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# Error Estimates for Three-Dimensional Singularly Perturbed Convection-Diffusion Problem

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**Abstract**

The main aim of the present work is to propose a posteriori error estimates for singularly perturbed convection-diffusion problem in three-dimensions. It has been observed that for small singular perturbation parameter, the problem under consideration displays nonphysical oscillations in the small subregions of the boundary layers. Hughes stabilization strategy along with the Streamline upwind/Petrov-Galerkin (SUPG) method has been proposed to approximate the solution of the problem. Reliable a posteriori error estimates in energy norm on anisotropic meshes have been developed for the proposed method.

**Keywords:**

Convection-diffusion equation, Streamline upwind/Petrov-Galerkin (SUPG) method, Hughes stabilization, singularly perturbed problems, a posteriori error estimation, finite element method.

## 1. Introduction

Many important mathematical models governing various physical phenomena occurring during the analysis of biological systems, heat transfer process, mass transfer process, etc., are represented by partial differential equations [14,16]. Very few researchers have developed finite element strategies for simulating three-dimensional partial differential equations. To name a few, Branco et al. [1] proposed three-dimensional finite element technique to analyse the shape evolution of fatigue cracks. Numerical tests have been performed and it has been shown that the numerical results agree with the experimental findings. Mola et al. [13] developed Streamline upwind Petrov-Galerkin (SUPG) technique for approximating unsteady three-dimensional non-linear water waves arising due to ship hull advancing in water based on semi-Lagrangian framework. SUPG projection has been considered to recover accurate estimates of position vector and potential gradients on free surface. The proposed technique results in stabilization of the transport dominated terms and robust adaptation of the spatial discretization on unstructured quadrilateral grids. Zhai et al. [17] analysed three-dimensional time fractional convection-diffusion equation by proposing an implicit compact finite difference scheme which uses fourth-order Pad$e'$ approximation for spatial discretization and central difference scheme for time discretization. Mohanty and Setia [11] developed high order compact finite difference scheme for approximating three-dimensional quasi-linear elliptic partial differential equation. In the present paper, we will focus on proposing the Hughes stabilized SUPG finite element technique for solving the singularly perturbed problems.

Consider the three-dimensional SPP given by

$$-\nabla.\epsilon\nabla u + \boldsymbol{b}.\nabla u + cu = f \qquad in \;\; \Gamma, \tag{1.1}$$

$$u = 0 \qquad on \;\; \partial\Gamma_D, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = g \qquad on \;\; \partial\Gamma_N, \tag{1.3}$$

where $\Gamma \subset R^3$ is a bounded domain with Lipschitz-continuous boundary $\partial\Gamma$ and $\epsilon$ is singular perturbation parameter satisfying $0 < \epsilon \ll 1$. We assume that $\partial\Gamma = \partial\Gamma_D \cup \partial\Gamma_N$ with $\partial\Gamma_D \cap \partial\Gamma_N = \emptyset$ , and $\boldsymbol{b}$, $c$ and $f$ are analytic. $\partial\Gamma_D$ and $\partial\Gamma_N$ represent the Dirichlet and Neumann boundaries of the domain respectively.

The considered problem is comprised of two basic phenomena-convection and diffusion. For the case when $\epsilon \ll ||\boldsymbol{b}||$, the above problem becomes convection dominant in nature. This results in singularities, such as shocks, interior and boundary layers which deteriorate the accuracy of numerical solutions obtained by various numerical schemes. Therefore, it becomes essential to obtain some reliable and efficient error estimates for the computed numerical solution to rely on.

Broadly, error estimates are of two different types, namely, a priori error estimates and a posteriori error estimates. It is seen that a priori error estimates provide crude information only about the asymptotic behavior of the solution and involves regularity conditions which are very difficult to achieve in case of singularities whereas a posteriori error estimates provide quantitative information of the computed numerical solution. Therefore, it is more expected to derive some reliable a posteriori error estimates based on the data of the problem and the computed numerical solution.

Stephansen [5] proposed robust a posteriori error estimates for convection-diffusion-reaction problems based on weighted interior-penalty discontinuous Galerkin methods. Lazarov [10] derived residual based a posteriori estimates for convection-diffusion-reaction equations using finite volume element approximations. Carstensen et al. [3] proposed residual-type explicit error estimators and averaging techniques for steady convection-diffusion-reaction problems using finite volume method. Further, the authors proposed adaptive mesh refining strategy and considered numerical examples to test the theoretical findings.

It has been seen that Streamline upwind/Petrov-Galerkin (SUPG) method provides good approximate solution in the region where there is no sharp change in the solution but fails badly in the small subregions of sharp boundary layers appearing in the sol. of singularly perturbed problems. It has been observed that occurrence of these nonphysical oscillations in the region of sharp boundary layers in discrete solution of SUPG method is based on the fact that this scheme is not monotonicity preserving. To overcome this hurdle, in the present work, an effort has been made by using Hughes stabilization strategy [6] alongwith Streamline upwind/Petrov-Galerkin (SUPG) method. This corresponds to addition of one more term in the SUPG discretization of the considered convection-diffusion problem. A posteriori error estimates have been obtained for the developed technique.

The paper is organised as follows:

In Section 2, some notations and standard norms have been discussed. Then the variational formulation and Hughes stabilized Streamline upwind finite element approximation of the continuous problem have been discussed. Section 3 deals with some important tools which are essential for deriving reliable error estimates.

In Section 4, residual based a posteriori error estimates have been derived on anisotropic meshes. In the last Section 5, concluding remarks have been made.

## 2 Hughes stabilized SUPG technique

Let $W^{1,\infty}(\Gamma)$and $L^{\infty}(\Gamma)$ denote the usual Sobolev space and Lebesgue space respectively. We will use the notation (. , .) for inner product $(.,.)_{\Gamma}$.

We assume that $-\frac{1}{2}\nabla.\boldsymbol{b}+c \geq c_0 > 0$ on $\overline{\Gamma}$.

For any open and bounded subset $T \subset \overline{\Gamma}$, , let $H^1(T)$ be the standard Sobolev space. Let

$$V_0 = \{w \in H^1(\Gamma), w = 0 \;\; on\; \partial\Gamma_D\}.$$

Since, our objective is to bound the global error $w - w_h$ in energy norm, so we define the energy norm on any bounded subset $T \subset \overline{\Gamma}$ as

$$|||w|||_T^2 = \epsilon.\left||\nabla w|\right|_T^2 + c_0\left||w|\right|_T^2. \qquad (2.1)$$

The finite element weak formulation of (1.1) is given as:

Find $v \in H^1(\Gamma)$ such that

$$B(v,w) =< F, w >, \qquad (2.2)$$

where

$$B(v,w) = \epsilon(\nabla v, \nabla w) + (\boldsymbol{b}.\nabla v, w) + (cv, w) \qquad (2.3)$$

$$< F, w >= (f,w) + (g,w)_{\partial\Omega_N}, \qquad \forall\, w \in V_0.$$

The existence and uniqueness of solution of above weak formulation (2.2) can be confirmed using the Lax Milgram lemma. Let $F = \{\Gamma_h\}$ denote the family of triangulations of $\Gamma$. Let $\Gamma_h$ be triangulation of domain $\Gamma$ consisting of tetrahedrons in three-dimensions, assuming only admissible and shape-regular triangulation in $\Gamma_h$. For any tetrahedron $T$ with face $E$, we define $n_{T,E}$ = $(n_x, n_y, n_z)$ to be the unit outward normal vector for face $E$ of the tetrahedron $T$. Let $n_E$ be the normal vector for face $E$ obtained from $n_{T,E}$ by fixing any of two normal components.

### 2.1 SUPG method

We define $V^h = \{w_h \in H^1 : w_h|_T \in P_1(T), \forall\, T \in \Gamma_h\}$, where $P_1(T)$ is the space of linear polynomials over tetrahedron $T$ and $V_0^h = \{w_h \in V^h\colon\; w_h|_{\partial\Gamma_D} = 0\}$. The SUPG method [2] for problem (1.1) is given by

Find $v_h \in V^h$ such that

$$\boldsymbol{B}_{\rho}(v_h, w_h) =< F, w_h > \quad \forall\, w_h \in V_0^h, \qquad (2.1.4)$$

where $\boldsymbol{B}_{\rho}(v_h, w_h) = \boldsymbol{B}(v_h, w_h) + < R_h(v_h), \rho\boldsymbol{b}.\nabla_h w_h >$ and $< R_h(v_h), \rho\boldsymbol{b}.\nabla_h w_h >= \sum_{T\in\Gamma_h} \rho_T\,(-\epsilon\Delta_h v_h + \boldsymbol{b}.\nabla_h v_h + c v_h - f, \boldsymbol{b}.\nabla_h w_h)_T$ , $\rho$ is nonnegative stabilization parameter, $\boldsymbol{B}(v,w)$and <*F*,*w*> are defined in (2.3).

**Hughes stabilized SUPG technique**

As is well known that convection dominated problems exhibit nonphysical oscillations at layers as $\epsilon$ decreases, the solution of singularly perturbed problem displays boundary layers. Since the Streamline upwind/Petrov-Galerkin (SUPG) method results in good approximate solution in the region where there is no sharp change in the solution but fails drastically in the subregions of sharp boundary layers, in the trial to overcome this hurdle, we propose Hughes stabilization technique [7] to SUPG method. It results in an addition of the term $<R_h(v_h), \sigma\boldsymbol{b}_h.\nabla_h w_h>$ in the SUPG finite element discretization of the convection-diffusion equation where

$$\boldsymbol{b}_h = \begin{cases} \dfrac{(\boldsymbol{b}.\nabla \boldsymbol{v}_h)\nabla v_h}{|\nabla v_h|^2} & , if\ |\nabla v_h| \neq 0, \\ 0, & if\ \ |\nabla v_h| = 0 \end{cases} \qquad (2.2.1)$$

and $\sigma$ is nonnegative stabilization parameter. This additional term increases the robustness of SUPG method in the boundary layer region by controlling oscillations. Using Hughes stabilization technique to SUPG finite element method, Eq.(1.1) is discretized as follows:

Find $v_h \in V^h$ such that

$$\boldsymbol{B}_{\rho,\sigma}(v_h, w_h) =< F, w_h > \quad \forall\, w_h \in V_0^h. \qquad (2.2.2)$$

Here $\boldsymbol{B}_{\rho,\sigma}(v_h, w_h) = \boldsymbol{B}(v_h, w_h) + < R_h(v_h), \rho\boldsymbol{b}.\nabla_h w_h > + < R_h(v_h), \sigma\boldsymbol{b}_h.\nabla_h w_h >$ and $< R_h(v_h), \sigma\boldsymbol{b}_h.\nabla_h w_h >= \sum_{T\in\Gamma_h} \sigma_T(-\epsilon\Delta_h v_h + \boldsymbol{b}.\nabla_h v_h + c v_h - f, \boldsymbol{b}_h.\nabla_h w_h)_T$.

Let $\rho_T$ and $\sigma_T$ be stabilization parameters over each element *T*. The existence and uniqueness of the finite element solution $v_h$ obtained using SUPG finite element discretization has been proved by Roos et al. [15]. The stabilization parameter $\rho_T$ satisfies

$$0 \leq \rho_T \leq \frac{1}{2}\min\{c_0 ||c||_{\infty,T}^{-2} \quad , \ (h_{min}^T)^{-2}\epsilon^{-1}\vartheta^{-2}\}, \qquad (2.2.3)$$

where $h_{min}^T$ is minimal length of element $T$ and the constant $\vartheta$ satisfies the inequality

$$||\nabla.\nabla v_h||_T \leq \vartheta (h_{min}^T)^{-1} ||\nabla v_h||_T \quad \forall v_h \in V_0^h. \qquad (2.2.4)$$

It can be observed that $\vartheta = 0$ for piecewise linear functions in $V_0^h$ . Therefore, the above bounds reduces to $0 \leq \rho_T \leq \frac{c_0}{2} ||c||_{\infty,T}^{-2}$. For easiness, we will use $c \lesssim d$ to denote that there exists a positive constant *A* independent of *c*,*d*, $\Gamma_h$ and $\epsilon$ such that $c \leq Ad$. Further, we assume that

$$\rho_T \lesssim h_{min}^T ||\boldsymbol{b}||_{\infty,\mathrm{T}}^{-1}\ \forall\ T \in \Gamma_h \quad . \qquad (2.2.5).$$

## 3 Some auxiliary tools

Since the considered singularly perturbed problem displays abrupt changes in the solution when Peclet number becomes very large as clear from (3.2), in such situations, elements with large aspect ratio i.e. anisotropic meshes are preferred. Therefore, in the present work, anisotropic mesh has been considered for domain discretization. In the present Section, some notations on anisotropic meshes have been discussed which will be used in later Sections.

**Notations:** Consider an arbitrary tetrahedron $T \in \Gamma_h$ with $Q_0Q_1$as longest edge (see Fig. 1). Represent three orthogonal vectors by $q_{i,T}$ with lengths $h_{i,T} = |q_{i,T}|$, where $q_{1,T}$ is taken along the largest edge. From Fig. 1, it can be be confirmed that $h_{1,T} > h_{2,T} \geq h_{3,T}$. Define $h_{min}^{T} = h_{3,T}$. These $q_{i,T}'s$ correspond to three anisotropic directions. Define an orthogonal matrix as $C_T$ = $(q_{1,T}, q_{2,T}, q_{3,T})$ $\in R^{3X3}$. Let $\alpha_T$ be scaling factor defined as

$$\alpha_T = \min\left\{c_0^{-\frac{1}{2}}, \epsilon^{-\frac{1}{2}}. h_{min}^{T}\right\}. \qquad (3.1)$$

We denote tetrahedron by $T$ or $T'$ or $T_i$, and its faces by $E$. Denote its height over face $E$ by

$$h_{E,T} = 3.\frac{|T|}{|E|},$$

where $|T|$ represents the volume of the tetrahedron and $|E|$ represents the area of the face $E$. Let $w_E$ be the bounded domain formed by using two tetrahedrons with common face $E$ and $w_T$ to be the domain consisting of tetrahedron $T$ and its face neighbouring tetrahedra. We denote the local mesh Peclet number as

$$Pe_T = \frac{||\boldsymbol{b}||_{\infty,T}\, h_{min}^{T}}{2\epsilon}, \qquad (3.2)$$

where $h_{min}^{T}$ is minimal length of element $T$. Let

$$Pe_{w_T} = max_{T' \subset w_T} Pe_{T'}$$

be mesh Peclet number on the domain $w_T$. For an interior face E = $T_1 \cap T_2$, define face based parameters $h_E$ = $(h_{E,T_1} + h_{E,T_2})/2$, $h_{min}^{E}$= $(h_{min}^{T_1} + h_{min}^{T_2})/2$ + and $\alpha_E$ = $(\alpha_{T_1} + \alpha_{T_2})/2$.

For boundary face E$\subset \partial T \cap \partial\Gamma$, we define $h_E$ = $h_{E,T}$, $h_{min}^{E} = h_{min}^{T}$ and $\alpha_E = \alpha_T$.

We assume

$$h_{i,T} \sim h_{i,T'} \quad \forall T, T' \text{ with } T \cap T' \neq \emptyset, i = 1,2,3,$$

and number of tetrahedra with node $y_j$ is bounded uniformly.

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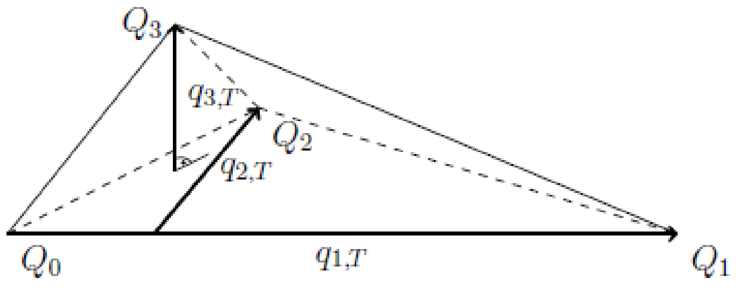


Figure 1: Tetrahedron $T$

## 4 Interpolation

Till now, very few researchers have proposed different a posteriori error estimates on anisotropic meshes for three-dimensional singularly perturbed problems. Since the focus is to obtain reliable upper error bounds, therefore, a suitable estimate or function called matching function [8] has been defined to measure alignment of anisotropic mesh $\Gamma_h$ and anisotropic function.

**Matching function**: Let $v \in H^1(\Gamma)$ ) and $\Gamma_h \in F$ be triangulation of $\Gamma$ . Then $M_1: H^1(\Gamma) \times F \to R$ is defined as

$$M_1(v,\Gamma_h) \coloneqq (\sum_{T\in\Gamma_h} (h_{min}^T)^{-2}.\,|C_T^T \nabla v||_T^2)^{\frac{1}{2}}/||\nabla v||, \qquad (4.1)$$

where $C_T \in R^{3\times 3}$ has been defined earlier.

In order to derive reliable error estimates in energy norm, we define Cl$e'$ment interpolation operator $I_c$ [4] for $v \in H^1(\Gamma)$.

**Lemma 1:** Let $v \in H_0^1(\Gamma)$ and $\alpha_T$ be the scaling factor defined by (3.1). Define Cl$e'$ment interpolation operator $I_c: H_0^1(\Gamma) \to V_0^h$ as defined in [4, 9]. Then it satisfies

$$|||I_C v||| \lesssim M_1(v,\Gamma_h).\,|||v|||, \qquad (4.2)$$

$$\sum_{T\in\Gamma_h} \alpha_T^{-2}.\,||v - I_c v||_T^2 \lesssim M_1(v,\Gamma_h)^2.\,|||v|||^2, \qquad (4.3)$$

$$\epsilon^{1/2} \sum_{E\subset\Gamma\setminus\partial\Gamma_D} \alpha_E^{-1}.\,||v - I_C\, v||_E^2 \lesssim M_1(v,\Gamma_h)^2.\,\big|\big|v\big|\big|^2. \qquad (4.4)$$

**Proof:** The proof of the Lemma has been discussed in [9].

### 4.1 Residual error estimates

In the present Section, first we will discuss exact and approximate residuals. Then, residual error estimator based on residuals has been defined to estimate error in energy norm. Further, reliable error bounds for the proposed scheme Hughes stabilized SUPG finite element method has been proposed on anisotropic meshes

**Exact residuals**: Let $R_T$ and $R_E$ denote the exact element residual and exact face residual over a general tetrahedron element $T$ respectively and are defined as

$$R_T = f - (-\epsilon\Delta u_h + \boldsymbol{b}.\nabla u_h \ + cu_h) \qquad \text{on } T,$$

$$R_E(x) = \begin{cases} lim_{s\to 0+}[\partial_{n_E} u_h(x+sn_E) - \partial_{n_E} u_h(x-sn_E)] & if\ E \subset \ \Gamma\backslash\partial\Gamma, \\ g - \partial_n u_h & if\ \ E \subset \partial\Gamma_N\ , \\ 0 & if\ \ E \subset \partial\Gamma_D, \end{cases}$$

where $n_E \perp E$ is unitary normal vector for face E$\subset \Gamma - \Gamma_N$ and $n \perp E \subset \partial\Gamma_N$ is outer unitary normal vector.

**Approximate residuals**: Let $Q$ be approximation operator, used to approximate the element residuals and the face residuals i.e.

$$r_T = Q(R_T) \in P^0(T) \qquad \forall\, T \in \Gamma_h,$$

$$r_E = Q(R_E) \in P^0(E) \ \ \forall\, E.$$

where $r_T$ denotes the approximate element residual and $r_E$ denotes the approximate face residual. Since the numerical solution $u_h$ is linear on $E$, therefore,

$$r_E = R_E \ \ \forall\, E \subset \Gamma\backslash\partial\Gamma_N.$$

**Residual error estimator**: Residual error estimator $n_T$ and the approximation term $\delta_T$ over any tetrahedron $T$ are defined as

$$n_T^2 = a_T^2 .\, ||r_T||_T^2 + \epsilon^{-\frac{1}{2}} .\, \alpha_T .\sum_{E\subset\partial T\backslash\partial\Gamma_D} ||r_E\,||_E^2 \qquad ,$$

$$\delta_T^2 = a_T^2 .\, ||r_T - R_T||_{w_T}^2 + \epsilon^{-\frac{1}{2}} .\, \alpha_T .\sum_{E\subset\partial T\cap\partial\Gamma_N} ||r_E - R_E||_E^2 \qquad ,$$

where $\alpha_T$ be scaling factor and $a_T$ = 3$\alpha_T$. Global error estimators are defined as

$$\Pi^2 = \sum_{T\in\Gamma_h} \Pi_T^2 \ \ \text{and } \pounds^2 = \sum_{T\in\Gamma_h} \pounds_T^2 \ \ . \qquad (4.1.1)$$

Before proposing reliable error upper bounds, we will prove the following two lemmas which will be used in the results to follow:

**Lemma 2**: Let $v \in H_0^1(\Gamma)$ be exact solution and $v_h \in V_0^h$ be numerical solution obtained by the proposed scheme. Then the bilinear form $\boldsymbol{B}_{\rho,\sigma}(.,.)$satisfies the bounds

$$\boldsymbol{B}_{\rho,\sigma}(v - v_h, w - I_c w) \lesssim [(\sum_{T\in\Gamma_h} \alpha_T^2\, ||R_T||_T^2)^{\frac{1}{2}} + \sum_{E\subset\Gamma\backslash\partial\Gamma_D} \epsilon^{-\frac{1}{2}} \alpha_E ||R_E\,||_E^2)^{1/2}].\, M_1(w,\Gamma_h).\left|||w|||\right|.$$

**Proof**: Using Cl$e^{'}$ment interpolation operator $I_c$, we can write the bilinear form $\boldsymbol{B}_{\rho,\sigma}(.,.)$ as

$\boldsymbol{B}_{\rho,\sigma}(v-v_h,w)=\boldsymbol{B}_{\rho,\sigma}(v-v_h,w-I_cw)+\boldsymbol{B}_{\rho,\sigma}(v-v_h,I_cw).$ (4.1.2)

Now, integrating by parts and using exact element residual $R_T$ and face residual $R_E$, the bilinear form $\boldsymbol{B}_{\rho,\sigma}(v-v_h,w)$ can be written as

$$\boldsymbol{B}_{\rho,\sigma}(v-v_h,w)=\sum_{T\in\Gamma_h}(R_T,w)_T+\sum_{E\subset\Gamma\setminus\partial\Gamma_D}(R_E,w)_E\ \forall\, w\in H_0^1(\Gamma).$$

Using the above expression for bilinear form, the middle term of (4.1.2) can be expressed as

$\boldsymbol{B}_{\rho,\sigma}(v-v_h,w-I_cw)=\sum_{T\in\Gamma_h}(R_T,w-I_cw)_T+\sum_{E\subset\Gamma\setminus\partial\Gamma_D}(R_E,w-I_cw)_E.$

Using Cauchy Schwarz inequality, we get

$$\sum_{T\in\Gamma_h}(R_T,w-I_cw)_T\le(\sum_{T\in\Gamma_h}\alpha_T^2\,||R_T||_T^2)^{\frac{1}{2}}.(\sum_{T\in\Gamma_h}\alpha_T^{-2}\,||w-I_cw||_T^2))^{\frac{1}{2}},$$

$$\sum_{E\subset\Gamma\setminus\partial\Gamma_D}(R_E,w-I_cw)_E\le((\sum_{E\subset\Gamma\setminus\partial\Gamma_D}\ \epsilon^{-\frac{1}{2}}\alpha_E||R_E|\,|_E^2)^{\frac{1}{2}}.(\sum_{E\subset\Gamma\setminus\partial\Gamma_D}\ \epsilon^{\frac{1}{2}}\alpha_E^{-1}||w-I_c\,w|\,|_E^2))^{1/2}\ .$$

Further, using Lemma 1, we get

$$\sum_{T\in\Gamma_h}(R_T,w-I_cw)_T\lesssim\sum_{T\in\Gamma_h}\alpha_T^2\,||R_T||_T^2)^{\frac{1}{2}}.M_1(w,\Gamma_h).\Big|||w|||\Big|,$$

$$\sum_{E\subset\Gamma\setminus\partial\Gamma_D}(R_E,w-I_cw)_E\lesssim(\sum_{E\subset\Gamma\setminus\partial\Gamma_D}\ \epsilon^{-\frac{1}{2}}\alpha_E||R_E\,||_E^2)^{1/2}.M_1(w,\Gamma_h)).\,|||w|||.$$

Therefore, the term $\boldsymbol{B}_{\rho,\sigma}(v-v_h,w-I_cw)$ is bounded above by

$$\boldsymbol{B}_{\rho,\sigma}(v-v_h,w-I_cw)\lesssim[\sum_{T\in\Gamma_h}\alpha_T^2\,||R_T||_T^2)^{\frac{1}{2}}+\sum_{E\subset\Gamma\setminus\partial\Gamma_D}\ \epsilon^{-\frac{1}{2}}\alpha_E||R_E|\,|_E^2)^{\frac{1}{2}}].\,M_1(w,\Gamma_h)).\,|||w|||.$$

**Lemma 3:** Let $v\in H_0^1(\Gamma)$ be exact solution and $v_h\in V_0^h$ be the approximate solution. Then Cle$'$ment interpolation operator $I_C\colon H_0^1(\Gamma)\ \to V_0^h$ satisfies the inequality

$$\boldsymbol{B}_{\rho,\sigma}(v-v_h,I_cw)\lesssim 2(\sum_{T\in\Gamma_h}\alpha_T^2\,||R_T||_T^2)^{\frac{1}{2}}.M_1(w,\Gamma_h).\Big|||w|||\Big|.$$

**Proof**: For any mesh function $v_h\in V_0^h$, using (2.2.4) and scaling arguments, we can get

$$||\nabla v_h||_T\le(h_{min}^T)^{-1}||v_h||_T.$$

From energy norm def.(2.1), we have

$$||v_h||_T\le c_0^{-\frac{1}{2}}|||v_h|||_T.$$

$$\to||\nabla v_h||_T\lesssim(h_{min}^T)^{-1}c_0^{-\frac{1}{2}}|||v_h|||_T. \qquad (4.1.3)$$

Again, from energy norm, we get

$$||\nabla v_h||_T\le\epsilon^{-1/2}|||v_h|||_T. \qquad (4.1.4)$$

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$$\rightarrow ||\nabla v_h||_T \lesssim \min\{(h_{min}^T)^{-1} c_0^{-\frac{1}{2}}, \epsilon^{-\frac{1}{2}}\}|||v_h|||_T. \quad (4.1.5)$$

On simplification, inequality (4.1.5) reduces to

$$||\nabla v_h||_T \lesssim \min\{(h_{min}^T)^{-1} c_0^{-\frac{1}{2}}, \epsilon^{-\frac{1}{2}}\}|||v_h|||_T = (h_{min}^T)^{-1}\alpha_T|||\, v_h|||_T \; for \;\; v_h \in V_0^h.$$

(Using Eq.(3.1))

$B_{\rho,\sigma}(v - v_h, I_c w) =< \epsilon\nabla v.\nabla I_c w > + < \boldsymbol{b}.\nabla v, I_c w > + < cv, I_c w >$ -{$< \epsilon\nabla v_h, \nabla I_c w>$ + $<\boldsymbol{b}.\nabla v_h, I_c w > + < cv_h, I_c w > + \sum_T \rho_T\,(R_T, \boldsymbol{b}.\nabla I_c w) + \sum_T \sigma_T\,(R_T, \boldsymbol{b_h}.\nabla I_c w)\}$.

Now based on the Galerkin orthogonal property and standard scaling results, the above equation can be rewritten as

$$B_{\rho,\sigma}(v - v_h, I_c w) = -\sum_T \rho_T\,(R_T, \boldsymbol{b}.\nabla I_c w) - \sum_T \sigma_T\,(R_T, \boldsymbol{b_h}.\nabla I_c w)$$

$$\leq \sum_{T\in\Gamma_h} \rho_T ||R_T\,||_T ||\boldsymbol{b}||_{\infty,T} ||\nabla I_c w||_T \quad + \sum_{T\in\Gamma_h} \sigma_T ||R_T\,||_T ||\boldsymbol{b_h}||_{\infty,T} ||\nabla I_c w||_T$$

$$\leq \sum_{T\in\Gamma_h} \rho_T ||R_T\,||_T ||\boldsymbol{b}||_{\infty,T} (h_{min}^T)^{-1}\alpha_T |||I_c w|||_T$$

$$+ \sum_{T\in\Gamma_h} \sigma_T ||R_T\,||_T ||\boldsymbol{b_h}||_{\infty,T}\, (h_{min}^T)^{-1}\alpha_T \Big|\Big|I_c w\Big|\Big|_T.$$

Using Lemma 1, we get

$$\Big|\Big|I_c w\Big|\Big| \leq M_1(w, \Gamma_h).\Big|\Big|w\Big|\Big| \;\forall\;\; w \in H_0^1(\Gamma).$$

Thus, we have

$$B_{\rho,\sigma}(v - v_h, I_c w) \leq (\sum_{T\in\Gamma_h} \rho_T^2\, ||R_T||^2 ||\boldsymbol{b}||_{\infty,T}^2 (h_{min}^T)^{-2}\alpha_T^2)^{\frac{1}{2}}. M_1(w, \Gamma_h).\Big|\Big|w\Big|\Big|$$

$$+(\textstyle\sum_{T\in\Gamma_h} \sigma_T^2\, ||R_T||^2 ||\boldsymbol{b_h}||_{\infty,T}^2 (h_{min}^T)^{-2}\alpha_T^2)^{\frac{1}{2}}. M_1(w, \Gamma_h).\Big|\Big|w\Big|\Big|.$$

It may be noted that the effect of nonlinear term $\boldsymbol{b}_h$ in the $L_\infty$ norm will be bounded by that of the term $||\boldsymbol{b}||_\infty$ as shown below i.e.

$$\boldsymbol{b_h} = \frac{(\boldsymbol{b}.\nabla v_h)\nabla v_h}{|\nabla v_h|^{\mathbf{2}}}, \quad |\nabla v_h| \neq 0$$

$$\boldsymbol{b_h} \leq \frac{\big||\boldsymbol{b}|\big|\big|\nabla v_h\big|\nabla v_h}{|\nabla v_h|^{\mathbf{2}}} \quad \{Using\ Cauchy - Schwarz\ inequality\}$$

$$||\boldsymbol{b_h}|| \leq \frac{||\boldsymbol{b}|| ||\nabla \boldsymbol{v_h}|| ||\nabla \boldsymbol{v_h}||}{|\nabla v_h|^2}$$

$$\lesssim ||\boldsymbol{b}||$$

From relation (2.2.5), we get

$$\rho_T \lesssim h^T_{min}/||\boldsymbol{b}||_{\infty,T} \quad \forall\, T \in \Gamma_h,$$

$$\sigma_T \lesssim h^T_{min}/||\boldsymbol{b_h}||_{\infty,T} \quad \forall\, T \in \Gamma_h.$$

Therefore,

$$B_{\rho,\sigma}(v - v_h, I_c w) \lesssim [(\sum_{T\in\Gamma_h} \alpha_T^2 ||R_T||_T^2)^{\frac{1}{2}} + (\sum_{T\in\Gamma_h} \alpha_T^2 ||R_T||_T^2)^{\frac{1}{2}}].M_I(w, \Gamma_h).\left|||w|||\right|$$

$$\lesssim 2(\sum_{T\in\Gamma_h} \alpha_T^2 ||R_T||_T^2)^{\frac{1}{2}}.M_I(w, \Gamma_h).\left|||w|||\right|.$$

**Theorem: (Residual error estimation)** Let $v \in H_0^1(\Gamma)$ be the exact solution and $v_h \in V_0^h$ be the Hughes stabilized SUPG finite element solution of (1.1)-(1.3). Then the error in energy norm is bounded above globally by

$$\left|||v - v_h|||\right| \leq M_I(v - v_h, \Gamma_h).[\mathit{\Pi} + \pounds].$$

**Proof:** We know that $\boldsymbol{B}_{\rho,\sigma}(v, v) \geq |||v|||^2 \quad \forall v \in H_0^1(\Gamma)$. Using this result, we get

$$\left|||v - v_h|||\right| \leq \frac{B_{\rho,\sigma}(\,v-v_h \quad )}{\left|||v-v_h|||\right|}, \tag{4.1.6}$$

where $w = v - v_h$. Putting $a_T = 3\alpha_T$ and using Lemma 2 and Lemma 3 in Eq.(4.1.2),

the above equation reduces to

$$|||v - v_h||| \leq [(\sum_{T\in\Gamma_h} \alpha_T^2 ||R_T\,||_T^2)^{\frac{1}{2}} + (\sum_{E\subset\Gamma\backslash\partial\Gamma_D} \epsilon^{-\frac{1}{2}} \alpha_E ||R_E\,||_E^2)^{1/2}].M_I(w, \Gamma_h). \tag{4.1.7}$$

Using triangle inequalities

$$||R_T||_T^2 \leq ||r_T - R_T||_T^2 + ||r_T||_T^2\,,$$

$$||R_E||_T^2 \leq ||r_E - R_E||_E^2 + ||r_E||_E^2\,,$$

Substituting these inequalities in Eq.(4.1.7), we get

$$\left|||v - v_h|||\right| \leq M_I(v - v_h, \Gamma_h).[\mathit{\Pi} + \pounds],$$

where $\mathit{\Pi}^2$ and $\pounds^2$ are defined in Eq.(4.1.1).

## 5 Conclusion

In the presented work, reliable a posteriori error estimates have been proposed for singularly perturbed convection-diffusion problem in three-dimensions. Hughes stabilization technique under

SUPG finite element framework has been proposed to approximate the solution of singularly perturbed problems. Anisotropic meshes have been considered for the domain discretization. At the end, residual based error estimates in energy norm on anisotropic meshes have been developed.

## References

[1] R. Branco and F.V. Antunes, Finite element modelling and analysis of crack shape evolution in mode-I fatigue Middle Cracked Tension specimens, Eng. Fract. Mech. 75:3020-3037, 2008.

[2] A.N. Brooks and T.J.R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Comput. Methods Appl. Mech. Engrg. 32:199-259, 1982.

[3] C. Carstensen, R. Lazarov and S. Tomov, Explicit and averaging a posteriori error estimates for adaptive finite volume methods, SIAM J. Numer. Anal. 42(6):2496-2521, 2006.

[4] P. Cl$e'$ment, Approximation by finite element functions using local regularization, RAIRO Anal. Numer. 2:77-84, 1975.

[5] A. Ern, A.F. Stephansen and M. Vohral$\acute{i}$k, Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection-diffusion-reaction problems, J. Comput. Appl. Math. 234:114-130, 2010.

[6] T.J.R. Hughes, M. Mallet, Akira and Mizukami, A new finite element formulation for computational fluid dynamics: II. Beyond SUPG, Comput. Methods Appl. Mech. Engrg. 54:341-355, 1986.

[7] V. John and K. Knobloch, On spurious oscillations at layers diminishing (SOLD) methods for convection-diffusion equations: Part I-A review, Comput. Methods Appl. Mech. Engrg. 196:2197-2215, 2007.

[8] G. Kunert, An a posteriori residual error estimator for the finite element method on anisotropic tetrahedral meshes, Numer. Math. 86:471-490, 2000.

[9] G. Kunert, Robust a posteriori error estimation for a singularly perturbed reaction-diffusion equation on anisotropic tetrahedral meshes, Adv. Comp. Math. 15:237-259, 2001.

[10] R. Lazarov and S. Tomov, A posteriori error estimates for finite volume element approximations of convection-diffusion-reaction equations, Computat Geosci. 6(3):483-503, 2002.

[11] R.K. Mohanty and N. Setia, A new high order compact off-step discretization for the system of 3D quasi-linear elliptic partial differential equations, Appl. Math. Model. 37:6870-6883, 2013.

[12] R.K. Mohanty, W. Dai and F. Han, Compact operator method of accuracy two in time and four in space for the numerical solution of coupled viscous Burger's equations, Appl. Math. Comput. 256:381-393, 2015.

[13] A. Mola, L. Heltai and A.D. Simone, A stable and adaptive semi-Lagrangian potential model for unsteady and nonlinear ship-wave interactions, Eng. Anal. Bound. Elem. 37:128-143, 2013.

[14] E.B. Postnikov and O.V. Titkova, A correspondence between the models of Hodgkin-Huxley and FitzHugh-Nagumo revisited, Eur. Phys. J. Plus. 131:1-9, 2016.

[15] H.G. Roos, M. Stynes and L. Tobiska, Numerical methods for Singularly Perturbed Differential Equations, Springer, Berlin, 1996.

[16] V. Sangwan, B.V.R. Kumar, S.V.S.S.N.V.G.K. Murthy and M. Nigam, Three-step Taylor Galerkin method for singularly perturbed generalized Hodgkin-Huxley equation, Int. J. Model. Simul. Sci. Comput. 1:257-276, 2010.

[17] S. Zhai, X. Feng and Y. He, An unconditionally stable compact ADI method for three-dimensional time-fractional convection-difusion equation. J. Comput. Phys. 269:138-155, 2014.