

Pebbling on Crisscross Sequence of m Complete Graphs

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Abstract:

This paper investigates the pebbling number, the two pebbling property, the t -pebbling number, and the $2t$ -pebbling property of a crisscross sequence comprised of m complete graphs. By analyzing these pebbling properties, we aim to gain insights into the efficiency and feasibility of pebbling operations within this specific graph structure.

Keywords: Graph pebbling, Graph theory, crisscross sequence of m complete graphs, Pebbling properties, Graph Optimization

Introduction

Graph pebbling is a mathematical concept that involves manipulating configurations of pebbles on the vertices of graph in order to achieve a specific target vertex with the desired amount of pebbles. Initially, Lagarias and Saks introduced graph pebbling while attempting to answer a number theoretic question posed by Erdos and Lemke concerning zero-sum sequences of finite group. The graph pebbling concept was formally introduced by Chung , who defined the pebbling number $f(G)$ for connected graph G . [11][12]Since then, the field of graph pebbling has become highly active, with numerous open problems and conjectures awaiting resolution. In the context of graph pebbling , *pebbling move* involves taking two pebbles from one vertex and placing one of them on a neighboring vertex. The second pebble is disregarded. If there are w pebbles distributed among the vertices of graph G , the distribution is considered *solvable*, if it is possible to manipulate the pebbles through a series of moves so that any given vertex v ends up with at least one pebble. On the other hand, if the distribution cannot be solved, it is referred to as *unsolvable*. The *pebbling number* , denoted as $f(G)$, - represents the smallest value of m for which all initial distributions of m pebbles on the graph G can be solved. The t -*pebbling number* , $f_t(G)$, of a connected graph G , represents the minimum positive integer such that given distribution of $f_t(G)$ pebbles, it is possible to move t pebbles to the chosen target vertex by performing series of pebbling moves. The connected graph G said to possess the *2-pebbling property* if for any distribution of pebbles in G where the number of pebbles exceeds $2f(G) - q$, where q , the number of vertices with at least one pebble, it is feasible, through the execution of

pebbling moves, to ensure the presence of two pebbles at the given vertex within G . [15]A graph G considered to possess the $2t$ –pebbling property if for any distribution of pebbles in G where the number of pebbles exceeds $2f_t(G) - q$, through the execution of pebbling moves, it is possible to move $2t$ pebbles to any arbitrary vertex in G . In section 2, we calculate the pebbling number and t –pebbling number of crisscross sequence of m complete graph and in section 3 and section 4, we demonstrate the graph $C_m(K_n)$ exhibits both 2-pebbling property and $2t$ –pebbling property[12].

Pebbling on crisscross sequence of m complete graphs

In this section, we aim to determine the pebbling number and the t –pebbling number of crisscross chain graph consisting m complete graphs. However, before getting into the calculations, it is crucial to comprehend the structure of crisscross chain graph of m complete graph. To aid in this understanding, we introduce $C_m(K_n)$ with the help of zig-zag sequence of n cycles which has been already studied in [8],[9],[10].

Definition 1. [14][15] *The zig-zag chain graph of even cycles denoted by $ZZ_n(C_{2k})$, is a graph which consists of zig-zag sequence of n even cycles, C_{2k} with $k \geq 3$. We have the following vertex set and edge set of $ZZ_n(C_{2k})$ for n even as follows.*

$$V(ZZ_n(C_{2k})) = \{a_i, b_i : 1 \leq i \leq n(k - 1)\} \cup \{x, y\} \text{ and}$$

$$E(ZZ_n(C_{2k})) = \{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n(k - 1) - 1\} \cup \{a_{(k+1)i-1} b_{(k+1)i-2}, a_{(k+1)j} b_{(k+1)j+1} : 1 \leq i \leq \frac{n}{2},$$

$$1 \leq j \leq (\frac{n}{2} - 1)\} \cup \{x a_1, x b_1, y a_{n(k-1)}, y b_{n(k-1)}\}$$

For n odd, we have the following vertex set and edge set.

$$V(ZZ_n(C_{2k})) = \{a_i, b_i : 1 \leq i \leq n(k - 1)\} \cup \{x, y\} \text{ and}$$

$$E(ZZ_n(C_{2k})) = \{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n(k - 1) - 1\} \cup$$

$$\{x a_1, x b_1, y a_{n(k-1)}, y b_{n(k-1)}\} \cup \{a_{(k+1)i-1} b_{(k+1)i-2}, a_{(k+1)j} b_{(k+1)j+1} : 1 \leq i, j \leq \frac{n-1}{2}\}.$$

The Figure 2.1(a) depicts the graph ZZ_3 for $k = 3$.

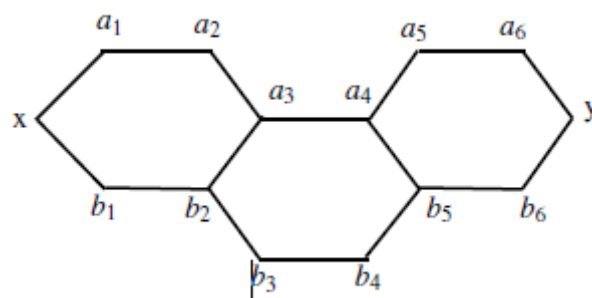


Figure 2.1(a)

Similarly, for the graph zig-zag chain graph of odd cycles, denoted by $ZZ_n(C_{2k+1})$ has defined in[4].

Definition 2.[5],[7],[9],[10]. *The crisscross sequence of m complete graphs is denoted by $C_m(K_n)$, is a graph which consists of crisscross sequence of m copies of K_n , with $n \geq 5$. We define the graph $C_m(K_n)$ as follows: The vertex set of $C_m(K_n)$ is $V(C_m(K_n)) = V(ZZ_m(C_n))$ and the edge set of $C_m(K_n)$ is the union of $E(ZZ_m(C_n))$ and each vertex $v \in C_i$ is adjacent to all the vertices of C_i . Here C_1, C_2, \dots, C_m is considered as m number of cells of $C_m(K_n)$. The figure depicts the the crisscross sequence of m copies of K_6 .*

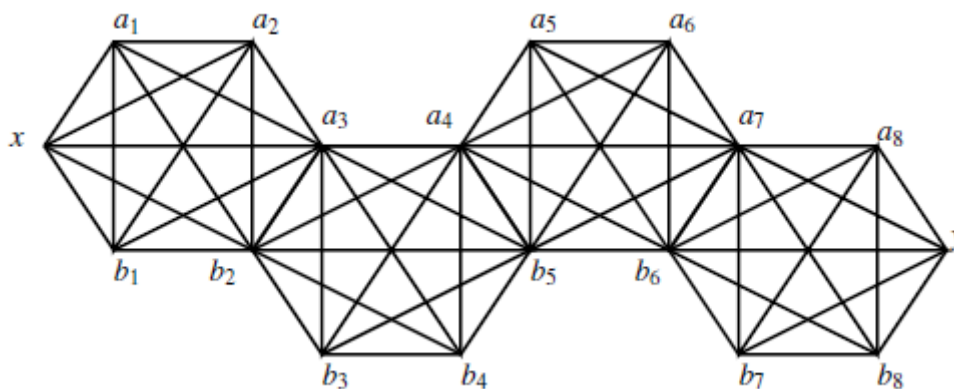


Figure 2.2(b)

The reader can easily view that the Figure 2.2(b) has 4 copies of K_6 . For $C_m(K_n)$ we have m copies of K_6 and we label each K_6 as C_1, C_2, \dots, C_m in order from left to right.

Pebbling number of crisscross sequence of m complete graphs

Theorem 1. *The pebbling number of the graph $C_2(K_n)$ is, $f(C_2(K_n)) = 2n - 2$.*

Proof. Put 3 pebbles on the vertex x and one pebble each on vertices a_i and b_i , excluding vertices a_k and b_{k-1} where $a_k, b_k \in V(C_1) \cap V(C_2)$. We are unable to transport a single pebble to the vertex y . So we have $f(C_2(K_n)) \geq 2n - 2$.

Think of a graph where the vertices are covered in at least $2n - 2$ pebbles. Suppose v is any target vertex. Let $p(v) = 0$. We consider the following scenarios:

Case (1) $v = a_k$ or $v = b_{k-1}$.

Take the supposition that $v = a_k$ without losing generality. Consider the case where $p(C_1) \geq n$ or $p(C_2) \geq n$. After that we may move a pebble to a_k . Considering the way $C_1 \cong C_2 \cong K_n$. Also $f(K_n) = n$. Assume that $p(C_1) \geq n - 1$ and $p(C_2) \geq n - 1$. If $p(V(C_1) - \{a_k\}) \geq n - 1$, at least two of the pebbles are present in one of the vertices of $V(C_2) - \{a_k, b_{k-1}\}$. Assume that $p(y) \geq 2$ to maintain the generality. Due to the fact that y is close to a_k , we can transfer a pebble there.

Case (2) $v \in V(C_2) - \{a_k\}$ or $v \in V(C_1) - \{b_{k-1}\}$.

Take $v = y$ and suppose that $v \in V(C_2) - \{a_k\}$. If $p(C_2) \geq n$, then we can reach the vertex y with one pebble. Given that $C_2 \cong K_n$. Thus we suppose that $p(C_2) < n$. Assume that $p(V(C_2) - \{a_k\}) \geq n - 1$. After that, we may move a pebble to the vertex y . Due to the fact that the graph induced by the vertices $V(C_2) - \{a_k\}$ is isomorphic to K_{n-1} . After that, we presumptively know that $p(V(C_2) - \{a_k\}) \leq n - 2$. We consider the following subcases:

Subcase (1a) $p(V(C_2) - \{a_k\}) = n - 2$.

Based on our assumptions, we obtain $p(V(C_1) - \{b_{k-1}\}) = n$. Then, at least any of the vertices belonging to $V(C_1) - \{b_{k-1}\}$ contain at least two pebbles. Assume that $p(x) \geq 2$ and we can move one pebble to b_{k-1} . This is because x is adjacent to b_{k-1} . The subgraph induced by the vertices $V(C_2) - \{a_k\}$ contains $n - 1$ pebbles. Therefore, the pebble can be moved to y . The subgraph induced by vertices $V(C_2) - \{a_k\}$ is isomorphic to K_{n-1} .

Subcase (1b) $p(V(C_2) - \{a_k\}) < n - 2$.

By assumption, $p(V(C_1) - \{b_{k-1}\}) \geq n + 1$ and any of the vertices in $V(C_1) - \{b_{k-1}\}$ has more than one pebble. Using exactly two pebbles, we can reach vertex a_k with one pebble. Thus, the number of pebbles retained in $V(C_1) - \{b_{k-1}\}$ was at least $n - 1$. We can then move an additional pebble to a_k . Therefore, one pebble can be moved to y from a_k .

Theorem 2. *The pebbling number of the graph $C_3(K_n)$ is $f(C_3(K_n)) = 3n - 2$.*

Proof. Put seven pebbles on vertex x and put one pebble on each a_i 's and b_i 's except $\{a_k, b_{k-1}, a_{2k-2}, b_{2k-1}\}$. Then, we cannot move the pebble to y . Thus, $f(C_3(K_n)) \geq 3n - 2$. Consider a graph with $3n - 2$ pebbles distributed at its vertices. Let v be a target vertex. Clearly $v \in C_i$, for any $i = 1, 2, 3$. We consider the following cases.

Case(1)

If $p(C_3) \geq n + 2$, then the two pebbles can be moved to a_{2k-2} and one pebble to v . $C_3 \cong K_n$ and the pebbling number of a complete graph K_n is n . Therefore, we assume $p(C_3) \leq n + 1$. This implies that at least $2n - 3$ pebbles were retained in $C_1 \cap C_2$. If $p(V(C_3) - \{a_{2k-2}, b_{2k-1}\}) \geq n$. Then, the two pebbles can be moved to y . We can reach vertex a_{k+1} using one pebble. This results in the subgraph $C_1 \cap C_2$ obtaining $2n - 2$ pebbles after moving one pebble to a_{k+1} . By Theorem 1 we move the pebble to v .

Case (2) $v \in C_1$ or $v \in C_3$.

Without loss of generality, assume that $v \in C_3$ and let us assume $v = y$. If $p(V(C_3) - \{a_{2k-2}, b_{2k-1}\}) \geq n - 2$, then we can move the pebble to y . The graph is induced by vertices $\langle V(C_3) - \{a_{2k-2}, b_{2k-1}\} \rangle \cong K_{n-2}$. Therefore, we assume $p(V(C_3) - \{a_{2k-2}, b_{2k-1}\}) \leq n - 3$. Thus, the number of pebbles retained in $V(C_1 \cap C_2)$ is at least $2n + 1$. We may take $p(V(C_2) - \{a_k, b_{k-1}\}) \geq n - 2$. We can then move the pebble to a_k . Additionally, the pebbles retained in C_1 are at least $n + 3$. In this case, we can move the two pebbles to a_k and move an additional pebble to a_{2k-2} . Thus, the pebble can be moved to y . Therefore, we take $p(C_2) \leq n - 4$, $p(a_{2k-2}) = 0$ and $p(b_{2k-1}) = 0$. This implies that the number of pebbles retained in $V(C_1)$ was at least $n + 5$. Suppose $p(a_k) \geq 2$ or $p(b_{k-1}) \geq 2$. Then, we can move one pebble to a_{2k-2} and use exactly two pebbles from a_k or b_{k-1} . In addition, $n + 3$ pebbles settled in C_1 . Because $C_1 \cong K_n$ and by using the pebbling number of complete graph we can move two pebbles to a_k and move an additional pebble to a_{2k-2} . We can then reach vertex y with one pebble. We assume that $p(a_k) \leq 1$ and $p(b_{k-1}) \leq 1$. If $p(a_k) = 1$, at least $n - 3$ pebbles are distributed on $n - 1$ vertices. Thus, using the pigeonhole principle, we conclude that at least one of those vertices must contain two pebbles, and we can then move an additional pebble to a_k . After moving a pebble to a_k we obtain that $p(V(C_1) - \{a_k\})$ is at least $n + 1$. We assume that $p(y) = 1$. Then, n pebbles are settled in $V(C_1) - \{a_k, b_{k-1}\}$ with $n - 1$ vertices. Again, using the pigeonhole principle, we can move the two pebbles to a_{2k-2} from a_k and b_{k-1} . Therefore, we assume that $p(a_k) = 0$ and $p(b_k) = 0$. Then, all $n + 4$ pebbles are distributed only on $V(C_1) - \{a_k, b_{k-1}\}$. Because the graph induced by vertex set $\langle V(C_1) - \{a_k, b_{k-1}\} \rangle$ is isomorphic to K_{n-2} , by referring the pebbling number of complete graph we can move four pebbles to a_k and move two pebbles to a_{2k-2} . Then, we move the pebble to y from a_{2k-2} .

Lemma 1. *Let $G = C_m(K_n)$ be a crisscross sequence of m complete graphs. Let us define $X_1 = C_{m_1}(K_n) = C_1 \cup \dots \cup C_{m_1}$ and $X_2 = C_{m_2}(K_n) = C_{m_1+1} \cup \dots \cup C_m$ be two subgraphs of G where $m_1 + m_2 = m$. Suppose that the number of pebbles distributed on X_2 is at least $2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2$. Then, move two pebbles to $a_{m_1(\frac{n}{2}-1)}$, for m_1 or $a_{m_1(\frac{n}{2}-1)+1}$, m_1 is odd.*

Proof. Consider graph $C_m(K_n)$ with $2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2$ pebbles distributed only on the vertices of X_2 . Two pebbles must be moved to $a_{m_1(\frac{n}{2}-1)}$, and we prove this result through induction on m_2 . For $m_2 = 1$, at least $2^{m_1} + n - 2$ pebbles were distributed on X_2 . Since $m \geq 4$. This implies that at least $n + 6$ pebbles were retained in X_2 . For $m_2 = 1$, we have $X_2 \cong K_n$. In addition, the two pebbling numbers of K_n are $f_2(K_n) = n + 2$ according to the pebbling number of complete graph. Thus, the two pebbles can move 2 pebbles to $a_{m_1(\frac{n}{2}-1)}$. Now, we assume that the result is true for all $m'_2 < m_2$. Let $2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2$ pebbles be distributed on X_2 . Suppose the number of pebbles distributed on C_{m_1+1} is at least $n + 2$. Then, the two pebbles can be moved to $a_{m_1(\frac{n}{2}-1)}$. Therefore, we assume that $p(C_{m_1+1}) \leq n + 1$. We have the following cases.

Case (1) $p(C_{m_1+1}) \geq n$.

Based on our assumptions, we can move one pebble to $a_{m_1(\frac{n}{2}-1)}$. This implies that at least $2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2 - n$ pebbles are retained in $\langle X_2 - C_{m_1+1} \rangle$. We need to make the following claim:

Claim (1) $p(\langle X_2 - C_{m_1+1} \rangle) \geq 2^{m_1}(2^{m_2-1} - 1) + (m_2 - 1)(n - 4) + 2$.

$$\begin{aligned} \text{We have, } p(\langle X_2 - C_{m_1+1} \rangle) &= 2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2 - n \\ &= 2^{m_1}(2 \cdot 2^{m_2-1}) + (m_2 - 1)(n - 4) + (n - 4) + 2 - n \\ &= 2^{m_1}(2^{m_2-1} - 1) + (m_2 - 1)(n - 4) - 2 \\ &\geq 2^{m_1}(2^{m_2-1} - 1) + (m_2 - 1)(n - 4) + 2, \text{ Since } m_1 + m_2 = n \geq 4. \end{aligned}$$

Thus, claim (1) allows us to move two pebbles to $a_{(m_1+1)(\frac{n}{2}-1)}$. Through induction, we can move an additional pebble to $a_{m_1(\frac{n}{2}-1)}$.

Case (2) $p(C_{m_1+1}) < n$. Suppose that any vertex belonging to C_{m_1+1} has at least two pebbles. Then, we can move a pebble to $a_{m_1(\frac{n}{2}-1)}$ and by Claim (1), we can move an additional pebble to $a_{m_1(\frac{n}{2}-1)}$.

Therefore, we assume that no vertices in C_{m_1+1} contain two pebbles. Without loss of generality, we assume that most pebbles are distributed on the vertices of C_1 . For the least case scenario, we consider that all the a_i 's and b_i 's in X_2 have one pebble on each and then place the remaining pebbles on y . This

$$\begin{aligned} \text{This implies at least} \\ 2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2 - m_2(n - 4) &= 2^{m_1}(2^{m_2} - 1) + 2 \\ = 2^{m_1+m_2} - 2^{m_1} + 2 > 2 \cdot 2^{m_2} \text{ pebbles retained in the vertex } y. \end{aligned}$$

Then, using 2^{m_2} pebbles twice, the two pebbles can move to $a_{m_1(\frac{n}{2}-1)}$.

Theorem 3. For any n, m , the pebbling number of graph $C_m(K_n)$ is $f(C_m(K_n)) = 2^m + 2(n - 3) + (m - 2)(n - 4)$.

Proof. Set $2^m - 1$ pebbles on vertex x and put one pebble on each a_i and b_i except for the vertices in $\{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n(k - 1) - l\}$. Thus, we cannot move one pebble to vertex y . So $f(C_m(K_n)) \geq 2^m + 2(n - 3) + (m - 2)(n - 4)$. Consider a graph with at least $2^m + 2(n - 3) + (m - 2)(n - 4)$ pebbles, Let $v \in C_i, i \in \{1, 2, \dots, m\}$ be any target vertex. We prove this result by induction on m . For $m = 2$ and $m = 3$, the results follow from Theorems 1 and 2 respectively. We assume that the result is true for all $m' < m$. We prove this result for all m . We consider the following cases:

Case (1) $v \in C_{m_1}$ for $2 < m_1 < m$. Let us assume that $X = C_1 \cup \dots \cup C_{m_1}$ and $Y = C_{m_1+1} \cup \dots \cup C_m$ are two disjoint subgraphs of $C_m(K_n)$. Here $X = C_{m_1}(K_n)$ and $Y = C_{m_2}(K_n)$, where $m = m_1 + m_2$. Clearly $v \in X$. Suppose $p(X) \geq 2^{m_1} + 2(n - 3) + (m_1 - 2)(n - 4)$. Then, we can move the pebble to v by induction on m . Therefore, we assume $p(X) < 2^{m_1} + 2(n - 3) + (m_1 - 2)(n - 4) - 1$. Also $p(C_{m_1}) \leq n - 2$. Suppose $p(X - C_{m_1}) \geq 2^{m_1+1}(2^1 - 1) + (n - 4) + 2$. From Lemma 1 we can move a pebble to v . So assume that $p(X - C_{m_1}) < 2^{m_1-1}(2^1 - 1) + (n - 4) + 2$ and $p(C_{m_1}) \leq n - 2$. Therefore $p(X) \leq 2^{m_1-1} + 2n - 4$. This implies that the number of pebbles distributed on Y was at least $2^{m_1}(2^{m_2} - 1/2) + (m_1 + m_2 - 2)(n - 4) - 1$. We have $p(Y) \geq 2^{m_1}(2^{m_2} - 1/2) + (m_1 + m_2 - 2)(n - 4) - 1 \geq 2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2$. Again, by Lemma 1, two pebbles can be moved to $a_{m_1(\frac{n}{2}-1)}$ or $a_{m_1(\frac{n}{2}-1)+1}$ when m_1 is even and m_1 is odd. Then, we can move one pebble to v .

Case (2) $v \in C_1$ or $v \in C_m$.

Without a loss of generality, we assume that $v \in C_m$ and $v = y$. Also $p(C_m) \leq n - 2$. Now take $X_1 = C_m \cong C_1(K_n)$ and $X_2 = C_1 \cup \dots \cup C_{m-1} \cong C_{m-1}(K_n)$.

Claim (1) $p(X_2) \geq 2(2^{m-1} - 1) + (m - 1)(n - 4) + 2$.

We have to prove

$$\begin{aligned} & p(X_2) - 2(2^{m-1} - 1) + (m - 1)(n - 4) + 2 \geq 0 \\ & = 2^m + 2(n - 3) + (m - 2)(n - 4) - n + 2 - 2^m + 2 - (m - 1)(n - 4) - 2 \\ & = 2(n - 3) - (n - 4) - n + 2 = 0. \end{aligned}$$

From Claim (1), we obtain $p(X_2) \geq 2(2^{m-1} - 1) + (m - 1)(n - 4) + 2$ pebbles. Again, by Lemma 1, two pebbles can be moved to $a_{(m-1)(\frac{n}{2}+1)}$ or $a_{(m-1)(\frac{n}{2}+1)+1}$ when m is odd or even, respectively. We can then move the pebble to v .

The t - pebbling number of crisscross sequence of m complete graphs

Theorem 4. The t -pebbling number of the graph $C_2(K_n)$ is $f_t(C_2(K_n)) = 4t + 2(n - 3)$.

Proof. Set $4t - 1$ pebbles on vertex x and one pebble on each a_i and b_i except for vertices a_k and b_{k-1} where $a_k, b_k \in V(C_1) \cup V(C_2)$. Therefore, we cannot move t pebbles to the vertex y . Therefore $f_t(C_2(K_n)) \geq 4t + 2n - 6$. Consider a graph with at least $4t + 2n - 6$ pebbles distributed at the vertices of $C_2(K_n)$. Let v be a target vertex and $p(v) = 0$. We prove this result by induction of t . For $t = 1$, the result follows from theorem 1. We assume that this result is true for all $t' < t$. Without a loss of generality, we assume that $v \in C_2$ and $v = y$. We consider the following cases:

Case (1) $n \leq p(C_2) < 2t + n - 2$. If $p(C_2) \geq 2t + n - 2$. By referring the pebbling number of complete graph, we can move t pebbles to y . Therefore, we assume that $p(C_2) \geq n$. We used n pebbles for y . Also $p(C_2) \leq 2t + n - 2$. Suppose $p(C_1) \leq 4t + n - 5$. Then $p(C_2) = n + 1$. This was because $C_2 \cong K_n$. Therefore, any of the vertices in C_2 contains two pebbles. Using exactly two pebbles, we can move a pebble to y . This leave at least $4t + 2n - 8$ pebbles retained in $C_2(K_n)$. Since $4t + 2n - 8 = 4(t - 1) + 2n - 6$. So by induction we can move additional $t - 1$ pebbles to y .

Case2: $p(C_2) < n$. It is assumed that vertices adjacent to y have at most one pebble. Suppose $p(a_k) = 1$ or $p(b_k) = 1$. By using at least two pebbles from C_1 and move an additional pebble to a_k . Then, we move the pebble to y . This left at least $4(t - 1) + 2n - 6$ pebbles on $C_2(K_n)$. Thus, by induction, we can move an additional $t - 1$ pebbles to y . So $p(C_2) \leq n - 3$. This implies that the number of unused

pebbles retained in C_1 was at least $4t + n - 2$. Thus by $f(K_n) = n$, we can move $2t$ pebbles to a_k and then move t pebbles to y .

Theorem 5. *The t -pebbling number of the graph $C_3(K_n)$ is $f_t(C_3(K_n)) = 8t + 3n - 10$.*

Proof. Set $8t - 1$ pebbles on vertex x and place one pebble on each a_i and b_i except a_k, b_{k-1}, a_{2k-2} and b_{2k-1} . Thus, we cannot move t pebbles to vertex y . So $f_t(C_3(K_n)) \geq 8t + 3n - 10$. Consider graph $C_3(K_n)$ with at least $8t + 3n - 10$ pebbles distributed on its vertices. Let $v \in C_i$ for $i = 1, 2, 3$ be any target vertex. We consider the following cases:

Case (1) $v \in C_2$.

Suppose $p(C_3) \geq 4t + n - 2$. We can move $2t$ pebbles to a_{2k-2} and move t pebbles to v by using pebbling number of complete graph. So assume that $p(C_3) \leq 4t + n - 3$. This results in at least $8t + 3n - 10 - (4t + n - 3) = 4t + 2n - 7$. We have $p(C_1 \cup C_2) \geq 4t + 2n - 7$. If $p(C_1 \cup C_2) \geq 4t + 2n - 6$, By previous theorem we can move t pebbles to v . Therefore, we assume that $p(C_1 \cup C_2) \leq 4t + 2n - 7$. If $p(C_1 \cup C_2) < 4t + 2n - 7$, then we get $p(C_3) \geq 4t + 3n - 3$. Otherwise, this contradicts the total number of pebbles distributed on $C_3(K_n)$. So we take $p(C_1 \cup C_2) = 4t + 2n - 7$ and $p(C_3) \geq 4t + n - 3 \geq 2(2(t - 1)) + n - 3 = 4t + n - 7$. Again, by using $f(K_n) = n$ we can move $2(t - 1)$ pebbles to a_{2k-2} and $t - 1$ pebble to v . Also $p(C_1 \cup C_2) = 4t + 2n - 7 \geq 2n - 2$, since $t \geq 2$ and $C_1 \cup C_2 \cong C_2(K_n)$. Thus, from Theorem 1 we can move an additional pebble to v .

Case (2) $v \in C_1$ or $v \in C_3$.

Fix $v \in C_3$ and $v = y$. We assume that $p(C_3) \geq n$. Subsequently, at least one of the vertices in C_3 contains at least two pebbles. Using exactly two pebbles, we can move one pebble to y . This results in at least $8t + 3n - 12 \geq 8(t - 1) + 3n - 10$ pebbles in $C_3(K_n)$. By induction, we can move an additional $t - 1$ pebbles to y . Thus, no vertices in C_3 contained two pebbles. Thus $p(V(C_3) - \{a_{2k-2}, b_{2k-1}\}) \leq n - 3$. This implies that at least $8t + 3n - 10 - n + 3 = 8t + 2n - 7$ pebbles are retained in $C_1 \cup C_2$ because $8t + 3n - 7 \geq 8t + 2n - 6$ for $n \geq 4$. Thus, from Theorem 4, we can move $2t$ pebbles to a_{2k-2} and then move t pebbles to y .

Theorem 6. *The t -pebbling number of the graph $C_m(K_n)$ is $f_t(C_m(K_n)) = t2^m + 2(n - 3) + (m - 2)(n - 4)$.*

Proof. Set $t2^m - 1$ pebbles on vertex x and put one pebble on each a_i and b_i except for the vertices in $\{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n(k - 1) - 1\}$. Thus, we cannot move t pebbles to y . So $f_t(K_n) \geq t \cdot 2^m + 2(n - 3) + (m - 2)(n - 4)$.

Consider a graph with at least $t2^m + 2(n - 3) + (m - 2)(n - 4)$ pebbles distributed at the vertices of $C_m(K_n)$. Let $v \in C_i$ for $i = 1, 2, \dots, m$ be any target vertex. We prove this result by induction of t and m . For $t = 1, m = 2$ and $m = 3$, the results follow from Theorems 3, 4 and 2. We assume that this result is true for all $t' < t$. We prove this result for all t . Let $v \in C_{m_1}$. Let $X_1 = C_1 \cup \dots \cup C_{m_1}$ and $X_2 = C_{m_1+1} \cup \dots \cup C_m$. Clearly $X_1 \cup X_2 \cong C_m(K_n)$. Suppose $p(X_2) \geq 2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2$. Then, the two pebbles can be moved to $a_{m_1} \left(\frac{n}{2} - 1\right)$ and move a pebble to v . This implies that the number of pebbles retained in X_1 was at least $t2^m + 2(n - 3) + (m - 2)(n - 4) - [2^{m_1}(2^{m_2} - 1) + m_1(n - 4) + 2] \geq (t - 1)2^m + 2(n - 3) + (m - 2)(n - 4)$. Thus, by induction, we can move an additional $t - 1$ pebbles to v . So assume that $p(X_2) < 2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2$.

Claim (1) $p(C_m(K_n)) - p(X_2) \geq f_t(C_{m_1}(K_n))$.

We have, $p(C_m(K_n)) - p(X_2)$

$$\begin{aligned}
 &= [t \cdot 2^m + 2(n-3) + (m-2)(n-4)] - [2^{m_1}(2^{m_2} - 1) + m_2(n-4) + 1] \\
 &= t \cdot 2^m + 2(n-3) + (m-2)(n-4) - 2^m + 2^{m_1} - m_2(n-4) - 1 \\
 &= t \cdot 2^{m_1}(2^{m_2} - 1) + t \cdot 2^{m_1} + 2(n-3) + (m_1 - 2)(n-4) - 1 \\
 &= f_t(C_{m_1}(K_n)) + t \cdot 2^m - t \cdot 2^{m_1} - 2^m - 1 \\
 &= f_t(C_{m_1}(K_n)) + 2^{m_1}(t \cdot 2^{m_2} - t - 1) - 1 \\
 &\geq f_t(C_{m_1}(K_n)), \text{ since } t \geq 2 \text{ and } m \geq 4.
 \end{aligned}$$

By above Claim (1), $p(X_1) \geq f_t(C_{m_1}(K_n))$, and by induction on m we can move t pebbles to v .

In the next sections, we show that the graph $C_m(K_n)$ has both the 2-pebbling and $2t$ -pebbling properties.

The 2-pebbling Property of crisscross Sequence of m – Complete Graphs

Theorem 7. *The graph $C_2(K_n)$ exhibits the two-pebbling property.*

Proof. Consider a graph with at least $2(2n - 2) - q + 1$ pebbles scattered at the vertices of $C_2(K_n)$. A minimum of two pebbles must be pushed to the target vertex. We assume that $v \in V(C_2)$ without losing generality. We look at the following scenarios:

Case (1) $v = a_k$ or $v = b_{k-1}$.

We assume that $v = a_k$ without losing generality. Because $a_k \in V(C_1) \cap V(C_2)$. We have $2f(C_2(K_n)) - +1 \geq 4n - 4 - (2n - 2) + 1 = 2n - 1 \geq |V(G)|$. According to the pigeonhole principle, one of the vertices in $V(C_2(K_n))$, say x , must have at least two pebbles. Thus, we can move one pebble to a_k using precisely two pebbles. This is due to the fact that a_k is adjacent to all of the graph's vertices. This means that as $q \leq 2n - 3$, at least $2(2n - 2) - q + 1 - 2$ pebbles are maintained in $V(C_2(K_n))$. As a result, $2(2n - 2) - q - 1 \geq 2n - 2 = f(C_2(K_n))$. We can transfer n more pebbles to a_k using Theorem 1.

Case (2) $v \in V(C_1) - \{a_k\}$ or $v \in V(C_2) - \{b_{k-1}\}$.

We assume that $v \in V(C_2) - \{b_{k-1}\}$ and $v = y$ and $v = y$ without losing generality. By the pigeonhole principle, if $p((C_2) - \{b_{k-1}\}) \geq n - 1$, at least one vertex, say a_{k+1} , contains two pebbles, and then pushes a pebble to y from a_{k+1} using precisely two pebbles. As a result, we suppose that $p((V_2) - \{b_{k-1}\}) \leq n - 2$. The following subcases must be present:

Subcase (1a) $p(V(C_2) - \{b_{k-1}\}) = n - 2$. As by our assumption

$$\begin{aligned}
 p(V(C_1) - \{b_{k-1}\}) &\geq 2(2n - 2) - q + 1 - n + 2 = 2(2n - 2) - q - n + 3 \\
 &= 3n - q - 1 \geq 3n - (2n - 3) - 1 = n + 2.
 \end{aligned}$$

The pigeonhole principle states that one of the vertices in $V(C_1) - \{b_{k-1}\}$ has at least two pebbles, i.e., $p(a_1) = 2$. As a result, we can transfer one pebble to a_k with only two pebbles. Hence, $p(V(C_2) - \{b_{k-1}\}) = n - 1$. This is due to the fact that $|V(C_2) - \{b_{k-1}\}| = n - 1$ and $p(y) = 0$. As a result, one of the vertices in $V(C_2) - \{b_{k-1}\}$ has two pebbles. Then, with precisely two pebbles, we can move one pebble to y . Let q_1 represent the number of occupied vertices in $V(C_1) - \{b_{k-1}\}$ and q_2 , and q_2 represent the number of occupied vertices in $V(C_2) - \{a_k\}$.

Claim (1) $p(C_2(K_n)) - p(V(C_2) - \{a_k\}) - q_2 - 2 \geq 2f(K_n) - q_2 + 1$. We have

$$\begin{aligned} p(C_2(K_n)) - [p(V(C_2) - \{a_k\}) - q_2] - 2 &= 2(2n - 2) - q + 1 - (n - 2) + q_2 - 2 \\ &= 4n - 4 - q_1 - n + q_2 + 1 \\ &= 3n - q_3 - 3 \\ &= 2n - q_1 + 1 + n - 4 \\ &\geq 2n - q_1 + 1, \text{ since } n \geq 5 = 2f(K_n) - q_1 + 1. \end{aligned}$$

Claim (1) states that at least $2f(K_n) - q_1 + 1$ pebbles are scattered in C_1 . This is due to the fact that $C_1 \cong K - n$ and K_n satisfy the 2-pebbling property. Because a_k is next to y , we can transfer two pebbles to a_k and one more pebble to y using the pebbling number of complete graph.

Subcase (1b) $p(V(C_2) - \{b_{k-1}\}) < n - 2$.

Assume that at most $n - 3$ pebbles are spread on $V(C_2) - \{b_{k-1}\}$ without losing generality.

Claim (2) $p(C_1) \geq 2f_2(K_n) - q_1 + 1$. Since $p(C_2) - q_2 = 0$, then we have,

$$2(2n - 2) - q_1 - q_2 + 1 = 4n - 4 - q_1 + 1 - (n - 3)$$

$3n - q_1 \geq 2n - q_1 + 5$, Since $n \geq 5$, $2(n + 2) - q_1 + 1$
 $= 2f_2(K_n) - q_1 + 1$. Because $C_1 \cong K_n$ and $p(C_1) \geq 2f_2(K_n) - q_1 + 1$. Then we may shift four pebbles to a_k and two pebbles to y .

Theorem 8. *The graph $C_3(K_n)$ exhibits the two-pebbling property[9][10].*

Proof. Consider a graph containing at least $2(3n - 2) - q + 1$ pebbles at each vertex. The two pebbles must be moved to any vertex of the target. Let $v \in C_i$ be the target vertex for $i = 1, 2, 3$. We look at the following scenarios:

Case (1) $v \in C_2$.

We will suppose that $p(C_2) \geq n + 2$. We can transfer two pebbles to v using the pebbling number of complete graph. As a result, we suppose that $p(C_2) < n + 2$. If $p(C_2) \geq n$, we can transfer a pebble to v for no more than two pebbles. Because $p(v) = 0$ and $q(C_2) \leq n - 1$, one of the vertices of C_2 holds at least two pebbles. This suggests that $V(C_3(K_n))$ contained at least $2(3n - 2) - q + 1 - 2$ pebbles. We have $2(3n - 2) - (3n - 6) = 3n + 2 \geq 3n - 2 = f(C_3(K_n))$ since $q \leq 3n - 5$. After that, the pebble may be transferred to v . Assume that $p(V(C_2) - \{a_k, b_{k-1}, a_{2k-2}, b_{2k-1}\}) \geq n - 4$. We can then transfer a pebble to v . As a result, we suppose that $p(V(C_2) - \{a_k, b_{k-1}, a_{2k-2}, b_{2k-1}\}) \leq n - 5$, which implies that at least $2(3n - 2) - q - 1 - ((n - 5) - q_2 - 1)$ pebbles are scattered on both C_1 and C_3 . $q = q_1 + q_2 + q_3$, where $q_1 = q(C_1)$, $q_2 = q(C_2 - \{a_k, b_{k-1}, a_{2k-2}, b_{2k-1}\})$, and $q_3 = q(C_3)$. As a result, in $V(C_1)$ or $V(C_3)$, which include at least $n + 2$ pebbles, at least $6n - 4 - q_1 - q_2 - q_3 + 1 - n + 5 + q_2 + 1 = 5n + 3 - q_1 - q_3$ pebbles are kept. Assume that $p(V(C_1)) \geq n + 2$. The two pebbles may then be moved to a_k , and the pebble can be moved to v in C_2 . This replaces at least $5n + 3 - q_1 - q_3 - (n + 2) = 4n + 1 - q_1 - q_3$ pebbles left in C_1 and C_3 . Assume $p(C_1) \geq 2n - q_1 + 1$. Then we may relocate the two pebbles to a_k and add another pebble to v . This was due to $C_1 \cong K_n$. If not, at least $2n - q_3 + 1$ pebbles remained in C_3 . This was due to $C_3 \cong K_n$. Using the pebbling number of the entire graph, we can add another pebble to v in C_2 . As a result, we suppose that $p(C_1) \leq n + 1$. This means that at least $p(V(C_3(K_n))) - p(C_1) = 6n - 4 - q + 1 - (n + 1) - q_1 = n - (q_2 + q_3) - 4$ is required that equals $f(C_2(K_n)) - q_2 - q_3 + 1$. Because $C_2 \cup C_3 \cong C_2(K_n)$. We can transfer two pebbles to v in C_2 using Theorem 7.

Case (2) $v \in C_1 - \{a_k, b_{k-1}\}$ or $v \in C_3 - \{a_{2k-2}, b_{2k-1}\}$.

We assume $v \in C_3$ and take $v = y$ without losing generality. Let $q = q_1 + q_2 + q_3 - 4$ and assume that all vertices of $\{a_k, b_{k-1}, a_{2k-2}, b_{2k-1}\}$ contain at least one pebble. If $p(C_1) \geq 2(n + 2) - q_1 + 1$, the four pebbles can transfer to a_k . The number of pebbles maintained in $C_3(K_n)$ must thus be at least $2(3n - 2) - q + 1 - (2(n + 2) - q_1 + 1) \geq f(C_2(K_n))$, and we may transfer an extra pebble to y using Theorem 1. As a result, we suppose that $p(C_1) < 2(n + 2) - q_1$. If $p(C_1) \geq n + 2$, two pebbles can be transferred to a_k and one pebble to a_{2k-2} . Then, using at least two pebbles from $V(C_2)$ or $V(C_3)$, we may transfer another pebble to a_{2k-2} and one to y . This means that in $V(C_2)$ and $V(C_3)$, at least $2n - 2$ pebbles were kept. As a result, we can add another pebble to y . As a result, we suppose that $p(C_1) < n + 2$.

Claim (1) $p(C_3(K_n)) - (n + 1) + q_1 \geq 2f(C_2(K_n)) - (q_2 + q_3) + 1$.

$$\begin{aligned} p(C_3(K_n)) - (n + 1) + q_1 &= 2(3n - 2) - q + 1 - n - 1 - q_1 \\ &= (4n - 4) - (q_2 + q_3) + 4 + n - 1 \\ &\geq (4n - 4) - (q_2 + q_3) + 1 \\ &= 2(2n - 2) - (q_2 + q_3) + 1 \end{aligned}$$

$$= 2f(C_2(K_n)) - (q_2 + q_3) + 1.$$

We can transfer two pebbles to y using the following Theorem 7. Because $C_2 \cup C_3 \cong C_2(K_n)$. Then we're finished.

Lemma 2.

Let $G = C_m(K_n)$ be a crisscross sequence of m complete graphs. Let $X_1 = C_{m_1}(K_n) = C_1 \cup \dots \cup C_{m_1}$ and $X_2 = C_{m_2}(K_n) = C_{m_1+1} \cup C_{m_1+2} \cup \dots \cup C_m$ be two subgraphs of G , where $m_1 + m_2 = m$. Assume the number of pebbles scattered on X_2 is more than $2(2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2) - q_2 + 1$. Then we may relocate four pebbles to $a_{m_1(\frac{n}{2}-1)}$ for m_1 is an even number, or $a_{m_1(\frac{n}{2}-1)+1}$ if m_1 is an odd number. Here, $q = q_1 + q_2$, with q_1 representing the number of occupied vertices of X_1 and q_2 representing the number of occupied vertices of X_2 .

Proof. Consider the graph $C_m(K_n)$ with the values $2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2 - q_2 + 1$ pebbles are exclusively found on the vertices of X_2 . We must relocate the four pebbles to $a_{m_1(\frac{n}{2}-1)}$. We

demonstrate this lemma using induction on m_2 . At least $2(2^{m_1} + n - 2) - q_2 + 1$ pebbles dispersed on X_2 for $m_2 = 1$. Because $m \geq 4$, at least $2(n + 6) - q_2 + 1$ pebbles must be kept in X_2 . If $m_2 = 1$, we get $X_2 \cong K_n$. Also $2(n + 6) - q_2 + 1 \geq 2f_2(K_n) - q_2 + 1$. Four pebbles can be moved to $a_{m_1(\frac{n}{2}-1)}$.

This result is assumed to be true for every $m_2' < m_2$. We demonstrate the findings for all m_2 . Allow $2(2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2) - q_2 + 1$ pebbles to be dispersed on $V(X_2)$. Assume the number of pebbles dispersed on C_{m_1+1} is more than $2(n + 2) - q_{m_1+1} + 1$. The four pebbles can then be transferred to $a_{m_1(\frac{n}{2}-1)}$. As a result, we suppose that $p(C_{m_1+1})$ is no more than $2(n + 2) - q_{m_1+1}$.

Case (1) $p(C_{m_1+1}) \geq 2n - q + 1$.

We may relocate the two pebbles to $a_{m_1(\frac{n}{2}-1)}$ based on this assumption. This means that at least $2^{m_1}(2^{m_2-1}) + m_2(n - 4) + 2 - q_2 - 2(n) + q_{m_1+1} - 1$ pebbles are maintained in $\langle X_2 - C_{m_1+1} \rangle$.

Claim(1)

$$p(X_2 - C_{m_1+1}) \geq 2(2^{m_1}(2^{m_2-1} - 1) + m_2(n - 4) + 2) - q + 1.$$

We have

$$\begin{aligned} p(X_2 - C_{m_1+1}) &= 2(2^{m_1}(2^{m_2} - 1) + m_2(n - 4) + 2) - q + 1 - 2n + q_{m_1+1} - 1 \\ &= 2(2^{m_1}(2^{m_2-1} - 1) + 2^{m_1(2^{m_2-1})+(m_2-1)(n-4)+2}) - q + 1 - 2n - q_{m_1+1} - 1 \\ &= 2(2^{m_1}(2^{m_2} - 1) + (m_2 - 1)(n - 4) + 2) - q + 1 + q_{m_1+1} + 2^{m_1+m_2} - q \\ &> 2(2^{m_1}(2^{m_2-1} - 1) + (m_2 - 1)(n - 4) + 2) - q + 1, \text{ since } m_1 + m_2 = n \geq 4 \text{ and we get the term } \\ & q_{m_1+1} + 2^{m_1+m_2} - q. \end{aligned}$$

We can move four pebbles to $a_{m_1+1}(\frac{n}{2} + 1)$ using Claim(1). Using induction on m_2 , two pebbles may be relocated to $a_{m_1}(\frac{n}{2} - 1)$.

Case (2) $p(C_{m_1+1}) < 2n - q_{m_1+1}$.

If at least two pebbles are present at the two vertices in C_{m_1+1} , they can be transferred to $a_{m_1}(\frac{n}{2}-1)$. We can move two additional pebbles to $a_{m_1}(\frac{n}{2} - 1)$ using Claim(1). Assume, for the sake of argument, that no two pebbles in C_{m_1+1} have two pebbles on each. We investigate the simplest example, in which the majority of the pebbles are placed on the vertices of C_1 . y , in particular, had a higher amount of pebbles. All other a_i and b_i have no more than one pebble. So the vertex y has $2(2^{m_1}(2^{m_2} - 1) + m_2(n - 4)) - q + 1 - m_2(n - 4) \geq 4(2^{m_2})$ pebbles, since $m_1 = 1$. This means that at least $4(2m_2)$ pebbles were kept in y . As a result, the four pebbles can shift to $a_{m_1}(\frac{n}{2} - 1)$.

Theorem 9. *The graph $C_m(K_n)$ satisfies two-pebbling property.*

Proof. Consider a graph that has at least $2^m + 2(n - 3) + (m - 2)(n - 4) - q + 1$ pebbles scattered across its vertices $C_m(K_n)$. The remaining two pebbles must be moved to any desired vertex. For $1 \leq i \leq m$, let $v \in C_i$. We demonstrate this result using induction on m . Theorems 7 and 8 provide the following findings for $m = 2$ and $m = 3$, respectively. We suppose the result holds true for every $m' < m$. We demonstrate the findings for all m . We have the following situations:

Case (1) $v \in C_{m_1}$ for $2 < m_1 < m$.

As the two subgraphs of $C_m(K_n)$, let us define $W = C_1 \cup \dots \cup C_{m_1}$ and $Z = C_{m_1+1} \cup \dots \cup C_m$. Assume $p(W) \geq 2(2^{m_1} + 2(n - 3) + (m_1 - 2)(n - 4)) - q_1 + 1$. The two pebbles can then be inducted to v . Assume $p(W) < 2(2^{m_1} + 2(n - 3) + (m_1 - 2)(n - 4) - q_1)$. If $p(W - C_{m_1}) \geq 2(2^{m_1-1}(2^1 - 1) + (n - 4 + 2)) - q_1 + 1$, We can shift four pebbles to $a_{(m_1-1)(\frac{n}{2}-1)}$ or $a_{(m_1-1)(\frac{n}{2}-1)+1}$ and then two pebbles to v using Lemma 2. So suppose $p(W - C_{m_1}) < 2(2^{m_1-1}(2^1 - 1) - (n - 4) + 2) - q_1 + 1$. This means that there were at least $2(2^m + 2(n - 3) + (m - 2(n - 4)) + 2) - q_2 + 1$ pebbles dispersed on Z . We can shift four pebbles using Lemma 2 to $a_{m_1(\frac{n}{2}-1)}$ or $a_{m_1(\frac{n}{2}-1)+1}$. The two pebbles can then be relocated to v .

Case (2) $v \in C_1$ or $v \in C_m$.

We assume that $v \in C_m$ and $v = y$ without losing generality. Also, $p(C_m) \leq n + 1$. We assume $W = C_m \cong C_1(K_n)$ and $Z = C_1 \cup \dots \cup C_{m-1} \cong C_{m-1}(K_n)$.

Claim(1) $p(Z) \geq 2(2(2^{m-1} - 1) + (m - 1)(n - 4) + 2) - q_2 + 1$. We Have to prove

$$\begin{aligned} p(Z) - [2(2(2^{m-1} - 1) + (m - 1)(n - 4) + 2) - q_2 + 1] &\geq 0 \\ &= 2(2^m + 2(n - 3) + (m - 2)(n - 4)) - q + 1 \\ &\quad - [2(2(2^{m-1} - 1) + (m - 1)(n - 4) + 2) - q_2 + 1] \\ &= 4(n - 3) + 2(m - 2)(n - 4) - q + 1 - 2(m - 1)(n - 4) \geq 4(n - 3 + 1) > 0. \end{aligned}$$

According to Claim(1), $p(Z) \geq 2(2(2^{m-1} - 1) + (m - 1)(n - 4) + 2) - q + 1$. Again, using Lemma 2, we can move four pebbles to $a_{(m-1)(\frac{n}{2}+1)}$ or $a_{(m-1)(\frac{n}{2}+1)+1}$, and we may then transfer two pebbles to v .

The 2t –pebbling property of $C_m(K_n)$

Theorem 10. *The graph $C_2(K_n)$ exhibits 2t – pebbling property.*

Proof. Consider a graph with at least $2(4t + 2n - 6) - q + 1$ pebbles scattered at the $C_2(K_n)$ vertices. Assume $v \in C_2$ and $v = y$. We demonstrate this result using induction on t . Theorem 7 yields the following result for $t = 1$. Assume that the outcome holds true for every $t' < t$. We take into account the following.

Case (1) $p(v) = 0$.

To prove Case(1), we investigate the following subcases.

Subcase (1) $v \in \{a_k, b_{k-1}\}$. Without loss of generality, assume that $v \in a_k$. Since $2(4t + 2n - 6) - q + 1 - 4 \geq 8t + 4n - 12 - (2n - 2) - 3 = 8t + 2n - 13 \geq 5$, since $t \geq 2$ and $n \geq 5$. Thus, we may transfer two pebbles to a_k by utilising precisely four pebbles from either C_1 or C_2 since a_k is adjacent to all the other vertices of $C_2(K_n)$. This yields at least $2(4t + 2n - 6) - q + 1 - 4$, which is greater than $2(4(t - 1) + 2n - 6) - q + 5$. As a result, we can induct an extra $2(t - 1)$ pebbles to a_k . As a result of induction, we can transfer one extra $2(t - 1)$ pebble to a_k .

Subcase (2) $v \in C_1$ or $v \in C_2$.

We assume that $v \in C_2$ and $v = y$ without losing generality. Assume $p(C_2) \geq n + 2$. At least two pebbles must be present on each of its two vertices. As a result, we can move two pebbles to y using precisely four pebbles. This means that in $V(C_2(K_n))$, at least $2(4(t - 1) + 2n - 6) - q + 1$ pebbles were kept. So suppose $p(C_2) = n$ or $p(C_2) = n + 1$. We can transfer one pebble to v for no more than two pebbles from $V(C_2)$. As a result, the number of pebbles kept in $V(C_1)$ must be more than $2(4t + 2n - 6) - q + 1 - (n + 1) > n + 5$. This means that any two of the vertices must have at least two pebbles, or one of them must have four pebbles. Then, using precisely four pebbles, two pebbles may be relocated to a_k , and the number of pebbles kept in $V(C_2(K_n))$ is at least $2(4t + 2n - 6) - q + 1 - 6 \geq 2(4(t - 1) + 2n - 6) - q + 3 > 2(4(t - 1) + 2n - 6) - q + 1$. As a result of induction, we can transfer more $2(t - 1)$ pebbles to v .

Case (2) $p(v) = x$ for $x \geq 1$.

We take into account the following subcases:

Subcase (2a) x is even.

Assume $p(v) = x = 2x'$. We need to transport more $2(t - x')$ pebbles to v . Because we have

$$\begin{aligned} p(C_2(K_n)) - 2x' &= 2(4t + 2n - 6) - q + 1 - 2x' \\ &= 2(4(t - x') + 4x' + 2n - 6) - q + 1 - 2x' \\ &= 2(4(t - x') + 2n - 6) - q + 1 + 6x' \end{aligned}$$

$$> 2f_{t-x'}(C_2(K_n)) - q + 1.$$

We can transfer more $2(t - x')$ pebbles to v via induction.

subcase (2b) x is odd.

Assume $p(v) = x = 2x' + 1$. We need to transport more $2(t - x') - 1$ pebbles to v . Since

$$\begin{aligned} p(C_2(K_n)) - 2x' - 1 &= 2(4t + 2n - 6) - q + 1 - 2x' - 1 \\ &= 2(4(t - x') + 4x' + 2n - 6) - q - 2x' \end{aligned}$$

$$= 2(4(t - x') + 2n - 6) - q + 6x' > 2f_{t-x'}(C_2(K_n)) - q + 1 \text{ Since } x' \geq 1$$

So we can add more $2(t - x') - 1$ pebbles to v .

Theorem 11. *The graph $C_3(K_n)$ exhibits $2t$ -pebbling property[13][16].*

Proof. Consider the graph $C_3(K_n)$, which has at least $2(8t + 3n - 10) - q + 1$ pebbles on its vertices. We must relocate $2t$ pebbles to any target vertex. For $i = 1, 2, 3$, let $v \in C_i$. We look at the following scenarios:

Case (1) $p(v) = 0$.

Theorem 8 yields the following result for $t = 1$. Assume that the outcome holds true for every $t' < t$. We take into account the following subcases:

Subcase (1a) $v \in C_2$.

Consider that $p(V(C_2) - \{a_k, b_{k-1}, a_{2k-2}, b_{2k-1}\}) \geq n - 2$. Then we may transfer two pebbles to v for a total of four pebbles. We have $p(V(C_2) - \{a_k, b_{k-1}, a_{2k-2}, b_{2k-1}\}) \cong K_{n-4}$. This indicates that $V(C_3(K_n))$ has at least $p(C_3(K_n)) - 4 \geq 2(8(t - 1) + 3n - 10)q + 1$ pebbles. We can transfer more $2(t - 1)$ pebbles to v via induction. We now suppose that $p(V_2(C_2) - \{a_k, b_{k-1}, a_{2k-2}, b_{2k-1}\}) \leq n - 3$. As a result, the total number of pebbles maintained in C_1 and C_2 was $p(C_3(K_n) - (n - 3) + q_2) \geq 4n + 12$. Then, at least one C_1 or C_3 contains at least $2n + 6$ pebbles. Make sure $p(C_1) \geq 2n + 6$. We can transfer four pebbles to a_k and two pebbles to $v \in C_2$ using at least eight pebbles from C_1 . Then we have $2(8t + 3n - 10) - q + 1 - 8 > 2(8(t - 1) + 3n - 10) - q + 1$. We can transfer more $2(t - 1)$ pebbles to v via induction.

Subcase (1b) $v \in C_1$ or $v \in C_3$. Allow $v \in C_3$ and set $v = y$. Assume that $p(C_3 - \{a_{2k-2}, b_{2k-1}\}) \geq n$. Then, for the expense of four pebbles, we may shift two pebbles to v . The number of pebbles kept in $V(C_3(K_n))$ was obviously at least $2f_{t-1}(C_3(K_n)) - q + 1$. With induction, we can transfer $2(t - 1)$ additional pebbles to y . As a result, we can shift zero pebbles to y by employing pebbles in $V(C_3) - \{a_{2k-2}, b_{2k-1}\}$. This implies that $p(V(C_3) - \{a_{2k-2}, b_{2k-1}\}) \leq n - 4$.

Claim(1) $p(C_3(K_n)) - p(C_3) \geq 2f_{2t}(C_2(K_n)) - q + 1$. We have $p(C_3(K - n)) - p(C_3) = 2(8t + 3n - 10) - q + 1 - n + 4$

$$\begin{aligned} &= 2(8t + 2n + n - 10) - q + 1 - n + 4 \\ &= 2(8t + 2n - 6) - q + 1 + 2n - 8 - n + 4 \\ &= 2(8t + 2n - 6) - q + 1 + n - 4 \end{aligned}$$

$$> 2(8t + 2n - 6) - q + 1, \text{ Since } n \geq 5 = 2f_{2t}(C_{(K_n)}) - q + 1 = p(C_1 \cup C_2).$$

According to Claim, we can transfer $4t$ pebbles to a_2k using Theorem 4 because $C_3(K_n) - C_3 \cong C_1 \cup C_2 \cong C_2(K_n)$ and then move $2t$ pebble to y .

Case (2) x is even.

We may express this as $x = 2x'$. We need to transport $2(t - x')$ more pebbles to v . Since, $p(C_3) - 2x' = 2(8t + 3n - 10) - q + 1 - 2x' = 2(8(t - x') + 8x' + 3n - 10) - q + 1 - 2x' > 2(8(t - x') + 3n - 10) - q + 1 + 16x' - 2x > 2f_{t-x'}(C_3(K_n)) - q + 1$.

Then we may add another $2(t - x')$ pebbles to v .

Subcase (2a) x is odd.

We may express this as $x = 2x' + 1$. We need to transport another $2t - 2x' - 1$ pebble to v . Because $p(C_3(K_n) - p(C_3)) \geq 2f_{t-x'}(C_2(K_n)) - q + 1$. We can transfer more $2(t - x')$ pebbles to v via induction.

Theorem 12. *The graph $C_m(K_n)$ exhibits the $2t -$ pebbling property.*

Proof. Consider a graph with at least $2(t2^m + 2(n - 3) + (m - 2)(n - 4)) - q + 1$ pebbles distributed at its vertices. The $2t$ pebbles must be moved to any target vertex. For $1 \leq i \leq m$, consider $v \in C_i$. We have identified the following cases.

Case (1) $p(v) = 0$. By demonstrating that this conclusion is true for all values of t and m , we can verify it. Due to the fact that it follows from Theorems 10 and 11, we know the conclusion is true for $t = 1, m = 2$, and $m = 3$. For any values of t' that are smaller than t , we assume that the conclusion is true. Then, we demonstrate that the conclusion must be true for t if it is true for all values of t' that are smaller than t .

Take the value $m = m_1 + m_2$. Undoubtedly, $v \in C_{m_1}$. Establish the definitions of $X_1 = C_1 \cup \dots \cup C_{m_1}$ and $X_2 = C_{m_1+1} \cup \dots \cup C_m$. Consider the scenario where $p(X_2) \geq 2(2m_1(2m_2 - 1) + m_2(n - 4) + 2) - q_2 + 1$. Based on Lemma 1, four pebbles can move to either $a_{m_1(\frac{n}{2}-1)}$ or $a_{m_1(\frac{n}{2})+1}$, from which we can move two pebbles to v . As a result, we must transfer an extra $2(t - 1)$ pebbles to v .

Claim (1) $p(C_m(K_n)) - p(X_2) \geq 2((t - 1)2^{m_1} + 2(n - 3) + (m_1 - 2)(n - 4)) - q_1 + 1$.

We have $p(C_m(K_n)) - p(X_2) = 2((t - 1)2^{m_1+m_2} + 2^{m_1} + 2(n - 3) + (m_1 + m_2 - 2)(n - 4) - m_2(n - 4) - 2) - q_1 > 2((t - 1)2^{m_1} + 2(n - 3) + (m_1 - 2)(n - 4)) - q_1 + 1$, since $m_1 \geq 2 > 2f_{t-1}(C_{m_1}(K_n)) - q_1 + 1$.

We may transfer the $2(t - 1)$ pebbles to v from claim (1). As a result, we presume that $p(X_2) \leq 2(2^{m_1}(2^{m_2-1}) + m_2(n - 4) + 2) - q_2$, which results in the claim that follows below.

Claim(2) $p(C_m(K_n)) - p(X_2) \geq 2f_t(C_{m_1}(K_n))$. We have

$p(C_m(K_n)) - p = 2(t2^{m_1+m_2} - 2^{m_1+m_2} + 2^{m_1} + 2(n - 3) + (m_1 - 2)(n - 4) - 2) - q_1 + 1 > 2(t2^{m_1} + 2(n - 3) + (m_1 - 2)(n - 4)) - q_1 + 1 > 2f_t(X_1) - q_1 + 1$.

We can transfer $2t$ pebbles to v through induction. We are then done.

Case (2) $p(v) = x$, for $x \geq 1$.

More $2t - x$ pebbles need to be moved to v . The following subcases exist:

Subcase (2a) x is even.

Now, x can be written as $2x'$. Moving the $2(t - x')$ pebbles to v is necessary. It is evident that $p(C_m(K_n)) - 2x' \geq 2f_{t-x'}(C_m(K_n)) - q + 1$. In order to use induction t , we can relocate the $2(t - x')$ pebbles to v .

Subcase (2b) x is odd.

Let x may be expressed as $2x' + 1$. Moving $2t - 2x' - 1$ pebbles to v is necessary. We obtain $p(C_m(K_n)) - 2x' - 1 \geq 2f_{t-x'}(C_m(K_n)) - q + 1$ without a doubt. Therefore, we may transfer $2x' - 1$ more pebbles to v .

Conclusion:

In conclusion, this paper delivers a comprehensive exploration of the pebbling number, the two-pebbling property, the t -pebbling number, and the $2t$ -pebbling property inside a crisscross sequence of m -complete graphs. By investigating these properties, we have expanded valuable perceptions into the productivity of pebbling operations in this structure. The study highlights the complications and variations in pebbling dynamics, offering a profounder understanding of how these properties impact the behavior of the graph. This research contributes to the broader field of graph theory, precisely in understanding resource distribution and optimization in networked systems.

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