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Locally Harmonious Chromatic Number of Certain Tree-Structured Networks

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Abstract:

Graph coloring is one of the oldest and best-known problems of graph theory. The locally harmoniouscoloring of G is a proper vertex coloring in which adjacent edges receive different color pairs [3]. In another way, all the vertices in $N[v]$ receive different colors for all v in G. The minimum number of colors required to obtain a locally harmonious coloring of a graph G is called the locally harmonious chromatic number of G and is denoted by $h_1(G)$. In this paper, we investigate the locally harmonious chromatic number of slim tree, hypertree, shuffle hypertree and l-complete binary tree.

Keywords: coloring; locally harmonious coloring; slim tree; hypertree; shuffle hypertree; l- complete binary tree.

1. Introduction

The introduction of a method for very high scale integration, which will allow the manufacture of a single silicon chip with one million active devices in five years, has also changed the limitson a multiprocessor interconnection network. With advancements in technology, active devices are becoming smaller, faster, and consuming less power. The speed gap between signals thatare wholly internal to the device and those that must transit through package pins, on the other hand, is expanding.

Consequently, a good VLSI building block could be a single integrated circuit standalone device [10]. Later, a wide range of networks, including lattices or trees, are considered because these structures seem to be appealing because the future computer systems are easily expandable. It’s not shocking, then, that tree-structured networks have recently sparked a lot of interest. Trees are a specific type of graph that are worth studying if only because they are used in so many different domains of practise and science. These graphs that do not have any cycles. They are naturally expandable, and even unbalanced trees retain the majority of the features that make them interesting.

As a result, the package periphery is still another physical hindrance, and connectivity topologies with few ports per node should be considered. A binary tree structure with only three ports per node appears to be particularly suitable if the number of ports per node must be limited to a minimum. Additional links, on the other hand, are required to provide redundant paths, which are the foundation for a fault-tolerant message routing system. These linkages can also be used to reduce the average distance between nodes and ensure that message density is consistent across allroutes. Because of their simple routing algorithms, the binary trees which contain half-ring and full ring emerged as potential competitors after an intense search for the ideal placement of these additional linkages.

Trees are essential for understanding the structure of graphs and have a far-reaching range of applications, which includes data processing. Additional properties such as roots and vertex-orderings are common in tree architectures. Trees play a crucial role in the structuring and scrutinizing of connected networks, as well as in the structural comprehension of graphs and the algorithms of information processing. Trees, in fact, form the foundation of optimally connected networks. One of the most important tasks in information management is determining how to store data in a space-efficient manner that simultaneously allows for quick retrieval and change. Tree-based systems are frequently the most effective means of reconciling these competing objectives.

The locally harmonious coloring [3] of a graph G is a proper vertex coloring in which every color-pair is present only once for every edge within distance one from any vertex v in $V(G)$. It leads to the constraint that all vertices in the closed neighborhood $N[v]$ receive different colors for every vertex v in G. The minimum number of colors required to obtain a locally harmonious coloring of a graph G is called the locally harmonious chromatic number of G and is denoted by $h_1(G)$. Given a graph G, the locally harmonious coloring problem is to determine $h_1(G)$. In 2015, W. Gao [3] proposed three algorithms to estimate $h_1(G)$. These algorithms have high impact ontime complexity due to an exhaustive search on vertices and branching rules. The concept of locally harmonious coloring is derived from d harmonious coloring [11], as a proper vertex coloring such that every color pair is present at most once for every edge within distance d; during the conferenceheld in Taiwan. The locally harmonious coloring is exactly same as d harmonious coloring when$d$=1.

2. lh-Coloring of Certain Tree-Derived Networks

In this section, we present the *lh*-Coloring Lemma in which we obtain the lower bound of locally harmonious coloring for Hypertree, Shuffle hypertree, *l*-Complete binary tree.

***lh*-Coloring Lemma.** [6] *Let G be a connected graph on n vertices and k be the number of vertices has* $\Delta(G)$ *number of common neighbors. Then* $h_1(G) \geq \Delta(G) + k$.

Theorem 2.1. [7] *Let G be a simple connected graph and H be any induced subgraph of G, then* $h_1(G) \geq h_1(H)$.

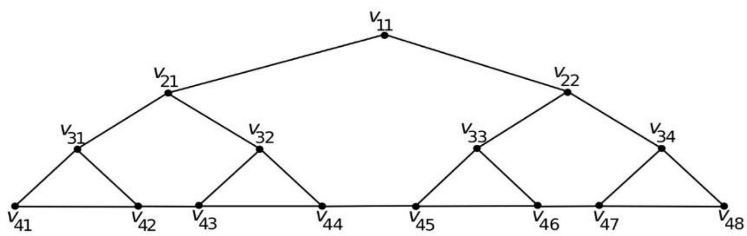

Figure 1: Vertex representation of $ST(4)$

3. Slim Tree

The n^{th} slim tree $ST(n)$, $n \geq 2$ is recursively defined as follows and is denoted by $ST(n) = (V, E, u, l, r)$, where V is the node set, E is the edge set and u, l, r are vertices addressed as root node, left node and right node respectively.

1. $ST(2)$ is the complete graph K_3 with its nodes labeled as u, l and r.
2. The s^{th} slim tree $ST(s)$, with $s \geq 3$, is composed of a root node u and two disjoint copies of $(s-1)^{th}$ slim trees as the left subtree and right sub-tree, denoted by $ST^l(n-1) = (V_1, E_1, u_1, l_1, r_1)$ and $ST^r(n-1) = (V_2, E_2, u_2, l_2, r_2)$, respectively. To be specific, $ST(n) = (V, E, u, l, r)$ is given by $V = V_1 \cup V_2 \cup \{u\}$, $E = E_1 \cup E_2 \cup \{(u, u_1), (u, u_2), (r_1, l_2)\}$, $l = l_1$, $r = r_2$.

To simplify our coloring scheme, here the vertices of the slim tree $ST(n)$ are labeled as v_{ij}, $1 \leq i \leq n$, $1 \leq j \leq 2^{i-1}$, where i represents the level and j represents the position of the vertices at a particular level as shown in Figure 1.

In this section we shall prove that $ST(n), n \geq 2$ admits locally harmonious coloring.

Theorem 3.1. *The locally harmonious chromatic number of slim tree network $ST(n)$, $n \geq 2$, is given by $h_1(ST(n)) = 5$.*

Proof. Since $ST(2)$ is the complete graph K_3, By *lh*-Coloring lemma, $h_1(ST(2)) = \Delta(ST(2)) + 1$. For $n > 2$, we now show that $ST(n)$ cannot be colored with $\Delta(ST(n)) + 1$ colors with the illustration by considering an induced subgraph $ST(3)$.

Without loss of generality, let us color the vertex v_{21} in $ST(3)$ with the color 1 and denoted by $c(v_{21}) = 1$. Then its neighboring vertices v_{11}, v_{31} and v_{32} can be colored from the set {2, 3, 4} in some order. Again, without loss of generality, assume $c(v_{11}) = 2$, $c(v_{31}) = 3$ and $c(v_{32}) = 4$. It follows, the vertices v_{33} and v_{34} can be colored in the following ways:

1. $c(v_{33}) = 2$
2. $c(v_{34}) = 1$ or 3.

Now, whatever may be the case, since $c(v_{33}) = 2$, it is easy to see that the vertex v_{22} is now adjacent with two vertices colored with 2. Therefore, $h_1(ST(3)) > 4$ as shown in Figure 2. By Theorem 2.1, we have $h_1(ST(n)) \geq h_1(ST(3)) \geq 5$.

We now color the vertices of ST (n) level-wise with the color order from left to right repeatedly using the colors from the set {1, 2, 3, 4, 5} as described in Table 1.

Table 1: Locally harmonious coloring order of $ST(n)$

S. No.	$(n - i + 1) \ (mod\ 4)$	Color order of level i
1	1	< 1, 2, 3, 4, 5 >
2	2	< 4, 1, 3, 5, 2 >
3	3	< 5, 4, 3, 2, 1 >
4	0	< 2, 5, 3, 1, 4 >

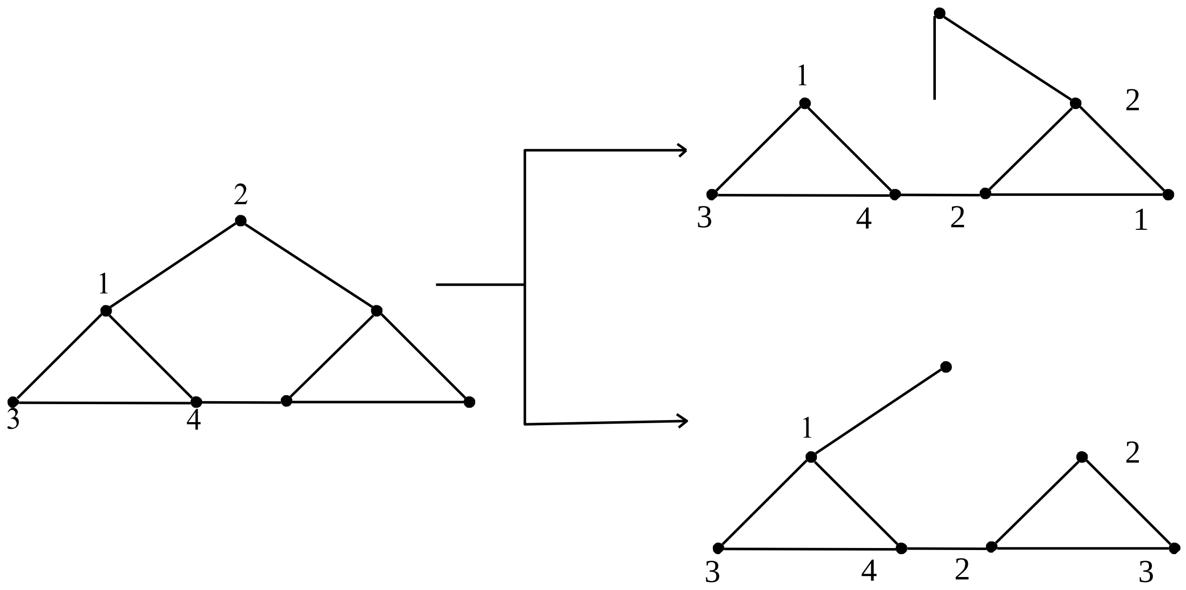

Figure 2: Illustration for $h_1(ST(3)) > 4$

From the construction of $ST(n)$, it is clear that all levels are independent except the level n. Three adjacent vertices at level $i = n$ have consecutive coloring say $k-1, k, k+1$.

Table 1 shows that for the vertex-colored k at any level i, $1 \leq i \leq n$, its neighboring ver- tices receives the distinct colors from the color-set $(k + 2)$, $(k + 3)$, $(k + 4)$, $(k + 6)$, $(k + 9)$ with the numbers taken modulo 5.

Therefore, no two incident edges share the same color-pair. By *lh*-Coloring Lemma, $h_1(ST(n))$ =5. See Figure 3.

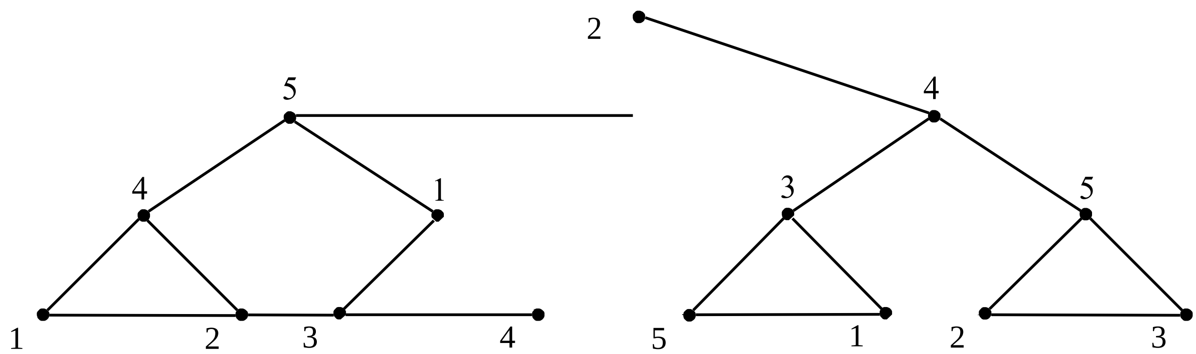

Figure 3: $h_1(ST(4)) = 5$

4. Hypertree

Definition 4.1. A complete binary tree Tn is the basic scaffolding of a hypertree. Here the vertices of the tree are labeled as v_{ij}, $0 \leq i \leq n$, $1 \leq j \leq 2n$, where i represents the level and j represents the position of vertices of the tree at any level i. The root node receives the label v_{01} and the root is said to be at level 0.

Horizontal links are added additionally in a hypertree, and two vertices of the tree are united in their own level i if their j's difference is 2^{i-1}. We denote an n-level hypertree as $HT(n)$. It has $2^{n+1}-$vertices and $3(2^n - 1)$ edges. See Figure 4.

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For $n \geq 2$, we now propose a locally harmonious coloring algorithm of $HT(n)$. For $2 \leq i \leq n$, the color order given in the algorithm implies to color the vertices v_{ij} from the left to the right repeatedly as construed in Table 2.

Algorithm 1: HYPERTREE, $HT(n)$

Initialize $n \geq 2$
$v_{01} \leftarrow 1$
$v_{11} \leftarrow 3$
$v_{12} \leftarrow 4$
$i \leftarrow 2$
for $j \leftarrow 1$ **to** 2^i **do**
  if $i \equiv 0(mod\ 3)$ **then**
    $v_{ij} \leftarrow \langle\langle 1,2\rangle_{2^{i-2}}, \langle 2,1\rangle_{2^{i-2}}\rangle$
  end if
  if $i \equiv 1(mod\ 3)$ **then**
    $v_{ij} \leftarrow \langle\langle 3,4\rangle_{2^{i-2}}, \langle 4,3\rangle_{2^{i-2}}\rangle$
  end if
  if $i \equiv 2(mod\ 3)$ **then**
    $v_{ij} \leftarrow \langle\langle 5,6\rangle_{2^{i-2}}, \langle 6,5\rangle_{2^{i-2}}\rangle$
  end if
  $i \leftarrow i+1$
end for

Algorithm 1 is illustrated in Figure 5.

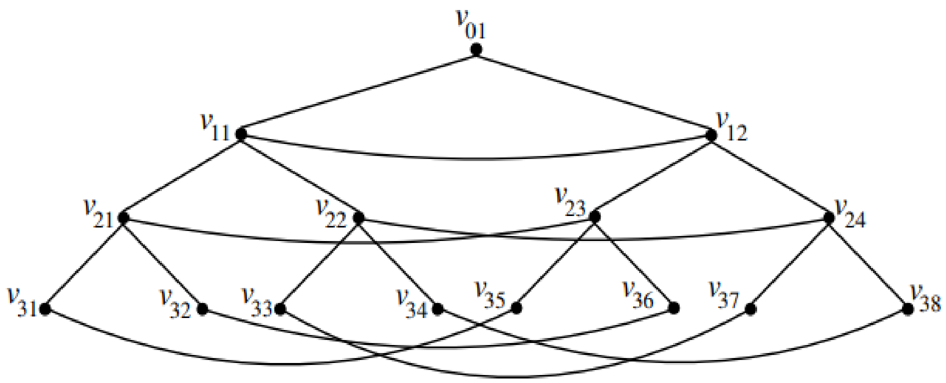

Figure 4: Vertex labeling of $HT(3)$

Table 2: Locally harmonious coloring order of $HT(n), n \geq 2$

S. No.	$i\ (mod\ 3)$	Color order of level i
1	0	$\langle\langle 1,2\rangle_{2^{i-2}}, \langle 2,1\rangle_{2^{i-2}}\rangle$
2	1	$\langle\langle 3,4\rangle_{2^{i-2}}, \langle 4,3\rangle_{2^{i-2}}\rangle$
3	2	$\langle\langle 5,6\rangle_{2^{i-2}}, \langle 6,5\rangle_{2^{i-2}}\rangle$

Proof of Correctness (Algorithm 1): It is to be noted that any vertex $v \in V(HT(n))$ at a particular level i have its neighborhood vertices $N(v)$ in the level $i-1, i, i+1$. Two vertices adjacent ata particular level 'i' have consecutive coloring say k and $k+1$ or vice versa. From the proposed algorithm, vertices of any three consecutive levels in $HT(n)$ have colored using three distinct color-pairs from the color set $\{1, 2, 3, 4, 5, 6\}$. Therefore, for any vertex $v \in V(HT(n))$, the vertices of $N[v]$ receives distinct colors. Thus, the coloring is locally harmonious coloring.

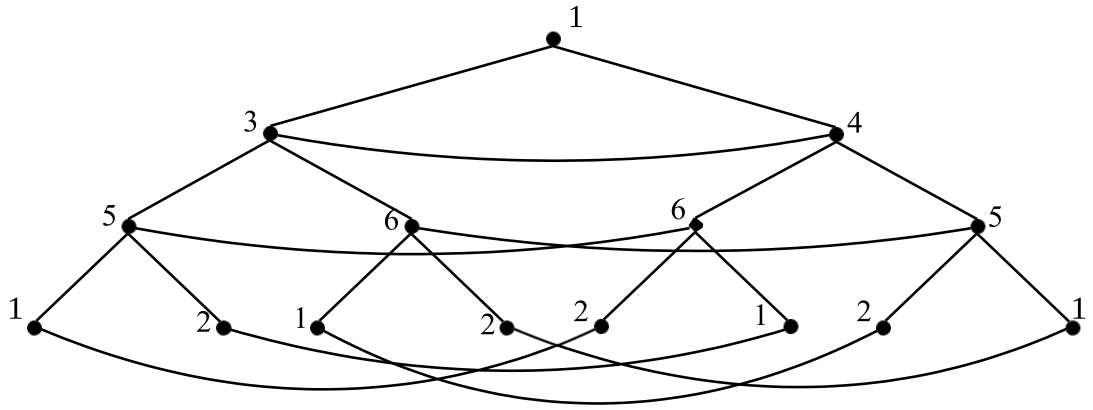

Figure 5: Locally harmonious coloring of $HT(3)$

Theorem 4.2. *Let G be the hypertree $HT(n), n \geq 2$. Then G admits locally harmonious coloringwith the chromatic number $h_1(G) = 6$.*

Proof. From the proposed algorithm, it is clear that no two adjacent edges share the same color-pairand the number of colors used to color G is 6. Hence, by *lh*-Coloring Lemma, $h_1(G) = 6$.

5. Shuffle Hypertree

Definition 5.1. Shuffle hypertree SHT (n) [8] have been constructed by doing certain modifications in HT (n), which is stated as follows: The deletion of hyper edges in HT (n) and replacing with the new hyper edges $(2^i + 2k - 1, 2^i + 2k)$ and $(2^i, 2^{i+1} - 1)$ where $1 \leq i \leq n$, $1 \leq k \leq 2^{i-1} - 1$, resultinginto a new graph called shuffle hyper tree. It is denoted by SHT (n). It is made up of $2^{n+1} - 1$, $3(2^n - 1)$ vertices and edges respectively. See Figure 6.

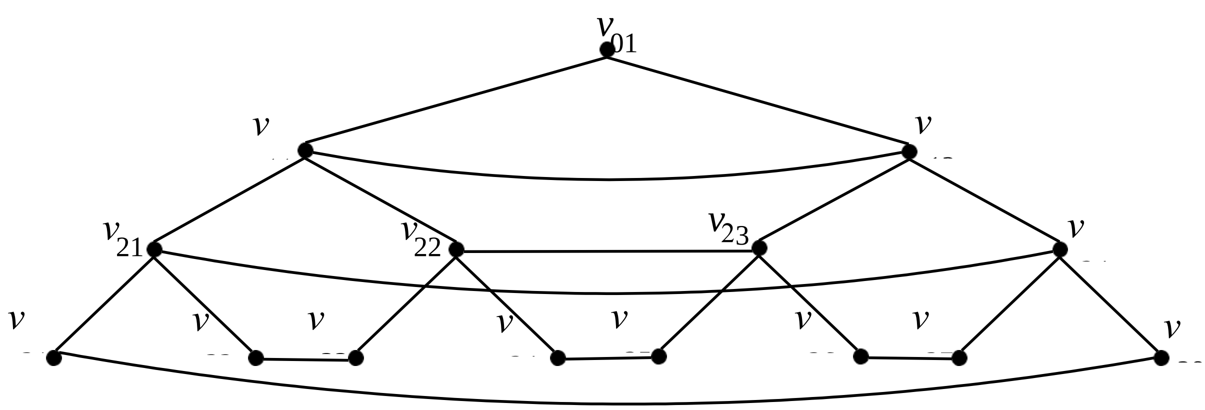

Figure 6: Vertex representation of $SHT(3)$

For $n \geq 2$, we now propose a locally harmonious coloring algorithm of $SHT(n)$. Color order given in an algorithm have to be color the vertices v_{ij} from the left to the right repeatedly in the particular level i using the color in the order as construed in Table 3.

Table 3: Locally harmonious coloring order of $SHT(n), n \geq 2$

S. No.	$i \pmod 3$	Color order of level i	
		$n = 2$	$n > 2$
1	0	$\langle 1 \rangle_{2^i}$	$\langle 1, 2 \rangle_{2^{i-1}}$
2	1	$\langle 2, 3 \rangle_{2^{i-1}}$	$\langle 3, 4 \rangle_{2^{i-1}}$
3	2	$\langle 4, 5 \rangle_{2^{i-1}}$	$\langle 5, 6 \rangle_{2^{i-1}}$

Theorem 5.2. *The shuffle hypertree network $SHT(n), n \geq 2$ admits locally harmonious coloring with the chromatic number*

$$h_1(SHT(n)) = \begin{cases} 5 & \text{if } n = 2 \\ 6 & \text{if } n > 2. \end{cases}$$

Proof. From the proposed algorithm, it is clear that the number of colors mandatory for solving the locally harmonious coloring problem of $SHT(2)$ is 5. Hence, by *lh*-Coloring Lemma, $h_1(SHT(2)) = 5$. Similarly, for the case $n > 2$, we arrive that $h_1(SHT(n)) = 6$. See Figure 7.

Algorithm 2: SHUFFLE HYPERTREE, $SHT(n)$

```
Initialize n ≥ 2
i ← 1
for j ← 1 to 2^i do
    if i ≡ 0(mod 3) then
        if n = 2 then
            v_ij ← ⟨1⟩_{2^i}
        else
            v_ij ← ⟨1, 2⟩_{2^{i-1}}
        end if
    end if
    if i ≡ 1(mod 3) then
        if n = 2 then
            v_ij ← ⟨2, 3⟩_{2^{i-1}}
        else
            v_ij ← ⟨3, 4⟩_{2^{i-1}}
        end if
    end if
    if i ≡ 2(mod 3) then
        if n = 2 then
            v_ij ← ⟨4, 5⟩_{2^{i-1}}
        else
            v_ij ← ⟨5, 6⟩_{2^{i-1}}
        end if
    end if
    i ← i + 1
end for
```

Algorithm 10 is illustrated in Figure 7.

Proof of Correctness (Algorithm 10): Two vertices adjacent at a particular level '*i*' have consecutive coloring say k and $k + 1$, vise versa. From the proposed algorithm, it is clear that the vertices of any three consecutive levels in $SHT(n)$ have colored using three distinct color-pairs from the colorset {1, 2, 3, 4, 5, 6}. Therefore, every color pair of colors is present only once for each edge within distance one of any vertex $v \in SHT(n)$. Thus, the coloring is locally harmonious coloring.

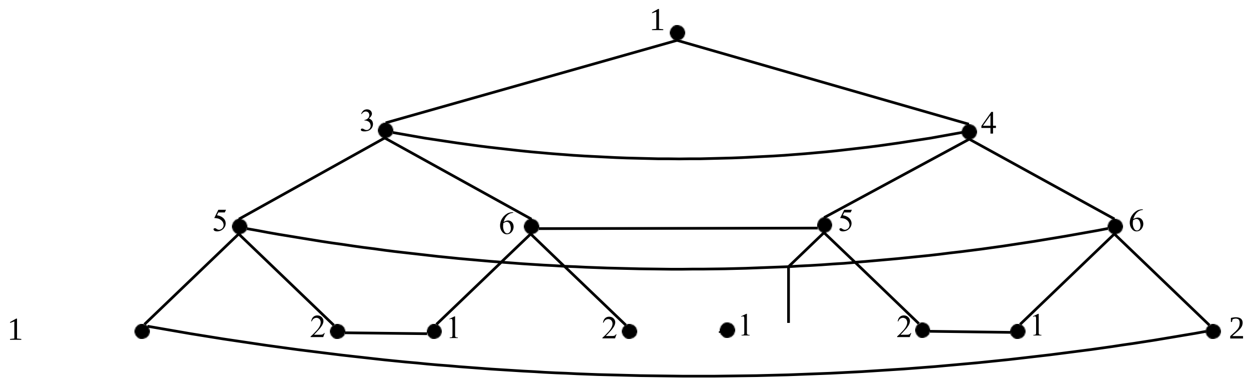

Figure 7: $h_1(SHT(3)) = 6$

6. *l*-Complete Binary Tree

Definition 6.1. A graph made up of 2-copies of the complete binary tree $T_n, n \geq 1$, say T_1, T_2 by merging all of the vertices in the last level of T_1 with the associated vertex of T_2 is called the *l*-complete binary tree, denoted by l-T_n. The vertices of l-T_n denoted as $v_{ij}, 0 \le i \le n, 1 \le j \le 2^i$ and $v_{ij}{}', n+1 \le i \le 2n, 1 \le j \le 2^{(n-i(mod\, n))mod\, n}$ where i representing a level and j represents the position of a vertex at any particular level i.

Formally, we define the vertex set as shown in Figure 8. The *l*-complete binary tree l-T_n, $n \geq 1$, contains 2^{n-1} number of vertex disjoint copies of C_4, where C_4 is the cycle on four vertices and the number of vertices in l-T_n is $3.2^{n-1} - 2$, $n \geq 1$.

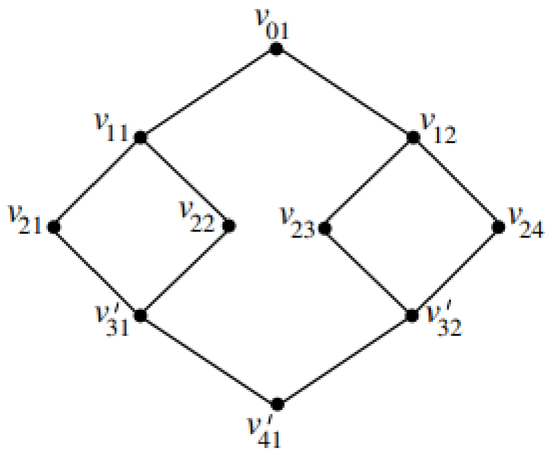

Figure 8: Vertex labeling of l-T_2

Theorem 6.2. *Let G be the l-complete binary tree l-*T_n*,* $n \geq 2$*. Then G admits locally harmoniouscoloring with the chromatic number* $h_1(G) = 6$*.*

Proof. W.l.g, let us color the vertex v_{01} at the level 0 of G with 1.

We shall show that $h_1(G) = 6$ by labeling the vertices v_{i1} and v'

Define a labeling function $l : v_1{}^i \to \{0, 1, 2\}, 1 \leq i \leq 2n$ such that

$$l\left(v_1{}^i\right) = (i + 1) mod\ 3$$

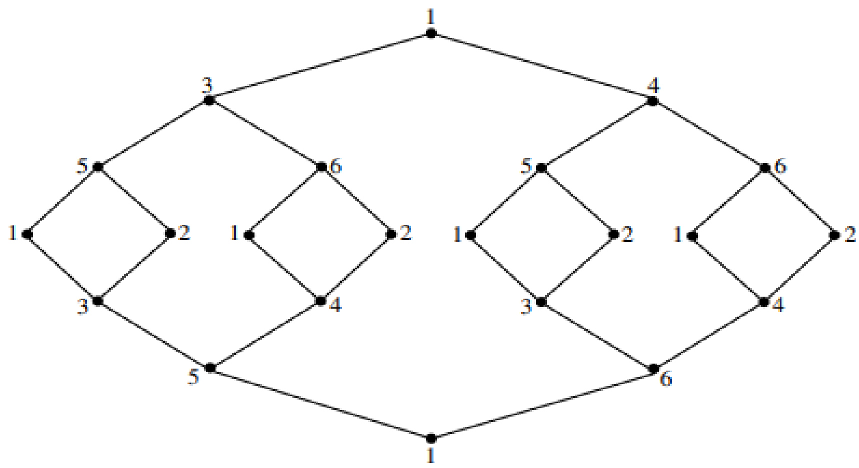

Figure 9: $h_1(l)$-$T_3 = 6$

Depending on the values the vertices of G can be colored level-wise using the color set $\{1, 2, 3, 4, 5, 6\}$ from the left to the right repeatedly as in the given order construed in Table 4.

The vertices in each level are independent and any vertex v_{ij} neighborhoods can be found only in the levels $i + 1$ and $i - 1$. Since the vertices of any three consecutive levels are colored with 3-different pair of colors $\{1, 2\}$, $\{3, 4\}$ and $\{5, 6\}$ respectively. By lh-Coloring Lemma, $h_1(G) = 6$. See Figure 9.

Table 4: The locally harmonious coloring order for the levels of G

S. No.	$l(v_1^i)$	Color order of level i from left to right		
		$1 \leq i \leq n$	$n+1 \leq i \leq 2n-1$	$i = 2n$
1	1	$\langle 1,2\rangle_{2^{i-1}}$	$\langle 1,2\rangle_{2^{(n-i(mod n))(mod n)-1}}$	1
2	2	$\langle 3,4\rangle_{2^{i-1}}$	$\langle 3,4\rangle_{2^{(n-i(mod n))(mod n)-1}}$	5
3	0	$\langle 5,6\rangle_{2^{i-1}}$	$\langle 5,6\rangle_{2^{(n-i(mod n))(mod n)-1}}$	3

7. Concluding Remarks

In this paper, we obtained the locally harmonious chromatic number for certain tree-derived networks which includes slim tree, hypertree, shuffle hypertree and l-complete binary tree. An elegant way of coloring algorithms has been proposed in finding the bounds of these graphs. Locally harmonious chromatic number of other interesting tree derived networks like sierpinski graph, sierpinskigasket, Christmas tree, X-tree are open and under study.

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