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Enhancing Network Security in Distributed Systems Using Middle Roman Dominating Functions

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Abstract:

A Middle Roman dominating function (MRDN) on a graph G = (V,E) is a function f:v→{0,1,2,3} satisfying the condition that every vertex u with f(u)=0 is adjacent to at most one vertex v with f(v)=2 or 3. Further if a vertex is assigned 2, then at most two of its vertices can be assigned 0 and if a vertex is assigned 3, then all its neighbours can be assigned 0. The weight of a MRDF is the value f(V(G))=∑_uϵV▒〖f(u)〗). The Middle Roman domination number γ_MR (G) is the minimum weight of a MRDF on G. In this paper, we introduce Middle Roman Domination number denoted as γ_MR (G), study the properties of the function, present some characterization and determine γ_MR (G)-value for some graphs. Middle Roman Dominating Functions (MRDFs) present a unique approach within graph theory, with significant implications for various fields in computer science. By assigning values to vertices under specific constraints, MRDFs enable the optimization of resources, enhancing network security, load balancing, and energy efficiency in distributed systems. This paper explores the application of MRDFs in scenarios such as intrusion detection, resilient network design, task scheduling, and sensor activation. By minimizing the overall weight while maintaining functional requirements, MRDFs provide an effective strategy for addressing challenges in network topology, resource allocation, and fault tolerance. The versatility of MRDFs makes them a valuable tool in the development of robust and efficient computing systems.

Keywords: Dominating function, Roman dominating function, Weak Roman dominating function, Middle Roman dominating function, Resource Allocation, Optimization, Network Security.

Mathematical Subject Classification: 05C69.

1. INTRODUCTION

A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of the dominating set of G . In 2004, Cockayne et al. published Roman Domination in graphs; as a result, numerous Roman domination parameters were introduced [2, 16, 17]. From then enormous work has been done in Roman domination. Consider a $G = (V, E)$ and define a function $f: V \to \{0, 1, 2\}$. Unguarded with regard to f is defined as a vertex u with $f(u) = 0$ that is not next to a vertex with 1 or 2. The function $f: V \to \{0, 1, 2\}$ satisfying the condition that each vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$, is referred to be a Roman dominant function, known as RDF of a graph $G = (V, E)$. Roman domination number (RDN) of G, which is represented by $\gamma_R(G)$, is the bare minimum number of guards that must be employed in any RDF. Let $G = (V, E)$ be a graph and f be a function $f: V \to \{0,1,2\}$. The function f is a weak Roman dominating function (WRDF) if each vertex

u with $f(u) = 0$ is adjacent to a vertex V with $f(v) > 0$ such that $f^{'}: V \to \{0,1,2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \epsilon V - u, v$ has no undefined vertex. The weight of f is $w(f) = \boldsymbol{\Sigma}_{v \epsilon V} f(v)$ the weak Roman domination number denoted by $\gamma_r(G)$ is the minimum weight of WRDF in G [6, 16, 17, 18, 28]. A Middle Roman dominating function (MRDN) on a graph $G = (V, E)$ is a function $f: V(G) \to \{0, 1, 2, 3\}$ satisfying the condition for every vertex $u \in V$, $f(u) = 0$, then vertex u has at least one neighbor v with $f(u) \geq 2$, If $f(u) = 1$, then $|N(u) \cap V_0| = 0$, $f(u) = 2$, then $|N(u) \cap V_0| \leq 2$, $f(u) = 3$, then $|N(u) \cap V_0| \geq 3$. Where $V_i, i = 1,2,3 \ldots, n$. The weight of a MRDF is the value $f(V(G)) = \sum_{u \epsilon V(G)} f(u)$. The Middle Roman domination number $\gamma_{MR}(G)$ is the minimum weight of a MRDF on G. We define the Middle Roman domination number based on the military strategy that if any region is un secured then it must be adjacent to a region with either *2* or *3* legions. If *2* legions are placed at a region then at most two neighboring regions can be undefended. If *3* legions are placed at a region then any number of neighboring legions can be undefended. The idea is that during an attack at any undefended region, the legions placed at the neighboring regions will move and protect this undefended region. Which provides a fruitful level of defence at a cheaper cost. In some highly populated areas of the city where emergency calls for police, ambulance, fire men service etc. are common. three units of servers can be placed, whereas at areas $V_i, i = 1,2,3 \ldots, n$ where only two neighbouring regions need service, exactly two units of servers can be placed. Similarly, in trust worthy areas, one unit of server can be placed. Such type of arrangements can give protection with minimum number of servers which also optimize the cost. Server placements will maximize the service of the servers with optimal cost [11,12,13,14]. Middle Roman Dominating Functions (MRDFs) offer a fascinating intersection of graph theory and computer science, providing powerful tools for addressing complex problems in network security, resource allocation, and optimization. These functions, defined by specific conditions on vertex assignments within a graph, allow for strategic placement and allocation of resources, ensuring that critical nodes are efficiently protected or empowered. By minimizing the overall cost or weight while satisfying the constraints of MRDFs, computer scientists can design robust systems that balance efficiency with security. This introduction explores the diverse applications of MRDFs in various domains, highlighting their significance in optimizing network structures, managing distributed systems, and enhancing the resilience of communication networks. Here we stated some of the characterization theorems of MRDFs followed by its applications.

Preposition 1:

If G a graph of order $n \geq 4$ with a vertex of degree $n - 1$ then $\gamma(G) = 1$ and $\gamma_{MR}(G) = 3$

Proposition 2:

Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{MR}-function then,

a. $G[V_1]$ the subgraph induced by V_1 has maximum degree *1*

b. No edge of G joins V_1 and V_3.

c. $\forall\, v \in V_0$, $|N(v) \cap V_1| \leq 1 = \{u, v_2\} \cap \{v_1, v_2, v_3\} = \{v_2\}$

d. $V_2 \cup V_3$ is a γ- set of V_0. $V_2 \cup V_3 \succ V_0$

Theorem 1: $\gamma_{MR}(P_n) = \gamma_{MR}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$

Proof is obvious for the above theorem.

Theorem 2: For any graph G, $\gamma_{MR}(G) = 3\gamma(G)$ if and only if $V_1 = \emptyset$ and $V_2 = \emptyset$

Proof: Let G be a middle roman graph and let $f = (V_0, V_1, V_2, V_3)$ be a γ_{MR}-function of G. From the proposition 3(d) $V_2, V_3 \succ V_0$ and $V_1 \cup V_2 \cup V_3 \succ V_0$ and hence, $\gamma(G) \leq |V_1 \cup V_2 \cup V_3| = |V_1| + |V_2| + |V_3| \leq |V_1| + 2|V_2| + 3|V_3| = \gamma_{MR}(G)$.

But since, G is Middle Roman, we know that, $3\gamma(G) = 3|V_1| + 3|V_2| + 3|V_3| = \gamma_{MR}(G) = |V_1| + |V_2| + 3|V_3|$. Hence $n_1 = |V_1| = |V_2| = 0$.

Conversely, let $f = (V_0, V_1, V_2, V_3)$ be a γ_{MR}-function of G $n_1 = |V_1| = |V_2| = 0$. Therefore $\gamma_{MR}(G) = 3|V_3|$ and since by the definition $V_1 \cup V_2 \cup V_3 \succ V_0$ it follows that V_3 is a dominating set of G. We know that V_3 is a γ- set of $G(V_2 \cup V_3)$, i.e., $|V_3| = \gamma(G)$ and $\gamma_{MR}(G) = 3\gamma(G)$ i.e., is a Middle Roman graph.

The theorem states that for any graph G, the Middle Roman domination number $\gamma_{MR}(G)$ equals three times the domination number $\gamma(G)$ if and only if two specific vertex sets, V_1 and V_2, are empty. These vertex sets likely represent particular configurations or substructures within the graph that affect the domination and Middle Roman domination properties and the applications of the theorem are as follows **Network Design and Optimization-Simplified Network Structures**: In the design and optimization of networks (such as communication or transportation networks), this theorem implies that achieving a specific domination-related property (where $\gamma_{MR}(G) = 3\gamma(G)$) requires the absence of certain substructures (V_1 and V_2). This can guide the design of networks to ensure they meet specific criteria, such as minimal redundancy or efficient coverage, by avoiding these substructures [20, 21]. **Fault-Tolerant Systems-Design of Fault-Tolerant Architectures**: In systems that require fault tolerance, ensuring that $\gamma_{MR}(G) = 3\gamma(G)$ can lead to designs where the system's resilience is maximized under specific conditions. The absence of the vertex sets V_1 and V_2 may correspond to configurations that prevent certain types of failures or ensure that every component is adequately backed up. **Graph-Based Modelling in Biological Networks-Stability in Ecological or Biological Systems**: In ecological networks or biological interaction networks, this theorem can be used to ensure that the network is stable and robust. For example, if $\gamma_{MR}(G) = 3\gamma(G)$ holds, it might indicate that the system is free from certain destabilizing interactions or species (represented by V_1 and V_2), leading to a more stable and resilient ecosystem or biological network [25]. **Algorithm Design in Graph Theory-Simplified Algorithms for Specific Graph Classes**: In algorithmic graph theory, this theorem provides a characterization that can be used to design simplified algorithms for specific classes of graphs. If $\gamma_{MR}(G) = 3\gamma(G)$ and $V_1 = \emptyset$ and $V_2 = \emptyset$, algorithms can be tailored to efficiently handle these graphs, taking advantage of their simplified structure.

This theorem provides a valuable tool for understanding when certain domination-related properties hold in a graph, with wide-ranging applications in network design, optimization, security, biological modelling, and more. The conditions $V_1 = \emptyset$ and $V_2 = \emptyset$ highlight specific configurations that need to be avoided to achieve these properties, offering practical guidance in various fields.

Characterization of $\gamma(G) = \gamma_{MR}(G)$

Theorem 3: For any graph G of order n, $\gamma(G) = \gamma_{MR}(G)$ if and only if $G = \overline{K_n}$

Proof: It is obvious that if $G = \overline{K_n}$ then $\gamma(G) = \gamma_{MR}(G)$. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{MR}(G)$- function. The equality $\gamma(G) = \gamma_{MR}(G)$ implies that we have equality in $\gamma(G) \leq |V_1| + |V_2| + |V_3| = |V_1| + |V_2| + 3|V_3| = \gamma_{MR}(G)$.

Hence $|V_3| = 0$, which implies that $V_0 = \emptyset$. Therefore $\gamma_{MR}(G) = |V_1| + |V_2| = |V| = n$. This implies that $G = \overline{K_n}$

The theorem states that for any graph G of order n, the equality $\gamma_{MR}(G)$ holds if and only if G is the complement of the complete graph K_n. This result has specific implications in areas where graph structure plays a crucial role and the applications are as follows **Network Security and Vulnerability Analysis-Detection of Isolated Nodes**: In network security, this theorem can be used to identify networks (or parts of networks) that are completely disconnected. The complement of a complete graph has no edges, meaning each node is isolated. If $\gamma(G) = \gamma_{MR}(G)$, it implies the network's vulnerability to isolation, which could be critical in detecting potential points of failure or attack. **Social Network Analysis-Analysis of Isolated Communities**: In social networks, where nodes represent individuals and edges represent interactions, this theorem helps identify when a community (represented by a graph G) has no internal interactions (as indicated by $G = \overline{K_n}$). This could be useful in sociological studies to identify isolated or inactive groups within a larger social structure. **Biological Networks-Study of Non-Interacting Species or Genes**: In biological networks, such as gene interaction or ecological networks, this theorem can be applied to understand when a set of species or genes are completely non-interacting. If a biological network is structured as $\overline{K_n}$, this suggests no direct interaction between the components, which could have implications for understanding certain biological phenomena or for the design of experiments. **Quantum Computing and Information Theory-Graph-Based Quantum State Analysis**: In quantum computing, where graph structures can represent quantum states or information flow, this theorem helps identify scenarios where quantum states or bits are completely independent, corresponding to the complement of a complete graph. This could be relevant in designing quantum algorithms or analyzing quantum entanglement properties. **Optimization of Isolated Systems-Design of Independent Subsystems** [33]: In systems design, particularly where subsystems must operate independently (such as in modular robotics or independent software components), this theorem can guide the design to ensure that no interaction occurs between subsystems, modelled by $G = \overline{K_n}$.

This theorem is significant in identifying when a graph structure corresponds to a completely independent set of vertices (no edges), which has applications across various fields where isolation or independence of components is a key consideration.

Theorem 4: For any complete bipartite graph $\gamma_{MR}(G) = \begin{cases} 4, \min(|X|,|Y|) = 2 \\ 5, \min(|X|,|Y|) = 3 \\ 6, \min(|X|,|Y|) = 4 \end{cases}$

The theorem describes the Middle Roman domination number $\gamma_{MR}(G)$ for complete bipartite graphs $G = K_{|X||Y|}$, where the vertex set is partitioned into two independent sets X and Y. The theorem states that $\gamma_{MR}(G)$ takes the following values based on the sizes of the partitions:

- $\gamma_{MR}(G) = 4, \min(|X|,|Y|) = 2$
- $\gamma_{MR}(G) = 5, \min(|X|,|Y|) = 3,$
- $\gamma_{MR}(G) = 6, \min(|X|,|Y|) = 4.$

Which can provide the following applications **Bipartite Network Design-Optimal Resource Allocation**: In designing bipartite networks, such as communication networks or supply chains, this theorem provides a clear guideline on the minimal resources required to dominate the entire network. For example, in a communication network modelled as a bipartite graph, knowing the Middle Roman domination number helps in optimizing the placement of communication nodes to ensure minimal resource usage while maintaining network coverage. **Bi-partitioned Social Networks-Influence and Control**: In social networks with two distinct groups (e.g., buyers and sellers in a marketplace), this theorem provides insights into the minimal influence needed to control or monitor the interactions between these groups. It can be useful in marketing strategies, where the goal is to dominate a bipartite network of consumers and products. **Sensor Networks-Energy-Efficient Sensor Placement**: In wireless sensor networks where sensors are deployed to monitor two different types of areas (e.g., indoor vs. outdoor), the theorem can be used to determine the minimal number of sensor nodes required to ensure full coverage and minimal energy consumption [31, 32]. **Market Research and Analysis**: **Consumer-Product Matching**: In market analysis, where consumer preferences and products are modelled as a bipartite graph, the theorem assists in understanding the minimal marketing effort needed to ensure that all product categories are effectively promoted to consumers.

This theorem is particularly useful in applications where bipartite graphs are prevalent, providing a direct method to evaluate the minimal resources or influence needed to dominate such graphs efficiently.

Theorem 5: For any double star $T, \gamma_{MR}(G) = \begin{cases} 4, \text{if } \deg(u_1) = \deg(u_2) = 2 \\ 5, \text{if } \deg(u_1) = 3, \deg(u_2) = 2 \\ 6, \text{if } \deg(u_1) = \deg(u_2) = 3 \end{cases}$

The theorem specifies the Middle Roman domination number $\gamma_{MR}(G)$ for a double star graph T, where T consists of two central vertices u_1 and u_2 with specified degrees. The values of $\gamma_{MR}(G)$ depend on the degrees of these central vertices:

- $\gamma_{MR}(G) = 4, \text{if } \deg(u_1) = \deg(u_2) = 2,$
- $\gamma_{MR}(G) = 5, \text{if } \deg(u_1) = 3, \deg(u_2) = 2,$
- $\gamma_{MR}(G) = 6, \text{if } \deg(u_1) = \deg(u_2) = 3.$

The above has various applications as follows **Network Design and Reliability-Optimal Node Placement**: In network design, where nodes represent key components or hubs, understanding the Middle Roman domination number of a double star graph helps in determining the minimum number of resources required to ensure network reliability and coverage, particularly in hub-and-spoke models where two main hubs are critical. **Telecommunications-Minimal Resource Deployment**: In telecommunications, where double star structures may represent network topologies with two central communication nodes, the theorem helps in deciding the minimal number of backup resources (e.g., routers, servers) needed to maintain network functionality in case of failures. **Biological Networks-Gene Interaction Networks**: In gene interaction networks where two key genes regulate other genes, modelled as a double star graph, the theorem provides insights into the minimum interventions required (e.g., genetic modifications or treatments) to influence or control the network effectively. **Data Centre Management-Redundancy Planning**: In data centres organized with two main server clusters (central vertices), the theorem aids in determining the minimal redundancy needed to ensure data security and availability across all connected storage units or servers. **Transportation and Logistics-Airport and Seaport Operations**: In transportation logistics, particularly for airports or seaports that function as central hubs, this theorem helps in planning the minimal yet effective allocation of resources like security, maintenance, or logistics personnel to ensure smooth operations across the network [4, 5].

This theorem is applicable in various domains where the network or structure resembles a double star graph, providing practical insights into the minimal resource allocation needed to maintain effective operations.

Proof is obvious for the above theorems.

Theorem 6: For any graph G with n vertices if there exists a vertex n with degree $n - 1$ then $\gamma_{MR}(G) = 2 \text{ or } 3$.

Proof: Let $v \in V(G)$ with $\deg(v) = n - 1$. We define a function $f = (V_0, V_1, V_2, V_3)$ as follows with $n = 3, V_1 = \emptyset, V_2 = \{v\}, V_3 = \emptyset$ and $V_0 = V - V_2$. Then clearly f is a MRDF and $\gamma_{MR}(G) = 2$. when $n \geq 4, V_1 = \emptyset, V_2 = \emptyset, V_3 = 3$ and $V_0 = V - V_3$. Then f is a MRDF with $\gamma_{MR}(G) = 3|V_3| = 3$.

The theorem states that for any graph G with n vertices, if there exists a vertex v with degree $n-1$(i.e., v is adjacent to every other vertex in the graph), then the Middle Roman domination number $\gamma_{MR}(G)$ is either 2 or 3 [7, 9]. This situation typically arises in graphs where a single vertex has a dominating influence over the rest of the graph, such as in complete graphs or star graphs. Here are some applications of this theorem are as follows **Network Hub Design-Optimal Placement of Central Nodes**: In network design, especially in star or hub-and-spoke topologies, the theorem can be used to determine the minimal effort needed to dominate the entire network. If a central hub node is connected to all other nodes, this configuration ensures that the Middle Roman domination number is minimized (either 2 or 3), leading to efficient network design and resource allocation. **Communication Networks-Designing Efficient Broadcast Networks**: In communication networks, where one central node needs to broadcast to all other nodes, this theorem implies that such a network is optimally dominated with minimal additional resources. This can be applied in the design of broadcasting stations, Wi-Fi networks, or cellular networks, where a central node serves as a primary transmitter. **Graph Theory Simplification-Characterization of Simple Graphs:** This theorem provides a straightforward criterion to identify graphs with low Middle Roman domination numbers. Graphs meeting the condition can be easily classified and analysed, simplifying the study of certain classes of graphs, such as complete or nearly complete graphs. **Social Network Analysis-Influential Nodes Identification**: In social networks, a vertex with degree $n-1$ represents an individual connected to every other person in the network. The theorem suggests that such a highly influential individual can dominate the network with minimal additional influence $\gamma_{MR}(G)$ being 2 or 3) [44, 45]. This insight can be used in identifying key influencers or leaders in social networks for marketing or information dissemination. **Security and Defence Applications: Strategic Positioning**: In scenarios where a central control or command node must oversee or secure an entire network, this theorem indicates that minimal resources are needed to ensure complete coverage and control. This can apply to military networks, cybersecurity frameworks, or surveillance systems, where a central node needs to ensure the security of the entire system [42, 43]. **Biological Networks-Central Nodes in Biological Systems**: In biological networks, such as protein interaction networks, a central protein (node) that interacts with nearly all others can ensure network stability or functionality with minimal intervention. The theorem highlights the significance of such central nodes in maintaining biological processes. **Urban Planning and Infrastructure-Centralized Service Locations**: In urban planning, where a central service location (e.g., a hospital or fire station) serves an entire community, this theorem can help in determining the minimal additional infrastructure required to ensure full coverage. This leads to cost-effective and efficient service provision in cities. **Data Centre and Cloud Computing-Efficient Resource Allocation**: In data centres or cloud computing environments, where one node acts as a central coordinator, the theorem indicates that minimal resources are needed to maintain overall system performance and redundancy. This can optimize the design and operation of cloud services. **Strategic Game Design-Central Control in Games**: In strategic games modelled by graphs, where one player or element controls all others (degree $n-1$), the theorem provides insights into minimal strategies needed to dominate the game, making it useful in game theory and AI.

This theorem is particularly valuable in scenarios where a single, highly connected node plays a critical role in the structure and function of a graph. It provides a clear understanding of how such a node can influence the entire system with minimal additional resources or effort.

Theorem 7: For any connected graph G, $\gamma_{MR}(G)=\gamma+1$ if and only if the following conditions holds (i) $\deg(v)\leq 2\ \forall\ v\in V$ (ii) $n\leq 4$

In order for a Middle Roman dominating function $f=(V_0,V_1,V_2,V_3)$ to have $\gamma_{MR}(G)=\gamma+1$ weight $\gamma(G)+1$, either (i) $|V_1|=\gamma(G)+1$, $|V_2|=0$ and $|V_3|=0$ or (ii) $|V_1|=\gamma(G)-1$, $|V_2|=1$ and $|V_3|=0$. Any other arrangement of weight $\gamma(G)+1$ would have $|V_1|+|V_2|<\gamma(G)$.

In case (i): since, $|V_2| = 0$ and $|V_3| = 0$, $|V_1| = V$. Then by Ore's theorem, for any connected graph with n vertices, $\gamma(G) \leq \frac{n}{2}$. Thus $n = \gamma(G) + 1 \leq \frac{n}{2} + 1$. Hence $n = 2$. Let $V_1, V_2 \in V$. If V_1 and V_2 are not adjacent, then $\gamma(G) = \gamma_{MR}(G)$.

Hence V_1 and V_2 are adjacent and hence G is a P_2. In case (ii) $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{MR}(G)$ function for G of weight $\gamma(G) + 1$ with $|V_1| = \gamma(G) - 1$, $|V_2| = 1$ and $|V_3| = 0$. Let $V_2 = \{u\}$. Let $v \in V_1$ be adjacent to u. Then $\{u\}$ is the γ-set of G and $\gamma_{MR}(G) = |V_1| + |V_2| = \gamma + 2$ which is a contradiction. Hence no edge of G joins $\{u\}$ and V_1 $\{u\} \succ V_0$, $\deg(u) = 2$. Let u_1 and u_2 be the vertices adjacent to u. Suppose $\deg(u_1) = \deg(u_2) = 2$.

Then $\{u_1, u_2\}$ is the γ-set of G, $V_2 = \{u_1, u_2\}$, $V_3 = \emptyset$ and $V_1 = \emptyset$, which implies that $\gamma_{MR}(G) = 2|V_2| = 0, \gamma + 2$ a contradiction. Suppose $\deg(u_1) = 2$ and $\deg(u_2) = 1$ and u_1, u_2 are not adjacent. Let w be the vertex adjacent to u_1. Then $\{u_1, u_2\}$ is the -set of G and $\gamma_{MR}(G) = |V_1| + |V_2| = \gamma(G) + 1$. Suppose $\deg(u_1) = 3, \deg(u_2) = 1$ and u_1, u_2 are adjacent.

Then $V_3 = \{u_1\}$, i.e., $|V_3| = 1$ which is a contradiction. Hence $n = 4$ and $\deg(v) \leq 2 \; \forall \; v \in V$. Conversely, assume that for every vertex of G deg $(v) \leq 2$ and $n \leq 4$. If $V_2 = \{u\}$, $V_1 = V - N[u]$ and $V_0 = V - V_1 - V_2$, suppose deg $(v) \leq 2$. Case (ii) $n = 2$, then since, G is a connected $\deg(v) = 1 \; \forall \; v \in V$.

Hence G is a P_2, $V_1 = \{v_1, v_2\}$ and $\{v_1\}$ is a γ- set. Hence $\gamma_{MR}(G) = \gamma + 1$. Case (ii) $n = 2$ Let V_1, V_2, V_3 be the vertices of G. If $\deg(v_i) = 2 \; \forall \; i = 1, 2, 3 \ldots$, then $G \cong C_3$ and if $\deg(v_i) = 2$ for some $i = 1$ say, then $G \cong P_3$. Hence $V_2 = \{v_1\}$, $V_1 = \emptyset$ and $\{v_1\}$ is the γ- set. Therefore $\gamma_{MR}(G) = \gamma + 1$.

The theorem provides a specific condition under which the Middle Roman domination number $\gamma_{MR}(G)$ of a connected graph G is equal to $\gamma + 1$, where γ is the domination number of the graph. The condition given is:

1. $\deg(v) \leq 2 \; \forall \; v \in V$ (i.e., each vertex has a degree of at most 2), and
2. $n \leq 4$ (the number of vertices in the graph is at most 4).

This characterization has several practical applications in various field such as, **Small-Scale Network Design-Design of Simple Networks**: In small networks (with up to 4 vertices), this theorem can simplify the process of determining optimal placements of resources or nodes to ensure full coverage. Since all nodes have a maximum degree of 2, the network structures are simple (such as paths or cycles), which can be used to design and analyse very small and specific network configurations. **Algorithmic Efficiency-Exact Computation for Small Graphs**: For graphs with up to 4 vertices, the theorem provides an exact and straightforward way to compute the Middle Roman domination number [34, 35]. This can be useful in algorithm design for small-scale problems where brute-force methods or exhaustive search can be feasible and efficient [41]. **Graph Theory Education-Teaching and Learning**: The theorem can serve as a pedagogical tool to illustrate the concepts of domination numbers and Middle Roman domination numbers in graph theory. Its simplicity makes it an excellent example for teaching these concepts to students in an introductory graph theory course. **Optimizing Small-Scale Graphs in Research-Specialized Applications**: In research areas that involve small, specialized networks (e.g., certain types of molecular networks, small-scale computational models), the theorem can be used to determine optimal configurations or understand the properties of these networks with precision. **Resource Allocation in Simple Systems-Efficient Resource Placement**: For systems or models with up to 4 nodes, where resources need to be allocated or placed optimally, the theorem provides a clear understanding of the relationship between the domination number and the Middle Roman domination number. Hence, the theorem provides a clear and specific framework for understanding and solving problems related to Middle Roman domination in very simple or small graphs, making it valuable in theoretical studies, educational contexts, and practical applications involving small-scale systems.

Theorem 8: For any connected graph G, $\gamma_{MR}(G) = \gamma + 2$ if and only if there exist a minimal dominating set S satisfying one of the following conditions: (i) there exist a vertex $v \epsilon S$ such that v is a support and $v \subseteq N[Pns(v,s)]$ (ii) there exist two vertices u and v in S such that $v \subseteq N[Pns(u,S)] \cup N[Pns(v,S)]$ G has a vertex v of degree $n - \gamma$ or G has two vertices v and w such that $|V[v] \cup N[w]| = n - \gamma(G) + 2$

Proof: If G has a vertex v of degree $n - \gamma \geq 3$, we define $V_0 = N(v)$, $V_1 = V - N[v], V_2 = \emptyset$ and $V_3 = \{v\}$, then $f = (V_0, V_1, V_2, V_3)$ is a MRDF with $f(v) = \gamma(G) + 2$ and hence is a γ_{MR}- function with $f(v) = \gamma(G) + 2$.

If there are two vertices u and v such that $N[u] \cup N[v] = n - \gamma(G) + 1, V_3 = \emptyset$ we define $V_2 = \{u,v\}, V_0 = N[u] \cup N[v] - \{u,v\}, V_1 = V - N[u] \cup N[v]$. Then $f = (V_0, V_1, V_2, V_3)$ is a MRDF with $f(v) = \gamma(G) + 2$ and hence is a γ_{MR}- function. In order for a MRDF $f = (V_0, V_1, V_2, V_3)$ to have weight $\gamma(G) + 2$ either (i) $|V_3| = 1, |V_2| = 0$ and $|V_1| = \gamma(G) - 1$ or (ii) $|V_3| = 0, |V_2| = 2$ and $|V_1| = \gamma(G) - 2$.

In case (i) let $f = (V_0, V_1, V_2, V_3)$ be a γ_{MR}- function for G of weight $\gamma(G) + 2$ with $|V_3| = 1, |V_1| = \gamma(G) - 1$. Let $V_3 = \{v\}$. Since, no edge of G join V_1 and v, $v \succ V_0$, it follows thatdeg$(v) = |V_0| = n - |V_1| - |V_3| = n - (\gamma(G) - 1) - 1 = n - \gamma(G)$.

In case (ii) let $f = (V_0, V_1, V_2, V_3)$ be a γ_{MR}- function for G of weight $\gamma(G) + 2$ with $|V_3| = 0, |V_2| = 2$ and $|V_1| = \gamma(G) - 2$. Let $V_2 = \{u,v\}$. Since, no edge of G join V_1 and to u or v, $\{u,v\} \succ V_0$. It follows that $|N[u] \cup N[v]| = n - |V_1| = n[\gamma(G) - 2] = n - \gamma(G) + 2$.

The theorem provides a characterization of the Middle Roman domination number $\gamma_{MR}(G)$ for a connected graph G in terms of its minimal dominating sets and specific conditions involving vertex degrees and neighborhoods. This result has several applications in various domains, particularly where network design, analysis, and optimization are critical. Here are some potential applications, **Network Design and Optimization-Optimal Resource Placement**: The theorem helps identify minimal dominating sets with specific properties that can be used to optimize the placement of resources or infrastructure within a network [23, 24]. For example, if a network (graph) needs to ensure full coverage with minimal redundancy, the theorem can guide the placement of resources such that the network is efficiently dominated. **Robustness and Fault Tolerance-Identifying Critical Nodes**: The conditions specified in the theorem help in identifying critical nodes that can be used to enhance the robustness of a network [14, 24, 42]. If a minimal dominating set satisfies one of the conditions, it suggests that certain nodes are crucial for maintaining network coverage, which can be useful for designing fault-tolerant networks. **Wireless Sensor Networks-Coverage and Connectivity**: In wireless sensor networks, ensuring coverage and connectivity with minimal nodes is crucial. The theorem can be applied to identify configurations of sensor nodes that provide optimal coverage and redundancy, which is important for maintaining network performance and reliability [39, 40]. **Biological Network Analysis-Understanding Biological Interactions**: In biological networks, such as protein-protein interaction networks, identifying minimal sets of proteins (nodes) that dominate the network can help in understanding key interactions and functional modules [26, 27, 29]. The theorem provides a framework for analyzing and optimizing these networks. **Social Network Analysis-Influence and Control**: In social networks, identifying key individuals or groups that can effectively influence or control the network is crucial [1, 3]. The theorem can be used to find minimal sets of influential nodes that meet certain criteria, aiding in strategic decision-making and intervention planning. **Security and Surveillance-Optimizing Coverage**: In security and surveillance systems, the theorem can be used to determine the optimal placement of cameras or sensors to ensure full coverage of an area with minimal equipment. The conditions help in selecting strategic locations for surveillance that provide comprehensive coverage [36,37].

Overall, the theorem provides a theoretical foundation for understanding and optimizing various network structures and configurations. It helps in solving practical problems related to network coverage, resource allocation, and system robustness across different applications.

Determination of γ_{MR} value of some graphs:

In this section we first determine the value of γ_{MR} for a caterpillar T. For this purpose, we discuss as follows:

Let $v_1, v_2, \dots, v_n$ be the consecutive super strong support of T. We call the graph induced by $N[v_i]$ as super strong support chain [SSS chain] for convenience. Next, we consider a moderate support u_r as an artificial SSS of the following conditions hold. Let u_{r-1} and u_{r+1} be the preceding and succeeding vertices of a moderate support u_r. Let n_i and n_j be the number of internal vertices of the (u_{r-1}, u_r) path and (u_r, u_{r+1}) respectively. Then u_r is said to be an artificial super strong support if one of the following conditions hold.

(i) If both u_{r-1} and u_{r+1} are weak support then $n_i \equiv 0, 2(mod\ 3)$ and $n_j \equiv 0, 2(mod\ 3)$.
(ii) If u_{r-1} is a moderate support and u_{r+1} is a weak support then $n_i \equiv 1, 2(mod\ 3)$ and $n_j \equiv 0, 2(mod\ 3)$ and vice versa.
(iii) If both u_{r-1} and u_{r+1} are moderate supports then $n_i \equiv 1, 2(mod\ 3)$ and $n_j \equiv 1, 2(mod\ 3)$.

Let u_s be the weak support on the spine of T. Then it will be considered as an artificial super strong support if one of the following conditions hold.

(i) If both u_{r-1} and u_{r+1} are weak support then $n_i \equiv 0, 2(mod\ 3)$ and $n_j \equiv 0, 2(mod\ 3)$.
(ii) If u_{r-1} is a moderate support and u_{r+1} is a weak support then $n_i \equiv 1, 2(mod\ 3)$ and $n_j \equiv 0, 2(mod\ 3)$.
(iii) If both u_{r-1} and u_{r+1} are moderate supports then $n_i \equiv 1, 2(mod\ 3)$ and $n_j \equiv 1, 2(mod\ 3)$.

Next, identify the following (i) All consecutive SSS say $s_{n1}, s_{n2}, s_{n3}, \dots, s_{nk}$ where $s_{ni} = \{s_1, s_2, s_3, \dots, s_{ni}\}$ such that each vertex of s_{ni} is a SSS. (ii) consecutive weak supports $w_{n1}, w_{n2}, w_{n3}, \dots, w_{nr}$ where each $w_{ni} = \{w_1, w_2, w_3, \dots, w_{ni}\}$ such that each vertex of w_{ni} is a weak support.

Now we determine the value of γ_{MR} for a caterpillar in the next theorem

Theorem 9: Let T be any caterpillar. let $S = \{S_{ni}\ i = 1, 2, \dots, k\ S_i\ is\ a\ SSS\ chain\}, S_1 = \{s/\ each\ vertex\ s\ is\ either\ a\ SSS\ or\ ASSS\}, S_2 = \{m/\ each\ m\ is\ either\ a\ MS\ or\ AMS\}, W = \{W_{nj},$ $J = 1, 2, \dots, l\ /\ each\ W_{nj}\ is\ a\ comb\ graph\ with\ nj\ weak\ supports\}$. Let $T_1 = [N[S] \cup N[S_1] \cup N[S_2] \cup N[W]]$. Let $T_2 = T - T_1$ and $r = 1, 2, \dots, t. R_1, R_2, \dots, R_t$ be the components of T_2. Then $\gamma_{MR} = 3[(x + a)^n] + \left[\sum_{i=1}^{k} n_i + |S_1|\right] + 2|S_2| + \frac{3}{4}\sum_{j=1}^{l} T_1 + 2|T_2|$

Proof: Let T be any caterpillar. identify the SSS chain, SSS, ASSS, MSS, AMSS and comb graphs using the above procedure. Let S, S_1, S_2, W be as defined in the theorem.

Let v be an artificial super strong support. Let v_1 and v_2 be the supports that precede and succeed v on the spine. Let P be the (v_1, v_2) path. Let $u_1, u_2, \dots, u_x$ be the internal vertices of (v_1, v)- path and $w_1, w_2, \dots, w_y$ be the internal vertices of (v, v_2)- path.

Case (i)

v_1 and $v_2$are the weak supports. If two legions are posted at v then $\left\lceil\frac{2(x-1)}{3}\right\rceil + \left\lceil\frac{2(y-1)}{3}\right\rceil + 2 = K_1$ legions are needed to safeguard the vertices on the path P. But, on the other hand if three legions are posted at v_1 then $\left\lceil\frac{2(x-2)}{3}\right\rceil + \left\lceil\frac{2(y-2)}{3}\right\rceil + 3$ legions are required to safeguard the path P, which is less than or equal to M_1. Hence, we assign three legions at v to safeguard $N[v]$.

Case (iii)

v_1 is a weak support and v_2 is a moderate support. If the legions are posted at v then $\left\lceil\frac{2(x-1)}{3}\right\rceil + \left\lceil\frac{2y}{3}\right\rceil + 2 = K_2$ legions are required to safeguard the vertices on the path P. But, on the other hand, if three legions are posted at v, then $\left\lceil\frac{2(x-2)}{3}\right\rceil + \left\lceil\frac{2(y-1)}{3}\right\rceil + 3$ legions are required to safeguard the path P which is less than K_2. Hence, we assign three legions at v.

Case (iv)

v_1 and v_2 are moderate supports. If legions are posted at v_1 then $\left\lceil\frac{2x}{3}\right\rceil + \left\lceil\frac{2y}{3}\right\rceil + 2 = K_3$ legions are required to safeguard the legions on the path. But, on the other hand if three legions are posted at v, then $\left\lceil\frac{2(x-1)}{3}\right\rceil + \left\lceil\frac{2y-1}{3}\right\rceil + 3$ legions are required to safeguard the path P, which is less than or equal to K_3. Hence, we assign three legions at v to safeguard $N[v]$.

Let w be an artificial strong support. Let w_1 and w_2 be the supports that precede and succeed w on the spine. Let P be the (w_1, w_2) path. Let $x_1, x_2, \dots, x_p$ and $y_1, y_2, \dots, y_q$ be the internal vertices of (w_1, w) path and (w, w_2) path respectively.

Case (v)

w_1 and w_2 are weak supports. If two legions are posted at w then, $\left\lceil\frac{2(p-1)}{3}\right\rceil + 2 = L_1$ legions are required to safeguard the internal vertices on the path (w_1, w). But on the other hand, if three legions are posted at w, then $\left\lceil\frac{2(p-2)}{3}\right\rceil + \left\lceil\frac{2q-2}{3}\right\rceil + 3$ legions are required to safeguard the path P, which is less than L_1. Hence, we assign three legions at w to safeguard $N[w]$.

Case (vi)

w_1 is a moderate support and w_2 is a weak support. If two legions are posted at w then $\left\lceil\frac{2p}{3}\right\rceil + \left\lceil\frac{2(q-1)}{3}\right\rceil + 2 = L_2$ legions are required to safeguard the vertices on the path P. But on the other hand, if three legions are posted at w, then $\left\lceil\frac{2(p-1)}{3}\right\rceil + \left\lceil\frac{2(q-2)}{3}\right\rceil + 3$ legions are required which is less than L_2. Hence, we assign three at w to safeguard $N[w]$.

Case (ii)

Both, w_{r-1} and w_{r+1} are moderate supports. If two legions are posted at w, then $\left\lceil\frac{2p}{3}\right\rceil + \left\lceil\frac{2q}{3}\right\rceil + 2 = L_3$ legions are required to safeguard the path P. But, on the other hand if three legions are placed at w then $\left\lceil\frac{2(p-1)}{3}\right\rceil + \left\lceil\frac{2(q-1)}{3}\right\rceil + 3$ legions are required to safeguard the vertices on the path (w_1, w_2). Which is less than L_3. Hence, we assign three legions at w to safeguard $N[w]$. Hence, in all the cases we see that three legions are required to safeguard $N[w]$.

Next, we consider a weak support u_r as an artificial moderate support, if any one of the following conditions hold.

(i) If both u_{r-1} and u_{r+1} are weak supports then $n_i \equiv 0, 2(mod\ 3)$ and $n_j \equiv 1(mod\ 3)$ or vice versa.
(ii) If both u_{r-1} and u_{r+1} are moderate supports then $n_i \equiv 1, 2(mod\ 3)$ and $n_j \equiv 0(mod\ 3)$ or vice versa.
(iii) It either u_{r-1} is moderate support and u_{r+1} is a weak support then $n_i \equiv 1, 2(mod\ 3)$ and $n_j \equiv 0(mod\ 3)$

Let u be an artificial moderate support. Let u_1 and u_2 be the supports that precede and succeed u on the spine. let P be the (u_1, u_2)- path. Let $t_1, t_2, \dots, t_m$ and $l_1, l_2, \dots, l_s$ be the internal vertices of (u_1, u)-path and (u, u_2)-path respectively.

Case (i)

Both, u_{r-1} and u_{r+1} are weak supports.

Claim: $n_i \equiv 0, 2(mod\ 3)$ and $n_j \equiv 1(mod\ 3)$

Sub case (i): $n_i \equiv 0, 2(mod\ 3)$

Suppose, $n_j \not\equiv 1(mod\ 3)$. Then either $n_j \equiv 0, (mod\ 3)$ or $n_j \equiv 2, (mod\ 3)$. In both cases u will become an artificial super strong support which is a contradiction to the artificial moderate support u_r.

Sub case (ii): $n_j \equiv 1(mod\ 3)$

Suppose $n_i \not\equiv 0, 2(mod\ 3)$. Then $n_i \equiv 1(mod\ 3)$ legions are posted at u, it can safeguard only two vertices u and the corresponding leaf vertices, where as by definition two legions placed at a region can safeguard at most two adjacent regions. Hence the middle Roman domination number in this case may not be minimum. Hence, $n_i \equiv 0,2(mod\ 3)$.

Case (ii)

Both, u_{r-1} and u_{r+1} are moderate supports

Claim: $n_i \equiv 1,2(mod\ 3)$ and $n_j \equiv 0(mod\ 3)$ or vice versa.

Subcase (a): suppose $n_i \equiv 1,2(mod\ 3)$ and $n_j \not\equiv 0(mod\ 3)$ then $n_j \equiv 1, 2(mod\ 3)$. Then in both cases u will become artificial SSS which is a contradiction to the artificial moderate support u_r. Hence, the claim.

Subcase (iii): u_{r-1} is a moderate support and u_{r+1} is a weak support.

Claim: $n_i \equiv 1,2(mod\ 3)$ and $n_j \not\equiv 1(mod\ 3)$.

Then $n_j \equiv 0(mod\ 3)$ or $n_j \equiv 2(mod\ 3)$. In both the cases, u_r will become artificial SSS a contradiction to the artificial moderate support u_r. $n_j \equiv 2(mod\ 3)$,

Subcase (b): $n_j \equiv 1(mod\ 3)$ and $n_i \not\equiv 1,2(mod\ 3)$ suppose not. Let $n_j \equiv 1(mod\ 3)$ and $n_i \equiv 0(mod\ 3)$. Then the two legions placed at u_r protect only two regions whereas, by definition two legions can protect at most three regions which may or may not minimize the middle Roman domination number. Hence, $n_i \equiv 0(mod\ 3)$. Similarly, we can bring $n_i \equiv 1, 2(mod\ 3)$ and $n_j \not\equiv 1\ (mod\ 3)$. Hence the claim.

Let $C_1, C_2, \dots C_{ni}$ be the support vertices of C'_{ni}. If ni is odd, then omit C_1 or C_j according to the following

(i) If u_{r-1} is a moderate support, then $n_i \equiv 1,2(mod\ 3)$

(ii) If u_{r-1} is a weak support, then $n_i \equiv 0,2(mod\ 3)$

(iii) If u_{r+1} is a moderate support, then omit C_n if $n_j \equiv 1,2(mod\ 3)$

(iv) If u_{r+1} is a weak support, then omit C_n $n_j \equiv 0,2(mod\ 3)$

let $\mathcal{S} = \{S_{m1}, S_{m1}, \ldots, S_{mr}\}$ be the SSS chain in T and with each chain $S_{mi}, i = 1,2,3,\ldots,m_r$ has m_i SSS. Let $x \in N(S_{m1})$ and $y \in N(S_{mr}), S = \{s_1, s_2, s_3, \ldots, s_n\}$ be the SSS or ASSS of T, $C = \{C_{n1}, C_{n2}, C_{n3}, \ldots, C_{np}\}$ be the comb graph in T, where $n_j, j = 1,2,3,\ldots,p$ are the even number of supports in C_j. Let $T_1 = T - \left[N[S] \cup N[s] \cup N[C] - \left\{x, \frac{y}{x} \in N(c_1), y \in N(c_n)\right\}\right] \cup L \cup A$. Where, $L = \left\{\frac{l}{l} is\ a\ leaf\ vertex\ in\ N(m_k), k = 1,2,3,\ldots,p\right\}$, $A = \{N[a] - \{h\}$, where a is an artificial SSS or h is the vertex on the internal path $(u_{r-1}, a)\ or\ (a, u_{r-1})\ with\ \{n_j \not\equiv 0,2\ mod\ 3\}$

Hence, $\gamma_{MR} = 3[(x + a)^n] + \left[\sum_{i=1}^{k} n_i + |S_1|\right] + 2|S_2| + \frac{3}{4}\sum_{j=1}^{l} T_1 + 2|T_2|$.

The above theorem has various applications in various applications including **Network Design and Optimization**: In telecommunications and computer networks, designing efficient network topologies and optimizing resource allocation often involves understanding the domination properties of the network. The theorem can be used to determine the optimal placement of network nodes or resources to achieve desired network performance or reliability. **VLSI Circuit Design**: In Very-Large-Scale Integration (VLSI) circuit design, the problem of minimizing the number of components (e.g., transistors, gates) while ensuring that all necessary connections and functionalities are present can be modelled using graph theory. The Middle Roman domination number can help in optimizing circuit layouts to minimize power consumption or improve performance. **Bioinformatics**: In bioinformatics, especially in the study of protein-protein interaction networks or gene regulatory networks, graph-theoretic models are used to understand the structure and function of biological networks. The theorem can be used to analyse and optimize these networks by providing insights into their dominating sets and efficient network configurations. **Urban Planning and Infrastructure**: For urban planning and the design of infrastructure networks (such as roads, utilities, and public services), the theorem can assist in optimizing the placement of facilities and services to ensure maximum coverage and efficiency. This can help in minimizing costs while ensuring adequate service provision. **Social Network Analysis**: In social network analysis, the Middle Roman domination number can be used to identify key individuals or groups that influence the entire network. This can be useful in marketing, spreading information, or managing social dynamics [46, 47]. **Algorithm Design**: The theorem provides a formula that can be used in the development of algorithms for graph-based problems. It can be applied to design algorithms for computing the Middle Roman domination number in practical scenarios where such graphs are encountered [38].

Conclusion:

The concept of a Middle Roman Dominating Function (MRDF) in graph theory has wide-ranging applications in computer science, particularly in optimizing resource allocation, enhancing network security, and improving efficiency in distributed systems. By strategically assigning values to nodes, MRDF enables efficient task scheduling, load balancing, and energy conservation in sensor networks. It also supports resilient network design and fault tolerance in communication networks. Furthermore, MRDF can be leveraged in robotics for path planning and resource deployment, and in game theory for strategic dominance. Overall, MRDF provides powerful solutions for balancing resource utilization, security, and efficiency across various fields.

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