

Elegant Fuzzy Labeling of Graphs

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Abstract:

Fuzzy graphs and fuzzy labelings are crucial in modeling real-time systems by accommodating varying degrees of information precision. In this paper, we introduce a novel fuzzy labeling called Elegant Fuzzy Labeling. We prove that a simple graph admits Elegant Fuzzy Labeling if and only if it admits elegant labeling. Furthermore, we prove that while a simple graph that admits Elegant Fuzzy Labeling also admits fuzzy labeling, but not conversely. We identify specific classes of simple graphs that admit Elegant Fuzzy Labeling and explore practical application of this new labeling approach.

Keywords: Elegant labeling, Fuzzy graph, Fuzzy labeling, Elegant Fuzzy Labeling.

Mathematics Subject Classification: 05C78, 05C72.

1. Introduction

Graph labeling refers to assigning labels to vertices, edges, or both, of a graph while adhering to specific conditions. Graph labeling trace their origin back to the one introduced by A. Rosa [20] in 1967 called β -labeling. S. W. Golomb [9] renamed β -labeling as graceful labeling. Gallian [6] in his survey paper has given an extensive account of various types of graph labelings. The graph labeling has wide range of applications such as x-ray crystallography, coding theory, radar astronomy, network design, circuit design, etc. In real life, we encounter many problems that are imprecise and ambiguous. To deal with such problems Zadeh [30] introduced the concept, fuzzy sets. Since the graphs are used to model many real world problems, fuzzy graph models are required to represent the vagueness in the objects and the vagueness in the relationship between them. The first definition of Fuzzy graphs was given by Kaufman [11] that was based on Zadeh's fuzzy relations. Later, Rosenfield [21] introduced the basic graph theoretic concepts such as bridges, paths, cycles, trees and connectedness in the fuzzy setting and established some of their properties. Fuzzy graphs are highly effective in modeling real-time systems, where varying levels of imprecision are inherently present. Gani et al. [7,15] introduced the concept of fuzzy labeling graphs and studied their properties. Fuzzy labeling is more appropriate than the classical labeling for many real-world problems. Several studies have explored different aspects of fuzzy graphs, highlighting their versatility and utility. For instance, Fathalian et al. [5] studied fuzzy magic labeling on simple graphs. Giri et al. [8] analyzed fermatean fuzzy graphs. Kosari et al. [13] studied perfectly regular fuzzy graphs. Selvarasu and Murugan [23] studied fuzzy anti-magic labeling in graphs. Shanmugapriya and Hemalatha [24] examined fuzzy vertex magic labeling. Borzooei and Rashmanlou [1] investigated cayley interval-valued fuzzy graphs. Shoaib et al. [29] studied pythagorean fuzzy graphs. Further extending the field, Shi et al. [28] expanded the concept of energy on the picture fuzzy graphs. Rao et al. [22] introduced intuitionistic fuzzy trees. Moreover, studies on vague graphs, an extension of fuzzy graphs, have been conducted by Kosari et al. [12], Rashmanlou et al. [19], and Shao et al. [25,26].

In 1980, harmonious graphs were introduced by Graham and Sloane [10]. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge uv is assigned the label $(f(u) + f(v)) \bmod q$, the resulting edge labels are distinct. Elegant labeling, a variation of harmonious labeling was defined by Chang et al. [3] in 1981. They defined, elegant labeling f of a graph G with q edges as an injective function from the vertices of G to the set $\{0,1, \dots, q\}$ such that when each edge uv is assigned the label $(f(u) + f(v)) \bmod (q + 1)$, the resulting edge labels are distinct and nonzero. Cahit [2], Elumalai and Sethuraman [4], Maulidia and Purwanto [14], Parwati and Purwanto [16], Prihandini et al. [17,18] and, Sherly and Purwanto [27] have produced several results on elegant graphs.

Motivated by the need to address uncertainties and impreciseness in problems more effectively, in this paper, we introduce a new type of fuzzy labeling called the Elegant Fuzzy Labeling. We prove that if a simple graph admits Elegant Fuzzy Labeling, then it admits elegant labeling and fuzzy labeling, and also prove that if a simple graph admits elegant labeling, then it admits Elegant Fuzzy Labeling. We prove that the path graphs P_n , $n \neq 4$ and cycles C_n , $n \equiv 0,3 \bmod 4$, admit Elegant Fuzzy Labeling. The line graph $L(G)$ of a simple graph G is the graph with edges of G as its vertices, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are incident in G . We prove that $L(P_n)$, where $n \neq 5$ and $L(C_n)$, where $n \equiv 0,3 \bmod 4$, admit Elegant Fuzzy Labeling. We also provide an application of Elegant Fuzzy Labeling. If G is a graph, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$.

2. Preliminaries

Definition 2.1. [3] Elegant labeling f of a graph G with q edges is an injective function from $V(G)$ into the set $\{0, 1, \dots, q\}$ such that the function g from $E(G)$ into the set $\{1, 2, \dots, q\}$ defined as $g(uv) = (f(u) + f(v)) \bmod (q + 1)$ for every uv in $E(G)$ is injective.

Example 2.2. Fig. 1 illustrates an elegant graph G with $V(G) = \{v_1, v_2, v_3\}$ and $E(G) = \{v_1v_2, v_2v_3, v_1v_3\}$. An elegant labeling $f: V(G) \rightarrow \{0, 1, 2, 3\}$ of G is defined as $f(v_1) = 0, f(v_2) = 1, f(v_3) = 2$. Hence the edge labels are $g(v_1v_2) = 1, g(v_2v_3) = 3, g(v_1v_3) = 2$.

Definition 2.3. [21] A graph $G = (\mu, \rho)$ with vertex set $V(G)$, edge set $E(G)$ and a pair of functions $\mu: V(G) \rightarrow [0,1]$ and $\rho: E(G) \rightarrow [0,1]$ is called fuzzy graph if for every uv in $E(G)$, $\rho(uv) \leq \mu(u) \wedge \mu(v)$ ($= \min\{\mu(u), \mu(v)\}$).

Example 2.4. Fig. 2 illustrates a fuzzy graph G with $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$. A function $\mu: V(G) \rightarrow [0,1]$ is defined as $\mu(v_1) = 0.5, \mu(v_2) = 0.3, \mu(v_3) = 0.4, \mu(v_4) = 0.6$, and $\rho: E(G) \rightarrow [0,1]$ is defined as $\rho(v_1v_2) = 0.2, \rho(v_2v_3) = 0.3, \rho(v_3v_4) = 0.4, \rho(v_1v_4) = 0.1$.

Definition 2.5. [7] A fuzzy graph $G = (\mu, \rho)$ is said to be a fuzzy labeling graph, if $\mu: V(G) \rightarrow [0,1]$ and $\rho: E(G) \rightarrow [0,1]$ are injective and for every uv in $E(G)$, $\rho(uv) < \mu(u) \wedge \mu(v)$.

Example 2.6. Fig. 3 illustrates a fuzzy labeling graph G with $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_2v_4\}$. A fuzzy labeling $\mu: V(G) \rightarrow [0,1]$ and $\rho: E(G) \rightarrow [0,1]$ are defined as follows: $\mu(v_1) = 0.5, \mu(v_2) = 0.6, \mu(v_3) = 0.7, \mu(v_4) = 0.8, \rho(v_1v_2) = 0.1, \rho(v_2v_3) = 0.2, \rho(v_3v_4) = 0.3, \rho(v_2v_4) = 0.4$.

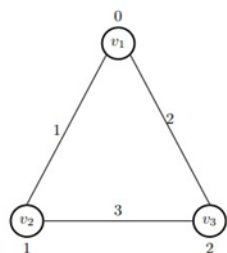


Fig. 1: Elegant graph

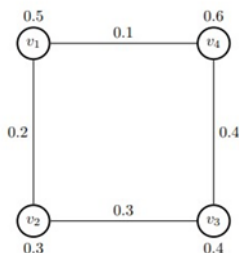


Fig. 2: Fuzzy graph

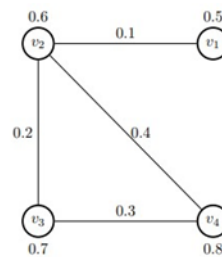


Fig. 3: Fuzzy labeling graph

Definition 2.7. Let G be a (p, q) graph. Let f be an injective function from $V(G)$ into the set $\{h', h' + 1, \dots, h' + q - 1, h' + q\}$ where h' is a suitable constant, and g be an injective function from $E(G)$ into the set $\{0, 1, \dots, q\}$ defined as, $g(uv) = (f(u) + f(v)) \bmod (q + 1)$ for every uv in $E(G)$. Define vertex labelling $\mu: V(G) \rightarrow [0, 1]$ as $\mu(v) = \frac{f(v)}{h' + q}$ and the edge labeling $\rho: E(G) \rightarrow [0, 1]$ as $\rho(uv) = \frac{g(uv)}{h' + q}$, where $h \leq \frac{1}{h' + q}$. If the edge labels are distinct and nonzero, and if the condition $\rho(uv) < \mu(u) \wedge \mu(v)$ is satisfied for all $uv \in E(G)$, then G is said to be an Elegant Fuzzy Labeling Graph and (μ, ρ) is an Elegant Fuzzy Labeling of G .

In the Definition 2.7, if $h' = 0$, then f is an elegant labeling. Suppose $f(v) = 0$ for some $v \in V(G)$ and $uv \in E(G)$ then, $g(uv) = (f(u) + f(v)) \bmod (q + 1)$, where $f(v) = 0$ and $f(u) \neq 0$, simplifying to $g(uv) = f(u) \bmod (q + 1)$. According to Definition 2.7, $0 < g(uv) < q + 1$, hence we have $g(uv) > f(u) \wedge f(v)$. Similarly, suppose $f(v') = 1$ for some $v' \in V(G)$ and $u'v' \in E(G)$, we get $g(u'v') \geq f(u') \wedge f(v')$. Since $g: E(G) \rightarrow \{1, 2, \dots, q\}$ is injective, by the Definition 2.7, for some $u''v'' \in G$, $g(u''v'') = q$ which implies, $g(u''v'') > f(u'') \wedge f(v'')$. However, we must have $g(uv) < f(u) \wedge f(v)$ for all $uv \in E(G)$ to attain $\rho(uv) < \mu(u) \wedge \mu(v)$ for all $uv \in E(G)$. Thus, we have translated the set $\{0, 1, \dots, q\}$ to $\{h', h' + 1, \dots, h' + q\}$, where h' is the suitable translation constant. To define labeling functions of $V(G)$ and $E(G)$ into $[0, 1]$, we have performed contraction with scaling factor h .

3. Elegant Fuzzy Labeling

In the sequel we take $h' = q + 1$ and $h = \frac{1}{2q+1}$. So we consider the following definition of Elegant Fuzzy Labeling of graphs, and obtain the results and prove that certain classes of graphs admit Elegant Fuzzy Labeling.

Definition 3.1. Let G be a (p, q) graph. Let f be an injective function from $V(G)$ into the set $\{q + 1, q + 2, \dots, 2q + 1\}$ and g be an injective function from $E(G)$ into the set $\{0, 1, \dots, q\}$ defined as, $g(uv) = (f(u) + f(v)) \bmod (q + 1)$ for every uv in $E(G)$. Define vertex labelling $\mu: V(G) \rightarrow [0, 1]$ as $\mu(v) = \frac{f(v)}{2q+1}$ and the edge labeling $\rho: E(G) \rightarrow [0, 1]$ as $\rho(uv) = \frac{g(uv)}{2q+1}$. If the edge labels are distinct and nonzero, and if the condition $\rho(uv) < \mu(u) \wedge \mu(v)$ is satisfied for all $uv \in E(G)$, then G is said to be an Elegant Fuzzy Labeling Graph and (μ, ρ) is an Elegant Fuzzy Labeling of G .

Example 3.2. Fig. 4 illustrates an Elegant Fuzzy Labeling Graph G , where $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$. The function f is defined as follows: $f(v_1) = 7$, $f(v_2) = 8$, $f(v_3) = 9$, $f(v_4) = 11$. Consequently, $g(v_1v_2) = 1$, $g(v_1v_3) = 2$, $g(v_1v_4) = 4$, $g(v_2v_3) = 3$, $g(v_2v_4) = 5$, $g(v_3v_4) = 6$. Therefore, an Elegant Fuzzy Labeling (μ, ρ) is given as

follows: $\mu(v_1) = 0.54$, $\mu(v_2) = 0.62$, $\mu(v_3) = 0.69$, $\mu(v_4) = 0.85$, $\rho(v_1v_2) = 0.077$, $\rho(v_1v_3) = 0.15$, $\rho(v_1v_4) = 0.31$, $\rho(v_2v_3) = 0.23$, $\rho(v_2v_4) = 0.38$, $\rho(v_3v_4) = 0.46$.

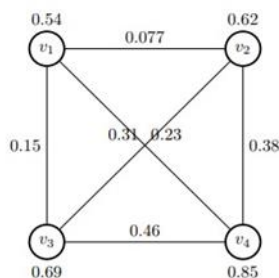


Fig. 4: Elegant Fuzzy Labeling Graph

3.1. General results on Elegant Fuzzy Labeling of simple graphs

Proposition 3.1.1. A simple graph G admits Elegant Fuzzy Labeling if and only if it admits elegant labeling.

Proof. Let G be a simple graph with p vertices and q edges. Let G admit Elegant Fuzzy Labeling. By Definition 3.1, G admits elegant labeling. Conversely, let G admit elegant labeling. By Definition 2.1, there exist an injective function $f: V(G) \rightarrow \{0, 1, \dots, q\}$ and an injective function $g: E(G) \rightarrow \{1, \dots, q\}$ defined as $g(uv) = (f(u) + f(v)) \bmod (q + 1)$. Now translate the function f by $(q + 1)$, So $f(u)$ increases by $q + 1$ and $g(uv)$ remains unchanged. Now define vertex labeling $\mu: V(G) \rightarrow [0, 1]$ as $\mu(u) = f(u).h$ and edge labeling $\rho: E(G) \rightarrow [0, 1]$ as $\rho(uv) = g(uv).h$, where $h = \frac{1}{2q+1}$. The functions f and g are injective. Hence, the vertex labels and edge labels are distinct and nonzero. Since $\min(f(u)) = q + 1$ and $\max(g(uv)) = q$, edge labels are less than the minimum of their respective endpoints labels. Therefore (μ, ρ) is an Elegant Fuzzy Labeling of G . \square

Proposition 3.1.2. If a simple graph G admits Elegant Fuzzy Labeling then it admits fuzzy labeling, but not conversely.

Proof. Let G be a simple graph with p vertices and q edges. Let G admit Elegant Fuzzy Labeling. By Definition 3.1, G admits fuzzy labeling. Conversely, if G admits fuzzy labeling then it is not necessary that G admits Elegant Fuzzy Labeling. For instance consider the graph P_4 with vertex set $V(P_4) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(P_4) = \{v_i v_{i+1}, 1 \leq i \leq 3\}$. Define vertex labeling $\mu: V(P_4) \rightarrow [0, 1]$ as $\mu(v_i) = \frac{i+3}{10}$ and edge labeling $\rho: E(P_4) \rightarrow [0, 1]$ as $\rho(v_i v_{i+1}) = \frac{i}{10}$. By Definition 2.5, P_4 admits fuzzy labeling. But P_4 is not elegant [2]. \square

3.2. Elegant Fuzzy Labeling of certain classes of simple graphs

Theorem 3.2.1. Path graphs P_n admit Elegant Fuzzy Labeling except for $n = 4$.

Proof. Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(P_n) = \{v_1 v_2, v_2 v_3, \dots, v_{n-1} v_n\}$. Let f be an injective function from $V(P_n)$ into the set $\{n, n + 1, \dots, 2n - 1\}$ and g be a function from $E(P_n)$ into the set $\{0, 1, \dots, n - 1\}$ defined as, $g(v_i v_{i+1}) = (f(v_i) + f(v_{i+1})) \bmod n$ for $1 \leq i \leq n - 1$.

Case 1: For P_{2m+1} , $m = 1, 2, 3, \dots$,

$f: V(P_{2m+1}) \rightarrow \{2m + 1, 2m + 2, 2m + 3, \dots, 4m + 1\}$ is defined as,

$$f(v_i) = \begin{cases} 3m + 1 + i & 1 \leq i \leq m \\ 2m + 1 & i = m + 1 \\ f(v_{i-1}) + 1 & m + 2 \leq i \leq 2m + 1. \end{cases}$$

For $1 \leq i \leq m - 1$,

$$\begin{aligned} g(v_i v_{i+1}) &= (f(v_i) + f(v_{i+1})) \bmod (2m + 1) \\ &= (3m + 1 + i + 3m + 1 + i + 1) \bmod (2m + 1) \\ &= (3(2m + 1) + 2i) \bmod (2m + 1) \\ &= 2i. \end{aligned}$$

For $i = m$,

$$\begin{aligned} g(v_m v_{m+1}) &= (f(v_m) + f(v_{m+1})) \bmod (2m + 1) \\ &= (3m + 1 + m + 2m + 1) \bmod (2m + 1) \\ &= 2m. \end{aligned}$$

For $m + 1 \leq i \leq 2m$,

$$\begin{aligned} g(v_i v_{i+1}) &= (f(v_i) + f(v_{i+1})) \bmod (2m + 1) \\ &= (2f(v_i) + 1) \bmod (2m + 1) \\ &= (2f(v_{i-1}) + 3) \bmod (2m + 1) \\ &\quad \vdots \\ &= (2f(v_{i-j}) + 2j + 1) \bmod (2m + 1), \quad j = 0, 1, 2, \dots, i - m - 1, \end{aligned}$$

and when $j = i - m - 1$,

$$\begin{aligned} g(v_i v_{i+1}) &= (2f(v_{m+1}) + 2(i - m - 1) + 1) \bmod (2m + 1) \\ &= (2(2m + 1) + 2(i - m - 1) + 1) \bmod (2m + 1) \\ &= 2(i - m - 1) + 1. \end{aligned}$$

Hence,

$$g(v_i v_{i+1}) = \begin{cases} 2i & 1 \leq i \leq m \\ 2(i - m - 1) + 1 & m + 1 \leq i \leq 2m. \end{cases}$$

Case 2: For P_{2m} , $m = 3, 5, 7, \dots$,

$f: V(P_{2m}) \rightarrow \{2m, 2m + 1, 2m + 2, \dots, 4m - 1\}$ is defined as,

$$f(v_i) = \begin{cases} 2m + 1 & i = 1 \\ f(v_{i-1}) + 2 & 2 \leq i \leq \lfloor \frac{m}{2} \rfloor \\ 3m & i = \lfloor \frac{m}{2} \rfloor + 1 \\ f(v_{i-1}) + 1 & \lfloor \frac{m}{2} \rfloor + 2 \leq i \leq m + \lfloor \frac{m}{2} \rfloor \\ 2m & i = m + \lfloor \frac{m}{2} \rfloor + 1 \\ f(v_{i-1}) + 2 & m + \lfloor \frac{m}{2} \rfloor + 2 \leq i \leq 2m. \end{cases}$$

Following a similar approach as in Case 1, we obtain,

$$g(v_i v_{i+1}) = \begin{cases} 4i & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor \\ 2\left(i - \lfloor \frac{m}{2} \rfloor - 1\right) + 1 & \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m + \lfloor \frac{m}{2} \rfloor - 1 \\ 2m - 1 & i = m + \lfloor \frac{m}{2} \rfloor \\ 4\left(i - m - \lfloor \frac{m}{2} \rfloor - 1\right) + 2 & m + \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq 2m - 1. \end{cases}$$

Case 3: For P_{4m} , $m = 3, 5, 7, \dots$,

$f: V(P_{4m}) \rightarrow \{4m, 4m + 1, 4m + 2, \dots, 8m - 1\}$ is defined as,

$$f(v_i) = \begin{cases} 4m & i = 1 \\ f(v_{i-1}) + 2 & 2 \leq i \leq m \\ 4m + 1 & i = m + 1 \\ f(v_{i-1}) + 2 & m + 2 \leq i \leq m + \lfloor \frac{m}{2} \rfloor \\ 7m & i = m + \lfloor \frac{m}{2} \rfloor + 1 \\ f(v_{i-1}) + 1 & m + \lfloor \frac{m}{2} \rfloor + 2 \leq i \leq 2m + \lfloor \frac{m}{2} \rfloor \\ 6m - 1 & i = 2m + \lfloor \frac{m}{2} \rfloor + 1 \\ f(v_{i-1}) - 2 & 2m + \lfloor \frac{m}{2} \rfloor + 2 \leq i \leq 3m \\ 7m - 1 & i = 3m + 1 \\ f(v_{i-1}) - 1 & 3m + 2 \leq i \leq 4m. \end{cases}$$

Following a similar approach as in Case 1, we obtain,

$$g(v_i v_{i+1}) = \begin{cases} 4i - 2 & 1 \leq i \leq m - 1 \\ 2m - 1 & i = m \\ 4i - 4m & m + 1 \leq i \leq m + \lfloor \frac{m}{2} \rfloor - 1 \\ 4m - 2 & i = m + \lfloor \frac{m}{2} \rfloor \\ 2i - m & m + \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq 2m + \lfloor \frac{m}{2} \rfloor - 1 \\ 2m - 2 & i = 2m + \lfloor \frac{m}{2} \rfloor \\ 14m - 4i - 2 & 2m + \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq 3m - 1 \\ 4m - 1 & i = 3m \\ 8m - 2i - 1 & 3m + 1 \leq i \leq 4m - 1. \end{cases}$$

Case 4: For P_{8m} , $m = 1, 2, 3, \dots$,

$f: V(P_{8m}) \rightarrow \{8m, 8m + 1, 8m + 2, \dots, 16m - 1\}$ is defined as,

$$f(v_i) = \begin{cases} 14m - 1 & i = 1 \\ 10m - 2 & i = 2 \\ f(v_{i-2}) - 2 & 3 \leq i \leq 2m \\ 12m - 1 & i = 2m + 1 \\ f(v_{i-1}) - 2 & 2m + 2 \leq i \leq 4m \\ 16m - 2 & i = 4m + 1 \\ f(v_{i-1}) - 2 & 4m + 2 \leq i \leq 6m + 1 \\ 16m - 1 & i = 6m + 2 \\ f(v_{i-2}) - 2 & 6m + 3 \leq i \leq 8m. \end{cases}$$

Following a similar approach as in Case 1, we obtain,

$$g(v_i v_{i+1}) = \begin{cases} 8m - 3 & i = 1 \\ 8m - 2i - 1 & 2 \leq i \leq 2m - 1 \\ 4m - 1 & i = 2m \\ 16m - 4i & 2m + 1 \leq i \leq 4m - 1 \\ 8m - 1 & i = 4m \\ 24m - (4i + 2) & 4m + 1 \leq i \leq 6m - 1 \\ 32m - (4i + 2) & i = 6m \\ 4m - 3 & i = 6m + 1 \\ 16m - 2i - 1 & 6m + 2 \leq i \leq 8m - 1. \end{cases}$$

Now we have $h = \frac{1}{2n-1}$. The vertex labeling $\mu: V(P_n) \rightarrow [0,1]$ is defined as, $\mu(v_i) = f(v_i) \cdot \frac{1}{2n-1}$ for every $1 \leq i \leq n$ and edge labeling $\rho: E(P_n) \rightarrow [0,1]$ is defined as, $\rho(v_i v_{i+1}) = g(v_i v_{i+1}) \cdot \frac{1}{2n-1}$ for every $1 \leq i \leq n - 1$. It can be verified that in all the cases $g(v_i v_{i+1})$ are distinct and nonzero, therefore $\rho(v_i v_{i+1})$ are distinct and nonzero. Further $\max\{g(v_i v_{i+1})\} = n - 1$ and $\min\{f(v_i)\} = n$. Therefore, $g(v_i v_{i+1}) < f(v_i) \wedge f(v_{i+1})$ for every $1 \leq i \leq n - 1$. Hence for every $1 \leq i \leq n - 1$,

$$\begin{aligned} \rho(v_i v_{i+1}) &= g(v_i v_{i+1}) \cdot \frac{1}{2n-1} \\ &< (f(v_i) \wedge f(v_{i+1})) \cdot \frac{1}{2n-1} \\ &< f(v_i) \cdot \frac{1}{2n-1} \wedge f(v_{i+1}) \cdot \frac{1}{2n-1} \\ &< \mu(v_i) \wedge \mu(v_{i+1}). \end{aligned}$$

This implies that, the edge labels are less than the minimum of their respective endpoints labels. Consequently, $P_n, n \neq 4$ are Elegant Fuzzy Labeling Graphs and (μ, ρ) is an Elegant Fuzzy Labeling of $P_n, n \neq 4$. □

Theorem 3.2.2. Line graph of path graphs $L(P_n)$ admit Elegant Fuzzy Labeling except for $n = 5$.

Proof. By Theorem 3.2.1, P_n admit Elegant Fuzzy Labeling except for $n = 4$. $L(P_n) \cong P_{n-1}$. Therefore $L(P_n)$ admit Elegant Fuzzy Labeling except for $n = 5$. □

Example 3.2.3. Consider a path graph P_5 in Fig. 5. By Theorem 3.2.1, a function $f: V(P_5) \rightarrow \{5, 6, 7, 8, 9\}$ is defined as follows: $f(v_1) = 8, f(v_2) = 9, f(v_3) = 5, f(v_4) = 6, f(v_5) = 7$. Consequently, $g(v_1 v_2) = 2, g(v_2 v_3) = 4, g(v_3 v_4) = 1, g(v_4 v_5) = 3$. Hence an Elegant Fuzzy Labeling (μ, ρ) of P_5 is given as follows: $\mu(v_1) = 0.89, \mu(v_2) = 1, \mu(v_3) = 0.56, \mu(v_4) = 0.67, \mu(v_5) = 0.78, \rho(v_1 v_2) = 0.22, \rho(v_2 v_3) = 0.44, \rho(v_3 v_4) = 0.11, \rho(v_4 v_5) = 0.33$.

Theorem 3.2.4. Cycles C_n admit Elegant Fuzzy Labeling for $n \equiv 0, 3 \pmod 4$.

Proof. Let C_n be a cycle with vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$. Let f be an injective function from $V(C_n)$ into the set $\{n + 1, n + 2, \dots, 2n + 1\}$ and g be a function from $E(C_n)$ into the set $\{0, 1, \dots, n\}$ defined as, $g(v_nv_1) = (f(v_n) + f(v_1)) \pmod{n + 1}$ and $g(v_iv_{i+1}) = (f(v_i) + f(v_{i+1})) \pmod{n + 1}$ for $1 \leq i \leq n - 1$.

Case 1: For C_{2m+1} , $m = 1, 3, 5, \dots$,

$f: V(C_{2m+1}) \rightarrow \{2m + 2, 2m + 3, 2m + 4, \dots, 4m + 3\}$ is defined as,

$$f(v_i) = \begin{cases} 2m + 2 & i = 1 \\ 3m + 1 + i & 2 \leq i \leq 3 \\ 2m + 3 & i = 4 \\ f(v_{i-2}) + 1 & 5 \leq i \leq m + 2 \\ f(v_{i-1}) + 1 & i = m + 3 \\ f(v_{i-3}) + 2 & i = m + 4 \\ f(v_{i-2}) + 1 & m + 5 \leq i \leq 2m + 1. \end{cases}$$

For $i = 1$,

$$\begin{aligned} g(v_1v_2) &= (2m + 2 + 3m + 3) \pmod{2m + 2} \\ &= (5m + 5) \pmod{2m + 2} \\ &= m + 1. \end{aligned}$$

For $i = 2$,

$$\begin{aligned} g(v_2v_3) &= (3m + 3 + 3m + 4) \pmod{2m + 2} \\ &= (6m + 7) \pmod{2m + 2} \\ &= 1. \end{aligned}$$

For $i = 3$,

$$\begin{aligned} g(v_3v_4) &= (3m + 4 + 2m + 3) \pmod{2m + 2} \\ &= (5m + 7) \pmod{2m + 2} \\ &= m + 3. \end{aligned}$$

For $4 \leq i \leq m + 1$,

$$\begin{aligned} g(v_iv_{i+1}) &= (f(v_i) + f(v_{i+1})) \pmod{2m + 2} \\ &= (f(v_{i-2}) + 1 + f(v_{i+1-2}) + 1) \pmod{2m + 2} \\ &= (f(v_{i-4}) + 2 + f(v_{i+1-4}) + 2) \pmod{2m + 2} \\ &\vdots \end{aligned}$$

Subcase 1: When $4 \leq i \leq m + 1$ and odd,

$$g(v_iv_{i+1}) = \left(f(v_{i-j}) + \frac{j}{2} + f(v_{i+1-j}) + \frac{j}{2} \right) \pmod{2m + 2}, \quad j = 2, 4, \dots, i - 3,$$

and when $j = i - 3$,

$$\begin{aligned}
 g(v_i v_{i+1}) &= \left(f(v_3) + f(v_4) + 2 \left(\frac{i-3}{2} \right) \right) \text{mod } (2m+2) \\
 &= (3m+4+2m+3+i-3) \text{mod } (2m+2) \\
 &= (2(2m+2) + m+i) \text{mod } (2m+2) \\
 &= m+i.
 \end{aligned}$$

Subcase 2: When $4 \leq i \leq m+1$ and even,

$$g(v_i v_{i+1}) = \left(f(v_{i-j}) + \frac{j}{2} + f(v_{i+1-j}) + \frac{j}{2} \right) \text{mod } (2m+2), \quad j = 0, 2, 4, \dots, i-4,$$

and when $j = i-4$,

$$\begin{aligned}
 g(v_i v_{i+1}) &= \left(f(v_4) + f(v_5) + 2 \left(\frac{i-4}{2} \right) \right) \text{mod } (2m+2) \\
 &= \left(f(v_4) + f(v_3) + 1 + 2 \left(\frac{i-4}{2} \right) \right) \text{mod } (2m+2) \\
 &= \left(f(v_3) + f(v_4) + 2 \left(\frac{i-3}{2} \right) \right) \text{mod } (2m+2) \\
 &= (3m+4+2m+3+i-3) \text{mod } (2m+2) \\
 &= (2(2m+2) + m+i) \text{mod } (2m+2) \\
 &= m+i.
 \end{aligned}$$

For $i = m+2$,

$$\begin{aligned}
 g(v_{m+2} v_{m+3}) &= (f(v_{m+2}) + f(v_{m+3})) \text{mod } (2m+2) \\
 &= (2f(v_{m+2}) + 1) \text{mod } (2m+2) \\
 &= (2f(v_m) + 2 + 1) \text{mod } (2m+2) \\
 &= (2f(v_{m-2}) + 4 + 1) \text{mod } (2m+2) \\
 &\quad \vdots \\
 &= (2f(v_{(m+2)-(m+2-3)}) + m+2-3+1) \text{mod } (2m+2) \\
 &= (2f(v_3) + m+2-2) \text{mod } (2m+2) \\
 &= (6m+6+2+m) \text{mod } (2m+2) \\
 &= m+2.
 \end{aligned}$$

For $i = m+3$,

$$\begin{aligned}
 g(v_{m+3} v_{m+4}) &= (f(v_{m+3}) + f(v_{m+4})) \text{mod } (2m+2) \\
 &= (f(v_{m+2}) + 1 + f(v_{m+1}) + 2) \text{mod } (2m+2) \\
 &= (f(v_m) + 1 + f(v_{m-1}) + 1 + 3) \text{mod } (2m+2) \\
 &= (f(v_{m-2}) + 2 + f(v_{m-3}) + 2 + 3) \text{mod } (2m+2) \\
 &\quad \vdots \\
 &= \left(f(v_3) + \frac{m-1}{2} + f(v_4) + \frac{m-3}{2} + 3 \right) \text{mod } (2m+2) \\
 &= \left(3m+4+2m+3 + \frac{m-1}{2} + \frac{m-3}{2} + 3 \right) \text{mod } (2m+2) \\
 &= (5m+10+m-2) \text{mod } (2m+2) \\
 &= 2.
 \end{aligned}$$

For $m + 4 \leq i \leq 2m$,

$$\begin{aligned} g(v_i v_{i+1}) &= (f(v_i) + f(v_{i+1})) \bmod (2m + 2) \\ &= (f(v_{i-2}) + 1 + f(v_{i+1-2}) + 1) \bmod (2m + 2) \\ &= (f(v_{i-4}) + 2 + f(v_{i+1-4}) + 2) \bmod (2m + 2) \\ &\vdots \end{aligned}$$

Subcase 1: When $m + 4 \leq i \leq 2m$ and even,

$$g(v_i v_{i+1}) = \left(f(v_{i-j}) + \frac{j}{2} + f(v_{i+1-j}) + \frac{j}{2} \right) \bmod (2m + 2), \quad j = 2, 4, \dots, i - m - 3,$$

and when $j = i - m - 3$,

$$\begin{aligned} g(v_i v_{i+1}) &= (f(v_{m+3}) + f(v_{m+4}) + (i - m - 3)) \bmod (2m + 2) \\ &= (6m + 6 + 2 + i - m - 3) \bmod (2m + 2) \\ &= i - m - 1. \end{aligned}$$

Subcase 2: When $m + 4 \leq i \leq 2m$ and odd,

$$g(v_i v_{i+1}) = \left(f(v_{i-j}) + \frac{j}{2} + f(v_{i+1-j}) + \frac{j}{2} \right) \bmod (2m + 2), \quad j = 0, 2, 4, \dots, i - m - 4,$$

and when $j = i - m - 4$,

$$\begin{aligned} g(v_i v_{i+1}) &= (f(v_{m+4}) + f(v_{m+5}) + (i - m - 4)) \bmod (2m + 2) \\ &= (f(v_{m+4}) + f(v_{m+3}) + 1 + (i - m - 4)) \bmod (2m + 2) \\ &= (f(v_{m+3}) + f(v_{m+4}) + (i - m - 3)) \bmod (2m + 2) \\ &= (6m + 6 + 2 + i - m - 3) \bmod (2m + 2) \\ &= i - m - 1. \end{aligned}$$

Now,

$$\begin{aligned} g(v_{2m+1} v_1) &= (f(v_{2m+1}) + f(v_1)) \bmod (2m + 2) \\ &= (f(v_{2m+1-2}) + 1 + 2m + 2) \bmod (2m + 2) \\ &= (f(v_{2m+1-4}) + 2 + 2m + 2) \bmod (2m + 2) \\ &\vdots \\ &= \left(f(v_{m+4}) + \frac{2m+1-m-4}{2} + 2m + 2 \right) \bmod (2m + 2) \\ &= \left(f(v_{m+4-3}) + 2 + \frac{m-3}{2} + 2m + 2 \right) \bmod (2m + 2) \\ &= \left(f(v_{m+1}) + \frac{m-3}{2} + 2m + 4 \right) \bmod (2m + 2) \\ &= \left(f(v_{m+1-2}) + 1 + \frac{m-3}{2} + 2m + 4 \right) \bmod (2m + 2) \\ &\vdots \\ &= \left(f(v_{m+1-m-1+4}) + \frac{m-3}{2} + \frac{m-3}{2} + 2m + 4 \right) \bmod (2m + 2) \\ &= \left(f(v_4) + \frac{m-3}{2} + \frac{m-3}{2} + 2m + 4 \right) \bmod (2m + 2) \\ &= (2m + 3 + m - 3 + 2m + 4) \bmod (2m + 2) \end{aligned}$$

$$\begin{aligned}
 &= (4m + 4 + m) \text{ mod}(2m + 2) \\
 &= m.
 \end{aligned}$$

Hence,

$$g(v_i v_{i+1}) = \begin{cases} m + 1 & i = 1 \\ 1 & i = 2 \\ m + 3 & i = 3 \\ m + i & 4 \leq i \leq m + 1 \\ m + 2 & i = m + 2 \\ 2 & i = m + 3 \\ i - m - 1 & m + 4 \leq i \leq 2m. \end{cases}$$

$$g(v_{2m+1} v_1) = m$$

Case 2: For C_{4m} , $m = 1, 2, 3, \dots$,

$f: V(C_{4m}) \rightarrow \{4m + 1, 4m + 2, 4m + 3, \dots, 8m + 1\}$ is defined as,

$$f(v_i) = \begin{cases} 4m + 2 & i = 1 \\ 8m + i - 2 & 2 \leq i \leq 3 \\ f(v_{i-2}) - 2 & 4 \leq i \leq 2m + 1 \\ f(v_{i-1}) - 2 & i = 2m + 2 \\ f(v_{i-1}) - 1 & 2m + 3 \leq i \leq 4m. \end{cases}$$

Following a similar approach as in Case 1, we obtain,

$$g(v_i v_{i+1}) = \begin{cases} 4m & i = 1 \\ 4m - 2 & i = 2 \\ 4m + 2 - 2i & 3 \leq i \leq 2m \\ 1 & i = 2m + 1 \\ 8m + 3 - 2i & 2m + 2 \leq i \leq 4m - 1. \end{cases}$$

$$g(v_{4m} v_1) = 3.$$

Now we have $h = \frac{1}{2n+1}$. The vertex labeling $\mu: V(C_n) \rightarrow [0,1]$ is defined as, $\mu(v_i) = f(v_i) \cdot \frac{1}{2n+1}$ for every $1 \leq i \leq n$ and the edge labeling $\rho: E(C_n) \rightarrow [0,1]$ is defined as, $\rho(v_i v_{i+1}) = g(v_i v_{i+1}) \cdot \frac{1}{2n+1}$ for every $1 \leq i \leq n - 1$ and $\rho(v_n v_1) = g(v_n v_1) \cdot \frac{1}{2n+1}$. It can be verified that in all the cases the elements of the set $\{g(v_n v_1), g(v_i v_{i+1}), 1 \leq i \leq n - 1\}$ are distinct and nonzero. Therefore, the edge labels are distinct and non zero. Further $\max\{g(v_n v_1), g(v_i v_{i+1}), 1 \leq i \leq n - 1\} = n$ and $\min\{f(v_i)\} = n + 1$. Therefore, $g(v_n v_1) < f(v_n) \wedge f(v_1)$ and $g(v_i v_{i+1}) < f(v_i) \wedge f(v_{i+1})$ for every $1 \leq i \leq n - 1$. Hence the edge labels are less than the minimum of their respective endpoints labels. Consequently, C_n for $n \equiv 0,3 \text{ mod } 4$ are Elegant Fuzzy Labeling Graphs and (μ, ρ) is an Elegant Fuzzy Labeling of C_n for $n \equiv 0,3 \text{ mod } 4$. \square

Remark 3.2.5. Cycles C_n where $n \equiv 1 \text{ mod } 4$ are not Elegant [3], therefore it does not admit Elegant Fuzzy Labeling.

Note. The case for cycles C_n where $n \equiv 2 \pmod 4$ remains an open question. Future work could investigate whether this class of cycles can admit Elegant Fuzzy Labeling or if modifications to the labeling scheme are required to accommodate these structures.

Theorem 3.2.6. Line graph of cycles $L(C_n)$ admit Elegant Fuzzy Labeling when $n \equiv 0,3 \pmod 4$.

Proof. By Theorem 3.2.4, C_n admit Elegant Fuzzy Labeling when $n \equiv 0,3 \pmod 4$. $L(C_n) \equiv C_n$. Therefore $L(C_n)$ admit Elegant Fuzzy Labeling when $n \equiv 0,3 \pmod 4$. \square

Example 3.2.7. Consider a cycle C_8 in Fig.6. By Theorem 3.2.4, the function $f:V(C_8) \rightarrow \{9, 10, 11, 12, 13, 14, 15, 16, 17\}$ is defined as follows: $f(v_1) = 10, f(v_2) = 16, f(v_3) = 17, f(v_4) = 14, f(v_5) = 15, f(v_6) = 13, f(v_7) = 12, f(v_8) = 11$. Consequently, $g(v_1v_2) = 8, g(v_2v_3) = 6, g(v_3v_4) = 4, g(v_4v_5) = 2, g(v_5v_6) = 1, g(v_6v_7) = 7, g(v_7v_8) = 5, g(v_8v_1) = 3$. Hence, an Elegant Fuzzy Labeling (μ, ρ) of C_8 is given as follows:
 $\mu(v_1) = 0.59, \mu(v_2) = 0.94, \mu(v_3) = 1, \mu(v_4) = 0.82, \mu(v_5) = 0.88, \mu(v_6) = 0.76, \mu(v_7) = 0.71, \mu(v_8) = 0.65,$
 $\rho(v_1v_2) = 0.47, \rho(v_2v_3) = 0.35, \rho(v_3v_4) = 0.24, \rho(v_4v_5) = 0.12, \rho(v_5v_6) = 0.06, \rho(v_6v_7) = 0.41, \rho(v_7v_8) = 0.29, \rho(v_8v_1) = 0.18.$

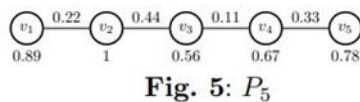


Fig. 5: P_5

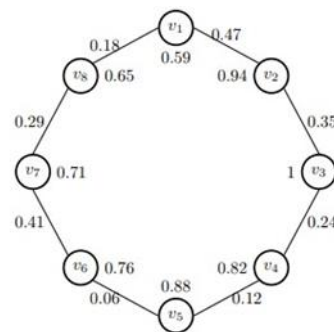


Fig. 6: C_8

4. Application

Consider a transit network as an Elegant Fuzzy Labeling Graph, where vertices represent stations and edges represent routes. Vertex labels denote passenger capacity at each station, while edge labels represent the passenger flow along the routes. The condition of Elegant Fuzzy Labeling, that labels are distinct and non-zero, guarantees that every station and route is utilized effectively. The condition that edge labels are less than their endpoints labels ensures that the passenger flow along the route does not exceed the station capacity, thus preventing congestion at stations and along the routes. This approach enables judicious allocation of resources, improves routing and scheduling, and ensures smooth movement of passengers.

5. Conclusion

In this paper, we have introduced a new type of fuzzy labeling called Elegant Fuzzy Labeling. We proved that a simple graph admits Elegant Fuzzy Labeling if and only if it admits elegant labeling. Additionally, we showed that while a simple graph admitting Elegant Fuzzy Labeling will also admit fuzzy labeling, the converse is not necessarily true. We have investigated certain classes of simple graphs to determine if they admit Elegant Fuzzy Labeling and provided an application using this labeling method. We employed translation and contraction of mapping to achieve the labeling. For future work, we plan to explore Elegant Fuzzy Labeling on other graph families, investigate whether

different translation constants and contraction scales can be used to define Elegant Fuzzy Labeling and extend the concept to various extensions of fuzzy graphs.

Declarations

Conflicts of Interest: None of the authors have any conflict of interest.

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