mho(m))$ be such that $\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)} = \sup\limits_{l \in \mho^{-1}(\mho(l))} \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)}$ and $\mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m)} = \sup\limits_{l \in \mho^{-1}(\mho(m))} \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)}$. Now, $\mu_\tau^{\mathscr{T}^+}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((\mho(l)\divideontimes_1\mho(m)))}$

$$
\begin{aligned}
&= \sup_{(l') \in \mho^{-1}(\mho(l)\divideontimes_1\mho(m))} \mu_L^{\mathscr{T}^+}(l') \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l')} \\
&= \sup_{(l') \in \mho^{-1}(\mho((l\Diamond_1 m))} \mu_L^{\mathscr{T}^+}(l') \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l')} \\
&= \mu_L^{\mathscr{T}^+}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\Diamond_1 m))} \\
&\geq \min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)}, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m)}\} \\
&= \min\{\mu_\tau^{\mathscr{T}^+}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}\mho(l)}, \mu_\tau^{\mathscr{T}^+}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}\mho(m)}\}.
\end{aligned}
$$

Thus, $\mu_\tau^{\mathscr{T}^+}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}((\mho(l)\divideontimes_1\mho(m)))} \geq \min\{\mu_\tau^{\mathscr{T}^+}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}\mho(l)}, \mu_\tau^{\mathscr{T}^+}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}\mho(m)}\}$.
Similarly, $\mu_\tau^{\mathscr{T}^+}((\mho(l) \divideontimes_2 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}((\mho(l)\divideontimes_2\mho(m)))} \geq \min\{\mu_\tau^{\mathscr{T}^+}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}\mho(l)}, \mu_\tau^{\mathscr{T}^+}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}\mho(m)}\}$
and
$\mu_\tau^{\mathscr{T}^+}((\mho(l) \divideontimes_3 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}((\mho(l)\divideontimes_3\mho(m)))} \geq \min\{\mu_\tau^{\mathscr{T}^+}\mho(l) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}\mho(l)}, \mu_\tau^{\mathscr{T}^+}\mho(m) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}\mho(m)}\}$.

Let $\mho(l), \mho(m) \in \mathscr{B}_2$. Let $l \in \mho^{-1}(\mho(l))$ and $m \in \mho^{-1}(\mho(m))$ be such that $\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l)} = \inf\limits_{l \in \mho^{-1}(\mho(l))} \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l)}$ and $\mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m)} = \inf\limits_{l \in \mho^{-1}(\mho(m))} \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l)}$.

Now, $\mu_{\tau}^{\mathscr{F}^{+}}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}((\mho(l)\divideontimes_1\mho(m)))}$

$$\begin{aligned}
&= \inf_{(l')\in\mho^{-1}(\mho(l)\divideontimes_1\mho(m))} \mu_L^{\mathscr{F}^{+}}(l') \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}(l')}\\
&= \inf_{(l')\in\mho^{-1}(\mho((l\Diamond_1 m))} \mu_L^{\mathscr{F}^{+}}(l') \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}(l')}\\
&= \mu_L^{\mathscr{F}^{+}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}((l\Diamond_1 m))}\\
&\leq \max\{\mu_L^{\mathscr{F}^{+}}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}(l)}, \mu_L^{\mathscr{F}^{+}}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}(m)}\}\\
&= \max\{\mu_{\tau}^{\mathscr{F}^{+}}\mho(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(l)}, \mu_{\tau}^{\mathscr{F}^{+}}\mho(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(m)}\}.
\end{aligned}$$

Thus, $\mu_{\tau}^{\mathscr{F}^{+}}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}((\mho(l)\divideontimes_1\mho(m)))} \leq \max\{\mu_{\tau}^{\mathscr{F}^{+}}\mho(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(l)}, \mu_{\tau}^{\mathscr{F}^{+}}\mho(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(m)}\}$.
Similarly, $\mu_{\tau}^{\mathscr{F}^{+}}((\mho(l)\divideontimes_2\mho(m)))\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}((\mho(l)\divideontimes_2\mho(m)))} \leq \max\{\mu_{\tau}^{\mathscr{F}^{+}}\mho(l)\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(l)}, \mu_{\tau}^{\mathscr{F}^{+}}\mho(m)\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(m)}\}$
and
$\mu_{\tau}^{\mathscr{F}^{+}}((\mho(l) \divideontimes_3 \mho(m)))\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}((\mho(l)\divideontimes_1\mho(m)))} \leq \max\{\mu_{\tau}^{\mathscr{F}^{+}}\mho(l)\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(m)}, \mu_{\tau}^{\mathscr{F}^{+}}\mho(m)\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(m)}\}$. Thus, τ is a CBIFSBS of $\mathscr{B}_2$.

Theorem 4.3. *The homomorphic preimage of every CBIFSBS is a CBIFSBS.*

Proof. The mapping $\mho : \mathscr{B}_1 \to \mathscr{B}_2$ be a homomorphism.
Now,$\mho((l\Diamond_1 m)) = \mho(l) \divideontimes_1 \mho(m), \mho((l\Diamond_2 m)) = \mho(l) \divideontimes_2 \mho(m)$ and
$\mho((l\Diamond_3 m)) = \mho(l) \divideontimes_3 \mho(m)$ for all $l, m \in \mathscr{B}_1$. Let $\tau = \mho(L)$, τ is a CBIFSBS of $\mathscr{B}_2$. Let $l, m \in \mathscr{B}_1$.
Now,$\mu_L^{\mathscr{T}^{-}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^{-}}((l\Diamond_1 m))} = \mu_{\tau}^{\mathscr{T}^{-}}(\mho((l\Diamond_1 m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{-}}(\mho((l\Diamond_1 m)))} = \mu_{\tau}^{\mathscr{T}^{-}}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{-}}((\mho(l)\divideontimes_1\mho(m)))} \leq \max\{\mu_{\tau}^{\mathscr{T}^{-}}\mho(l)\cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{-}}\mho(l)}, \mu_{\tau}^{\mathscr{T}^{-}}\mho(m)\cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{-}}\mho(m)}\} = \max\{\mu_L^{\mathscr{T}^{-}}(l)\cdot e^{\delta\chi_L^{\mathscr{T}^{-}}(l)}, \mu_L^{\mathscr{T}^{-}}(m)\cdot e^{\delta\chi_L^{\mathscr{T}^{-}}(m)}\}$. Thus,$\mu_L^{\mathscr{T}^{-}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^{-}}((l\Diamond_1 m))} \leq \max\{\mu_L^{\mathscr{T}^{-}}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^{-}}(l)}, \mu_L^{\mathscr{T}^{-}}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^{-}}(m)}\}$.
Now, $\mu_L^{\mathscr{F}^{-}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^{-}}((l\Diamond_1 m))} = \mu_{\tau}^{\mathscr{F}^{-}}(\mho((l\Diamond_1 m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{-}}(\mho((l\Diamond_1 m)))} =$
$\mu_{\tau}^{\mathscr{F}^{-}}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{-}}((\mho(l)\divideontimes_1\mho(m)))} \geq \min\{\mu_{\tau}^{\mathscr{F}^{-}}\mho(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{-}}\mho(l)}, \mu_{\tau}^{\mathscr{F}^{-}}\mho(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{-}}\mho(m)}\} =$
$\min\{\mu_L^{\mathscr{F}^{-}}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^{-}}(l)}, \mu_L^{\mathscr{F}^{-}}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^{-}}(m)}\}$.
Thus,$\mu_L^{\mathscr{F}^{-}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^{-}}((l\Diamond_1 m))} \geq \min\{\mu_L^{\mathscr{F}^{-}}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^{-}}(l)}, \mu_L^{\mathscr{F}^{-}}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^{-}}(m)}\}$.
The mapping $\mho : \mathscr{B}_1 \to \mathscr{B}_2$ be a homomorphism.
Now, $\mho((l\Diamond_1 m)) = \mho(l) \divideontimes_1 \mho(m), \mho((l\Diamond_2 m)) = \mho(l) \divideontimes_2 \mho(m)$ and
$\mho((l\Diamond_3 m)) = \mho(l) \divideontimes_3 \mho(m)$ for all $l, m \in \mathscr{B}_1$. Let $\tau = \mho(L)$,τ is a CBIFSBS of $\mathscr{B}_2$. Let $l, m \in \mathscr{B}_1$.
Now,$\mu_L^{\mathscr{T}^{+}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^{+}}((l\Diamond_1 m))} = \mu_{\tau}^{\mathscr{T}^{+}}(\mho((l\Diamond_1 m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{+}}(\mho((l\Diamond_1 m)))} = \mu_{\tau}^{\mathscr{T}^{+}}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{+}}((\mho(l)\divideontimes_1\mho(m)))} \geq \min\{\mu_{\tau}^{\mathscr{T}^{+}}\mho(l)\cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{+}}\mho(l)}, \mu_{\tau}^{\mathscr{T}^{+}}\mho(m)\cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{+}}\mho(m)}\} = \min\{\mu_L^{\mathscr{T}^{+}}(l)\cdot e^{\delta\chi_L^{\mathscr{T}^{+}}(l)}, \mu_L^{\mathscr{T}^{+}}(m)\cdot e^{\delta\chi_L^{\mathscr{T}^{+}}(m)}\}$. Thus,$\mu_L^{\mathscr{T}^{+}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^{+}}((l\Diamond_1 m))} \geq \min\{\mu_L^{\mathscr{T}^{+}}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^{+}}(l)}, \mu_L^{\mathscr{T}^{+}}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^{+}}(m)}\}$.
Now, $\mu_L^{\mathscr{F}^{+}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}((l\Diamond_1 m))} = \mu_{\tau}^{\mathscr{F}^{+}}(\mho((l\Diamond_1 m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}(\mho((l\Diamond_1 m)))}$
$= \mu_{\tau}^{\mathscr{F}^{+}}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}((\mho(l)\divideontimes_1\mho(m)))} \leq \max\{\mu_{\tau}^{\mathscr{F}^{+}}\mho(l) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(l)}, \mu_{\tau}^{\mathscr{F}^{+}}\mho(m) \cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{+}}\mho(m)}\}$
$= \max\{\mu_L^{\mathscr{F}^{+}}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}(l)}, \mu_L^{\mathscr{F}^{+}}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}(m)}\}$.
Thus, $\mu_L^{\mathscr{F}^{+}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}((l\Diamond_1 m))} \leq \max\{\mu_L^{\mathscr{F}^{+}}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}(l)}, \mu_L^{\mathscr{F}^{+}}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^{+}}(m)}\}$.

Theorem 4.4. *If $\mho : \mathscr{B}_1 \to \mathscr{B}_2$ is a homomorphism,then $\mho(L_{(t,s)})$ is a SBS of CBIFSBS τ of $\mathscr{B}_2$.*

Proof. The mapping $\mho : \mathscr{B}_1 \to \mathscr{B}_2$ be a homomorphism.
Now,$\mho((l\Diamond_1 m)) = \mho(l) \divideontimes_1 \mho(m), \mho((l\Diamond_2 m)) = \mho(l) \divideontimes_2 \mho(m)$
and $\mho((l\Diamond_3 m)) = \mho(l) \divideontimes_3 \mho(m)$ for all $l, m \in \mathscr{B}_1$. Let $\tau = \mho(L)$,L is a CBIFSBS of $\mathscr{B}_1$. By Theorem 4.2, τ is a CBIFSBS of $\mathscr{B}_2$. Let $L_{(t,s)}$ be any SBS of L. Suppose that $l, m \in L_{(t,s)}$. Then $l\Diamond_1 m, l\Diamond_2 m$ and $l\Diamond_3 m \in L_{(t,s)}$. Now,$\mu_{\tau}^{\mathscr{T}^{-}}(\mho(l)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{-}}(\mho(l))} = \mu_L^{\mathscr{T}^{-}}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^{-}}(l)} \leq t, \mu_{\tau}^{\mathscr{T}^{-}}(\mho(m)) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{-}}(\mho(m))} = \mu_L^{\mathscr{T}^{-}}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^{-}}(m)} \leq t$. Thus, $\mu_{\tau}^{\mathscr{T}^{-}}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_{\tau}^{\mathscr{T}^{-}}((\mho(l)\divideontimes_1\mho(m)))} \leq \mu_L^{\mathscr{T}^{-}}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^{-}}((l\Diamond_1 m))} \leq t$.
Now, $\mu_{\tau}^{\mathscr{F}^{-}}(\mho(l))\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{-}}(\mho(l))} = \mu_L^{\mathscr{F}^{-}}(l)\cdot e^{\delta\chi_L^{\mathscr{F}^{-}}(l)} \geq s, \mu_{\tau}^{\mathscr{F}^{-}}(\mho(m))\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{-}}(\mho(m))} = \mu_L^{\mathscr{F}^{-}}(m)\cdot e^{\delta\chi_L^{\mathscr{F}^{-}}(m)} \geq s$. Thus,$\mu_{\tau}^{\mathscr{F}^{-}}((\mho(l)\divideontimes_1\mho(m)))\cdot e^{\delta\chi_{\tau}^{\mathscr{F}^{-}}((\mho(l)\divideontimes_1\mho(m)))} \geq \mu_L^{\mathscr{F}^{-}}((l\Diamond_1 m))\cdot e^{\delta\chi_L^{\mathscr{F}^{-}}((l\Diamond_1 m))} \geq s$,for all $\mho(l), \mho(m) \in \mathscr{B}_2$. Similarly other operations,$\mho(L_{(t,s)})$ is a SBS of CBIFSBS τ of $\mathscr{B}_2$.

The mapping $\mho : \mathscr{B}_1 \rightarrow \mathscr{B}_2$ be a homomorphism.
Now, $\mho((l\Diamond_1 m)) = \mho(l) \divideontimes_1 \mho(m), \mho((l\Diamond_2 m)) = \mho(l) \divideontimes_2 \mho(m)$
and $\mho((l\Diamond_3 m)) = \mho(l) \divideontimes_3 \mho(m)$ for all $l, m \in \mathscr{B}_1$. Let $\tau = \mho(L)$,L is a CBIFSBS of $\mathscr{B}_1$. By Theorem 4.2, τ is a CBIFSBS of $\mathscr{B}_2$. Let $L_{(t,s)}$ be any SBS of L. Suppose that $l, m \in L_{(t,s)}$. Then $l\Diamond_1 m, l\Diamond_2 m$ and $l\Diamond_3 m \in L_{(t,s)}$. Now,$\mu_\tau^{\mathscr{T}^+}(\mho(l)) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}(\mho(l))} = \mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)} \geq t, \mu_\tau^{\mathscr{T}^+}(\mho(m)) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}(\mho(m))} = \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m)} \geq t$. Thus,$\mu_\tau^{\mathscr{T}^+}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}((\mho(l)\divideontimes_1\mho(m)))} \geq \mu_L^{\mathscr{T}^+}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\Diamond_1 m))} \geq t$.

Now, $\mu_\tau^{\mathscr{F}^+}(\mho(l)) \cdot e^{\delta\chi_\tau^{\mathscr{F}^+}(\mho(l))} = \mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l)} \leq s, \mu_\tau^{\mathscr{F}^+}(\mho(m)) \cdot e^{\delta\chi_\tau^{\mathscr{F}^+}(\mho(m))} = \mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m)} \leq s$. Thus,$\mu_\tau^{\mathscr{F}^+}((\mho(l)\divideontimes_1\mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{F}^+}((\mho(l)\divideontimes_1\mho(m)))} \leq \mu_L^{\mathscr{F}^+}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\Diamond_1 m))} \leq s$,for all $\mho(l), \mho(m) \in \mathscr{B}_2$. Similarly other operations, $\mho(L_{(t,s)})$ is a SBS of CBIFSBS τ of $\mathscr{B}_2$.

Theorem 4.5. *If $\mho : \mathscr{B}_1 \rightarrow \mathscr{B}_2$ is any homomorphism,then $L_{(t,s)}$ is a SBS of CBIFSBS L of $\mathscr{B}_1$.*

Proof. The mapping $\mho : \mathscr{B}_1 \rightarrow \mathscr{B}_2$ be any homomorphism.
We have $\mho((l\Diamond_1 m)) = \mho(l) \divideontimes_1 \mho(m), \mho((l\Diamond_2 m)) = \mho(l) \divideontimes_2 \mho(m)$ and $\mho((l\Diamond_3 m)) = \mho(l) \divideontimes_3 \mho(m)$ for all $l, m \in \mathscr{B}_1$. Let $\tau = \mho(L)$,τ is a CBIFSBS of $\mathscr{B}_2$. By Theorem 4.3,L is a CBIFSBS of $\mathscr{B}_1$. Let $\mho(L_{(t,s)})$ be a SBS of τ. Suppose that $\mho(l), \mho(m) \in \mho(L_{(t,s)})$. Now,$\mho((l\Diamond_1 m)), \mho((l\Diamond_2 m))$ and $\mho((l\Diamond_3 m)) \in \mho(L_{(t,s)})$.
Now, $\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)} = \mu_\tau^{\mathscr{T}^-}(\mho(l)) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}(\mho(l))} \leq t, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m)} = \mu_\tau^{\mathscr{T}^-}(\mho(m)) \cdot e^{\delta\chi_\tau^{\mathscr{T}^-}(\mho(m))} \leq t$. Thus,$\mu_L^{\mathscr{T}^-}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^-}((l\Diamond_1 m))} \leq \max\{\mu_L^{\mathscr{T}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(l)}, \mu_L^{\mathscr{T}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^-}(m)}\} \leq t$.

Now, $\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)} = \mu_\tau^{\mathscr{F}^-}(\mho(l)) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}(\mho(l))} \geq s, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m)} = \mu_\tau^{\mathscr{F}^-}(\mho(m)) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}(\mho(m))} \geq s$.
Thus, $\mu_L^{\mathscr{F}^-}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^-}((l\Diamond_1 m))} = \mu_\tau^{\mathscr{F}^-}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{F}^-}((\mho(l)\divideontimes_1\mho(m)))}$
$\geq \min\{\mu_L^{\mathscr{F}^-}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(l)}, \mu_L^{\mathscr{F}^-}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^-}(m)}\} \geq s$,for all $l, m \in \mathscr{B}_1$.
Similarly other operations,$L_{(t,s)}$ is a SBS of CBIFSBS L of $\mathscr{B}_1$.

The mapping $\mho : \mathscr{B}_1 \rightarrow \mathscr{B}_2$ be any homomorphism.
We have $\mho((l\Diamond_1 m)) = \mho(l) \divideontimes_1 \mho(m), \mho((l\Diamond_2 m)) = \mho(l) \divideontimes_2 \mho(m)$ and
$\mho((l\Diamond_3 m)) = \mho(l) \divideontimes_3 \mho(m)$ for all $l, m \in \mathscr{B}_1$. Let $\tau = \mho(L)$,τ is a CBIFSBS of $\mathscr{B}_2$. By Theorem 4.3,L is a CBIFSBS of $\mathscr{B}_1$. Let $\mho(L_{(t,s)})$ be a SBS of τ. Suppose that $\mho(l), \mho(m) \in \mho(L_{(t,s)})$. Now,$\mho((l\Diamond_1 m)), \mho((l\Diamond_2 m))$ and $\mho((l\Diamond_3 m)) \in \mho(L_{(t,s)})$.
Now,$\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)} = \mu_\tau^{\mathscr{T}^+}(\mho(l)) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}(\mho(l))} \geq t, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m)} = \mu_\tau^{\mathscr{T}^+}(\mho(m)) \cdot e^{\delta\chi_\tau^{\mathscr{T}^+}(\mho(m))} \geq t$.
Thus, $\mu_L^{\mathscr{T}^+}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{T}^+}((l\Diamond_1 m))} \geq \min\{\mu_L^{\mathscr{T}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(l)}, \mu_L^{\mathscr{T}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{T}^+}(m)}\} \geq t$.

Now, $\mu_L^{\mathscr{F}^+}(l) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(l)} = \mu_\tau^{\mathscr{F}^+}(\mho(l)) \cdot e^{\delta\chi_\tau^{\mathscr{F}^+}(\mho(l))} \leq s, \mu_L^{\mathscr{F}^+}(m) \cdot e^{\delta\chi_L^{\mathscr{F}^+}(m)} = \mu_\tau^{\mathscr{F}^+}(\mho(m)) \cdot e^{\delta\chi_\tau^{\mathscr{F}^+}(\mho(m))} \leq s$.
Thus, $\mu_L^{\mathscr{F}^+}((l\Diamond_1 m)) \cdot e^{\delta\chi_L^{\mathscr{F}^+}((l\Diamond_1 m))} = \mu_\tau^{\mathscr{F}^+}((\mho(l) \divideontimes_1 \mho(m))) \cdot e^{\delta\chi_\tau^{\mathscr{F}^+}((\mho(l)\divideontimes_1\mho(m)))} \leq s$, for all $l, m \in \mathscr{B}_1$.
Similarly other operations, $L_{(t,s)}$ is a SBS of CBIFSBS L of $\mathscr{B}_1$.

Acknowledgment. This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2025).

Conflicts of Interest The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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