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Some Properties of RICCI Solitons in LP-Kenmotsu Manifolds

Bidyabati Thangjam[1], M. S. Devi[*2]
[1,2] Department of Mathematics, Mizoram University, Tanhril, Aizawl-796004, India.
devi_saroja@rediffmail.com

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Abstract:

Examining Ricci solitons in Lorentzian para-Kenmotsu manifolds is the goal of this paper. We have established that a symmetric parallel second-order covariant tensor in a Lorentzian para-Kenmotsu manifold is a constant multiple of the metric tensor. We have shown that if L_V g+2S is parallel to the Levi-Civita connection associated with g, where V is a given vector field, then (g,V,λ) is a Ricci soliton. We have observed that a Ricci soliton in a W_2-semi-symmetric Lorentzian para-Kenmotsu manifold is shrinking. Furthermore, certain curvature properties of Lorentzian para-Kenmotsu manifolds admitting Ricci solitons are studied. Finally, we have provided an example of a 3-dimensional Lorentzian para-Kenmotsu manifold.

Keywords: Lorentzian Para-Kenmotsu manifolds, Ricci solitons, Symmetric second order tensors.

Mathematics Subject Classification (2020): 53C21, 53C25, 53E20.

Introduction

Hamilton [5] introduced the concept of Ricci solitons, which is a natural generalization of an Einstein metric and is defined on a Riemannian manifold M. A Ricci soliton is a tripled (g, V, λ) such that

$$\mathscr{L}_V + 2S + 2\lambda g = 0, \qquad (1.1)$$

where g is a Riemannian metric, V is a vector field, λ is a real scalar, S is a Ricci tensor of M and $\mathscr{L}_V$ denotes the Lie derivative operator along the vector field V. A Ricci soliton is said to be shrinking if λ is negative, steady if λ is zero, and expanding if λ is positive.

In recent years, many geometers have studied Ricci solitons. Ingalahalli and Bagewadi [8] studied Ricci solitons on α-Sasakian manifolds. Pokhariyal et al. [12] found some results on Trans-Sasakian manifolds. Ayar and Demirhan [1] provided basic information about Ricci solitons on nearly Kenmotsu manifolds and obtained some structures on this manifold satisfying a semi-symmetric metric connection. Later, Shah [15] studied Ricci solitons in Lorentzian Para-Sasakian manifolds, while Chen et al. [2] studied Ricci solitons and certain related metrics on a three-dimensional Trans-Sasakian manifolds. Many other geometers had also studied Ricci solitons on various manifolds.

Sato[14] introduced the concept of an almost para-contact Riemannian manifold. Subsequently, Sinha and Prasad [7] defined a specific class of almost para-contact metric manifolds, namely para Kenmotsu and special para Kenmotsu manifolds. Another related structure is Lorentzian para-Sasakian manifold, which were introduced by Matsumoto [9]. Several other researchers had also studied this manifold ([4], [10], [16], [18]). Recently, Haseeb and Prasad [6, 7] focused on studying the properties of Lorentzian para-Kenmotsu manifolds, particularly in terms of Ricci-pseudosymmetricity and Riccigeneralized pseudosymmetricity conditions. Pandey et al. [11] investigated the geometric properties of η-Ricci solitons on Lorentzian para-Kenmotsu manifolds. Based on these studies, our motivation is to investigate Ricci solitons on Lorentzian para-Kenmotsu manifolds.

The paper is structured as follows: Section 1 is the introduction, then there is a preliminaries section. In section 3, parallel symmetric second-order tensors and Ricci solitons in Lorentzian para-Kenmotsu manifolds are studied. The next section investigates the properties of Ricci solitons in W_2-semi-symmetric Lorentzian para-Kenmotsu manifolds of dimensions $(2n+1)$. In section 5, we study Ricci tensor of a Lorentzian para-Kenmotsu manifold admitting a Ricci soliton. Then, in the next section, curvature properties of Lorentzian para-Kenmotsu manifolds admitting Ricci solitons are examined. The last section provides an example of a Lorentzian para-Kenmotsu manifold. The paper concludes with a summary of the findings.

1. Preliminaries

A $(2n+1)$ $-$differentiable manifold M with a $(1,1)$ tensor field ϕ, contravariant vector field ξ, a 1-form η, and a Lorentzian metric g is referred as a Lorentzian almost para-contact manifold[9] if g satisfies the following conditions:

$$\phi^2 X_1 = X_1 + \eta(X_1), \qquad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2) \tag{2.2}$$

and

$$\eta(\xi) = -1, \qquad g(X_1, \xi) = \eta(X_1), \tag{2.3}$$

$$\eta(\phi X_1) = 0. \tag{2.4}$$

A Lorentzian almost para-contact manifold M is a Lorentzian para-Kenmotsu manifold if it satisfies [6]

$$(\nabla_{X_1} \phi) = -g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1, \tag{2.5}$$

for any vector fields X_1 and X_2 on M and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g.

In Lorentzian para-Kenmotsu manifolds, the following relations hold [11]:

$$\nabla_{X_1}\xi = -X_1 - \eta(X_1)\xi, \tag{2.6}$$

and

$$(\nabla_{X_1} \eta)X_2 = -g(X_1, X_2) - \eta(X_1)\eta(X_2). \tag{2.7}$$

In addition to these, the following relations also hold [6]:

$$\eta(R(X_1, X_2)\, X_3) = g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2), \tag{2.8}$$

$$R(\xi, X_1)\, X_2 = g(X_1, X_2)\xi - \eta(X_1)\, X_2, \tag{2.9}$$

$$R(X_1, X_2)\xi = \eta(X_2)\, X_1 - \eta(X_1) X_2, \tag{2.10}$$

$$S(X_1, \xi) = 2n\eta(X_1), \quad Q(\xi) = 2n, \tag{2.11}$$

$$S(\phi X_1, \phi X_2) = S(X_1, X_2) + 2n\eta(X_1)\eta(X_2), \quad (2.12)$$

for all vector fields X_1 and X_2 on M.

Let (g, V, λ) be a Ricci soliton in a $(2n+1)$-dimensional Lorentzian para-Kenmotsu manifold M. Then, we have

$$(\mathscr{L}_\xi g)(X_1, X_2) = g(\nabla_\xi X_1, X_2) + g(X_1, \nabla_\xi X_2). \quad (2.13)$$

Using [2.6] in [2.13], we have

$$(\mathscr{L}_\xi g)(X_1, X_2) = -2g(X_1, X_2) - 2\eta(X_1)\eta(X_2). \quad (2.14)$$

From [1.1] and [2.14], we get

$$S(X_1, X_2) = (1-\lambda)g(X_1, X_2) + \eta(X_1)\eta(X_2), \quad (2.15)$$

$$QX_1 = (1-\lambda)X_1 + \eta(X_1)\xi, \qquad r = 2n - \lambda(2n+1). \quad (2.16)$$

In view of [2.3] and [2.15], we have

$$S(X_1, \xi) = -\lambda\eta(X_1), \quad (2.17)$$

$$Q\xi = -\lambda\xi. \quad (2.18)$$

Definition 2.1 [7]**:** A $(2n+1)$-dimensional Lorentzian para-Kenmotsu manifold M is called an η-Einstein manifold if its Ricci tensor S satisfies the following equation:

$$S(X_1, X_2) = \alpha g(X_1, X_2) + \beta\eta(X_1)\eta(X_2), \quad (2.19)$$

where α and β are scalars.

2. Parallel Symmetric Second Order Tensor and Ricci Solitons in Lorentzian para-Kenmotsu manifolds

In this section, we study parallel symmetric second order tensor and Ricci solitons in Lorentzian para-Kenmotsu manifolds.

Theorem 3.1: A symmetric parallel second order covariant tensor in a Lorentzian para-Kenmotsu manifold is a constant multiple of the metric tensor.

Proof: Let h be a symmetric tensor field of $(0,2)$-type which is parallel with respect to ∇ that is $\nabla h = 0$.

Then by applying the Ricci identity [8], we obtain

$$\nabla^2 h(X_1, X_2; X_3, X_4) - \nabla^2 h(X_1, X_2; X_4, X_3) = 0, \quad (3.1)$$

which implies that

$$h(R(X_1, X_2)X_3, X_4) + h(X_3, R(X_1, X_2)X_4) = 0. \quad (3.2)$$

Replacing $X_3 = X_4 = \xi$ in (3.2) and using (2.10) and the symmetry of h, we get

$$2[\eta(X_2)h(X_1, \xi) - \eta(X_1)h(X_2, \xi)] = 0. \quad (3.3)$$

Putting $X_1 = \xi$ in (3.3), we have

$$h(X_2,\xi) = -\eta(X_2)h(\xi,\xi). \quad (3.4)$$

Now, differentiating (3.4) covariantly with respect to X_1, we obtain

$$(\nabla_{X_1}h)(X_2,\xi) + h(\nabla_{X_1}X_2,\xi) + h(X_2,\nabla_{X_1}\xi) = -[(\nabla_{X_1}\eta)(X_2) + \eta(\nabla_{X_1}X_2)]h(\xi,\xi)$$
$$+\eta(X_2)[(\nabla_{X_1}h)(\xi,\xi) + 2h(\nabla_{X_1}\xi,\xi)]. \quad (3.5)$$

Using (2.6), (3.4) and the parallel condition $\nabla h = 0$ in (3.5), we get

$$h(X_2,\nabla_{X_1}\xi) = -(\nabla_{X_1}\eta)(X_2)h(\xi,\xi). \quad (3.6)$$

In consequences of (2.6), (2.7), (3.4) and (3.6), it yields

$$h(X_1,X_2) = -g(X_1,X_2)h(\xi,\xi). \quad (3.7)$$

From the above (3.7) and (3.4), we can conclude that $h(\xi,\xi)$ is a constant.

This completes the proof of the theorem.

Theorem 3.2. Let M be a Lorentzian para-Kenmotsu manifold. Assume that a symmetric metric tensor field $h = \mathscr{L}_V\, g + 2S$ is parallel with respect to the Levi-Civita connection associated with g. Then (g, V, λ) yields a Ricci-soliton on M.

Proof: Let us assume that a symmetric tensor field $h = \mathscr{L}_V g + 2S$ is parallel with respect to the Levi-Civita connection associated with g.

Then, $h(\xi,\xi) = 2\lambda$, this shows that $\lambda = \frac{1}{2}h(\xi,\xi)$.

Now, as h is parallel with respect to g, then from (3.7) we get

$$H(X_1,X_2) = -2\lambda g(X_1,X_2), \quad (3.8)$$

for all vector fields X_1 and X_2 on M, which leads to

$$\mathscr{L}_V g(X_1,X_2) = -2\lambda g(X_1,X_2) - 2S(X_1,X_2). \quad (3.9)$$

Hence, we complete the proof of the theorem.

Theorem 3.3. A Ricci semi-symmetric Lorentzian para-Kenmotsu manifold is an Einstein manifold.

Proof: We consider a Ricci semi-symmetric Lorentzian para-Kenmotsu manifold, i.e., $R \circ S = 0$.

We have [8],

$$(R(X_1,X_2)\circ S)(X_3,X_4) = -S(R(X_1,X_2)X_3,X_4) - S(X_3,R(X_1,X_2)X_4). \quad (3.10)$$

Putting $X_1 = \xi$ and using $R \circ S = 0$ in (3.10), we have

$$S(R(\xi,X_2)X_3,X_4) + S(X_3,R(\xi,X_2),X_4) = 0. \quad (3.11)$$

Using (2.9) in (3.11), we obtain

$$-g(X_2,X_3)S(\xi,X_4) - \eta(X_3)S(X_2,X_4) + S(X_3,\xi)g(X_2,X_4) + \eta(X_4)S(X_3,X_2) = 0. \quad (3.12)$$

Setting $X_3 = \xi$ in (3.12) and by making used of (2.3) and (2.11), we have

$$S(X_2, X_4) = 2ng(X_2, X_4). \tag{3.13}$$

Thus, we complete the proof.

Proposition 3.1. If a $(2n+1)$-dimensional Lorentzian para-Kenmotsu manifold is an η-Einstein manifold, then the Ricci soliton with constant scalar curvature is shrinking.

Proof:

Suppose that the Lorentzian para-Kenmotsu manifold is an η-Einstein manifold. Then, we will find the values of α and β.

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis of the tangent at any point of the manifold. Putting $X_1 = X_2 = e_i$ in (2.19) and taking summation over i, we get

$$r = (2n+1)\alpha - \beta. \tag{3.14}$$

Again, setting $X_1 = X_2 = \xi$ in (2.19), and using (2.11), we have

$$-2n = -\alpha + \beta. \tag{3.15}$$

Then from (3.14) and (3.15), we get

$$\alpha = \left[\frac{r}{2n} - 1\right],\ \beta = [-2n - 1 + \frac{r}{2n}]. \tag{3.16}$$

Substituting the value of α and β in (2.19), we have

$$S(X_1, X_2) = \left[\frac{r}{2n} - 1\right] g(X_1, X_2) + \left[-2n - 1 + \frac{r}{2n}\right] \eta(X_1)\eta(X_2). \tag{3.17}$$

For a $(2n+1)$-dimensional Lorentzian para-Kenmotsu manifold the symmetric parallel covariant tensor $h(X_1, X_2)$ of type $(0,2)$ is given by

$$h(X_1, X_2) = (\mathscr{L}_\xi\, g)(X_1, X_2) + 2S(X_1, X_2). \tag{3.18}$$

Using (2.13) and (3.17) in (3.18), we get

$$h(X_1, X_2) = \left[\frac{2r}{2n} - 4\right] g(X_1, X_2) + \left[-4n + \frac{2r}{2n} - 4\right] \eta(X_1)\eta(X_2). \tag{3.19}$$

Taking covariant derivative of (3.19) with respect to X_3, we have

$$(\nabla_{X_3} h)(X_1, X_2) = \left[\frac{2(\nabla_{X_3} r)}{2n}\right] [g(X_1, X_2) + \eta(X_1)\eta(X_2)]$$

$$+ \left[\frac{2r}{2n} - 4n - 4\right] \left[g(X_1, \nabla_{X_3}\xi)\eta(X_2) + g(X_2, \nabla_{X_3}\xi)\eta(X_1)\right]. \tag{3.20}$$

By putting $X_3 = \xi$ and $X_1 = X_2 \in (span\xi)^\perp$ in (3.20) and by using $\nabla h = 0$, we obtain

$$2\nabla_\xi r = 0. \tag{3.21}$$

On integrating (3.21), we get

$$r = c, \tag{3.22}$$

where c is some integral constant.

Thus, from (3.22) we have r is constant scalar curvature.

Finally, we will check the nature of the Ricci-soliton.

From (3.18), we have $h(X_1, X_2) = -2\lambda g(X_1, X_2)$, then putting $X_1 = X_2 = \xi$, we have

$$h(\xi, \xi) = 2\lambda. \tag{3.23}$$

If we put $X_1 = X_2 = \xi$ in (3.19), that is

$$h(\xi, \xi) = -\left[\frac{2r}{2n} - 4\right] + \left[-4n + \frac{2r}{2n} - 4\right]. \tag{3.24}$$

The above equation is reduced as

$$h(\xi, \xi) = -4n. \tag{3.25}$$

Equating (3.23) and (3.25), we obtain

$$\lambda = -2n < 0, \tag{3.26}$$

that is the Ricci soliton in a Lorentzian para-Kenmotsu manifold is shrinking.

Hence, the theorem is proved.

3. Ricci solitons in a W_2-semisymmetric Lorentzian para-Kenmotsu manifold

Here, we study the conditions of Ricci solitons in a W_2- semisymmetric Lorentzian para-Kenmotsu manifold.

Definition 4.1. [13] In a $(2n+1)$-dimensional Lorentzian para-Kenmotsu manifold M, the W_2-curvature tensor is defined as

$$W_2(X_1, X_2)X_3 = R(X_1, X_2)X_3 + \frac{1}{2n}[g(X_1, X_3)QX_2 - g(X_2, X_3)QX_1], \tag{4.1}$$

for all X_1, X_2 and X_3 in M.

Theorem 4.1. A Ricci soliton in a W_2-semi symmetric Lorentzian para-Kenmotsu manifold M of dimension $(2n+1)$ is shrinking.

Proof: Putting $X_1 = \xi$ in (4.1) and using (2.3) and (2.9), we have

$$W_2(\xi, X_2)X_3 = g(X_2, X_3)\xi - \eta(X_3)X_2 + \frac{1}{2n}[\eta(X_3)QX_2 - g(X_2, X_3)Q\xi]. \tag{4.2}$$

Taking inner product on both sides of (4.1) with respect to ξ, we get

$$\eta(W_2(X_1, X_2)X_3) = \eta(R(X_1, X_2)X_3) + \frac{1}{2n}[g(X_1, X_3)g(QX_2, \xi) - g(X_2, X_3)g(QX_1, \xi)]. \tag{4.3}$$

Using (2.8) and (2.17) in (4.3), we obtain

$$\eta(W_2(X_1, X_2)X_3) = \left(1 + \frac{\lambda}{2n}\right)[g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)]. \tag{4.4}$$

Suppose that the condition, $R(\xi, X_1) \circ W_2(X_2, X_3)X_4 = 0$ holds in M. Then by definition, we have

$$R(\xi, X_1)W_2(X_2, X_3)X_4 - W_2(R(\xi, X_1)X_2, X_3)X_4 - W_2(X_2, R(\xi, X_1)X_3)X_4$$

$-W_2(X_2,X_3)R(\xi,X_1)X_4=0,$ (4.5)

for all vector fields X_1,X_2,X_3 and X_4 on M.

In view of (2.9) and (4.5), we get

$$g(X_1,W_2(X_2,X_3)X_4)\xi-\eta(W_2(X_2,X_3)X_4)X_1-g(X_1,X_2)W_2(\xi,X_3)X_4$$
$$+\eta(X_2)W_2(X_1,X_3)X_4-g(X_1,X_3)W_2(X_2,\xi)X_4+\eta(X_3)W_2(X_2,X_1)X_4$$
$$-g(X_1,X_4)W_2(X_2,X_3)\xi+\eta(X_4)W_2(X_2,X_3)X_1=0. \tag{4.6}$$

Again, taking inner product on both sides of (4.6) with ξ and using (2.3), we have

$$-g(X_1,W_2(X_2,X_3)X_4)-\eta(W_2(X_2,X_3)X_4)\eta(X_1)-g(X_1,X_2)\eta(W_2(\xi,X_3)X_4)$$
$$+\eta(X_2)\eta(W_2(X_1,X_3)X_4)-g(X_1,X_3)\eta(W_2(X_2,\xi)X_4)+\eta(X_3)\eta(W_2(X_2,X_1)X_4)$$
$$-g(X_1,X_4)\eta(W_2(X_2,X_3)\xi)+\eta(X_4)\eta(W_2(X_2,X_3)X_1)=0. \tag{4.7}$$

In consequence of (4.2), (4.4) and (4.7), it yields

$$g(R(X_2,X_3)X_4,X_1)+\frac{1}{2n}[g(X_2,X_4)S(X_1,X_3)-g(X_3,X_4)S(X_1,X_2)]$$
$$-\left(1+\frac{\lambda}{2n}\right)[\eta(X_1)\{g(X_4,X_3)\eta(X_2)-g(X_2,X_4)\eta(X_3)\}$$
$$+g(X_1,X_2)\{\eta(X_4)\eta(X_3)+g(X_3,X_4)\}$$
$$+\eta(X_2)\{g(X_3,X_4)\eta(X_1)-g(X_1,X_4)\eta(X_3)\}$$
$$-g(X_1,X_3)\{\eta(X_4)\eta(X_2)+g(X_2,X_4)\}$$
$$+\eta(X_3)\{g(X_1,X_4)\eta(X_2)-g(X_2,X_4)\eta(X_1)\}$$
$$+\eta(X_4)\{g(X_3,X_1)\eta(X_2)-g(X_2,X_1)\eta(X_3)\}]=0. \tag{4.8}$$

Let $\{e_1,e_2,\dots,e_{2n+1}\}$ be an orthonormal basis. Putting $X_1=X_2=e_i$ in (4.8) and taking summation over i, where $1\le i\le(2n+1)$, we have

$$\left[\frac{2n+1}{2n}\right]S(X_3,X_4)=\frac{r}{2n}g(X_3,X_4)-2\left(1+\frac{\lambda}{2n}\right)[ng(X_3,X_4)$$
$$-(2n+1)\eta(X_3)\eta(X_4)]. \tag{4.9}$$

Again, taking orthonormal frame field over X_3 and X_4, we get

$$\lambda=-2n<0,$$

which implies that the soliton is shrinking.

Hence, the proof is completed.

Theorem 4.2. Let M be a $(2n+1)$-dimensional Lorentzian para-Kenmotsu manifold and (g,V,λ) be a Ricci soliton satisfying the condition $W_2(\xi,X_1)\circ S=0$ in M, then the Ricci soliton is steady.

Proof: Let M be a $(2n+1)$-dimensional Lorentzian para-Kenmotsu manifold and (g,V,λ) be a Ricci soliton in M.

We assume that the condition $W_2(\xi, X_1) \circ S = 0$ holds in M, then we have

$$S(W_2(\xi, X_1)X_2, X_3) + S(X_2, W_2(\xi, X_1)X_3) = 0. \quad (4.10)$$

Using (2.17), (4.2) and (4.10), we obtain

$$-\lambda g(X_1, X_2)\eta(X_3) - \lambda g(X_1, X_3)\eta(X_2) + \frac{1}{2n}[S(QX_1, X_3)\eta(X_2) + S(QX_1, X_2)\eta(X_3)]$$
$$-S(X_1, X_3)\eta(X_2) - S(X_1, X_2)\eta(X_3) - \frac{1}{2n}[g(X_1, X_2)S(Q\xi, X_3)$$
$$+g(X_1, X_3)S(Q\xi, X_2)] = 0. \quad (4.11)$$

Setting $X_3 = \xi$ in (4.11) and using (2.1), (2.3) and (2.17), we get

$$\left(\lambda + \frac{1}{2n}\lambda^2\right) g(X_1, X_2) + S(X_1, X_2) + \frac{1}{2n}[\lambda^2\eta(X_1)\eta(X_2) - S(QX_1, X_2)]$$
$$-\frac{1}{2n}\eta(X_1)S(Q\xi, X_2) = 0. \quad (4.12)$$

Again, putting $X_2 = \xi$ in (4.12) and using (2.1) and (2.17), we have

$$\lambda = 0.$$

The above equation implies that the Ricci soliton is steady.

Hence, we complete the proof.

4. Ricci tensor of a Lorentzian para-Kenmotsu manifold admitting a Ricci soliton

In this section, we study Ricci tensor of a Lorentzian para-Kenmotsu manifold admitting a Ricci soliton.

Theorem 5.1. Let M be a Lorentzian para-Kenmotsu manifold admitting a Ricci soliton (g, V, λ). If the Ricci tensor S of the manifold is η-recurrent, then the Ricci soliton is steady.

Proof: Suppose that the Ricci tensor of the Lorentzian para-Kenmotsu manifold is η-recurrent, i.e.,

$$\left(\nabla_{X_1} S\right)(X_2, X_3) = \eta(X_1)S(X_2, X_3), \quad (5.1)$$

for all vector fields X_1, X_2, X_3 on M.

Then, by using (2.15), we have

$$\left(\nabla_{X_1} S\right)(X_2, X_3) = -2\eta(X_1)\eta(X_2)\eta(X_3) - g(X_1, X_2)\eta(X_3) - g(X_1, X_3)\eta(X_2). \quad (5.2)$$

Using (2.15) in (5.1) and comparing with (5.2), we get

$$-g(X_1, X_2)\eta(X_3) - g(X_1, X_3)\eta(X_2) - (1 - \lambda)g(X_2, X_3)\eta(X_1) = 3\eta(X_1)\eta(X_2)\eta(X_3). \quad (5.3)$$

Setting $X_2 = X_3 = \xi$ in (5.3), we obtain

$$-\lambda\eta(X_1) = 0. \quad (5.4)$$

Since $\eta(X_1) \neq 0$, we have $\lambda = 0$.

Therefore, the Ricci soliton is steady.

Theorem 5.2. Let M be an Lorentzian para-Kenmotsu manifold, admitting a Ricci soliton (g, V, λ). Then Q and S are parallel along ξ, where Q is the Ricci operator, defined by $S(X_1, X_2) = g(QX_1, X_2)$ and S is the Ricci tensor of M.

Proof: We can express the equations for the Ricci operator and Ricci tensor along ξ as follows,

$$(\nabla_\xi Q)X_1 = \nabla_\xi Q(X_1) - Q(\nabla_\xi X_1) \tag{5.5}$$

and

$$(\nabla_\xi S)(X_1, X_2) = \nabla_\xi S(X_1, X_2) - S(\nabla_\xi X_1, X_2) - S(X_1, \nabla_\xi X_2). \tag{5.6}$$

Using (2.16) in (5.5), we can simplify to obtain

$$(\nabla_\xi Q)X_1 = 0. \tag{5.7}$$

Similarly, applying (2.15) to (5.6), we get

$$(\nabla_\xi S)(X_1, X_2) = 0. \tag{5.8}$$

Therefore, from (5.7) and (5.8), we can conclude that Q and S are parallel along ξ, which completes the proof.

5. Ricci soliton in a Lorentzian para-Kenmotsu manifold and its curvature properties

In this section, we explore some curvature properties of a Lorentzian para-Kenmotsu manifold admitting a Ricci soliton.

Definition 6.1. [19] In a Lorentzian para-Kenmotsu manifold M, the projective curvature P of the manifold is defined as

$$P(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{2n}[S(X_2, X_3)\, X_1 - S(X_1, X_3)\, X_2], \tag{6.1}$$

for all vector fields X_1, X_2 and X_3 on M.

Proposition 6.1. A Lorentzian para-Kenmotsu manifold M, admitting a Ricci soliton (g, V, λ) is ξ-projectively flat iff the soliton is shrinking.

Proof: Putting $X_3 = \xi$ in (6.1) and by using (2.10) and (2.11), we get

$$P(X_1, X_2)\xi = \left[\frac{2n+\lambda}{2n}\right][\eta(X_2)\, X_1 - \eta(X_1)\, X_2]. \tag{6.2}$$

This implies that $P(X_1, X_2)\xi = 0$ if and only if $\lambda = -2n$, which proves the proposition.

Definition 6.2. [13] In a Lorentzian para-Kenmotsu manifold M, the concircular curvature C of the manifold is defined as

$$C(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{r}{2n(2n+1)}[g(X_2, X_3)X_1 - g(X_1, X_3)X_2]. \tag{6.3}$$

Proposition 6.2. A Lorentzian para-Kenmotsu manifold M, admitting a Ricci soliton (g, V, λ) is ξ-concircularly flat if the soliton is shrinking.

Proof: Setting $X_3 = \xi$ in (6.3) and using the equations (2.3) and (2.16), we get

$$C(X_1,X_2)X_3 = \left[\frac{4n^2+\lambda(2n+1)}{2n(2n+1)}\right][\eta(X_2)X_1 - \eta(X_1)X_2]. \tag{6.4}$$

This shows that $C(X_1,X_2)\xi = 0$ if and only if $\lambda = -\frac{4n^2}{2n+1}$.

Hence, we prove the proposition.

Definition 6.3. [9] In a Lorentzian para-Kenmotsu manifold M, the conharmonic curvature tensor H is defined as

$$H(X_1,X_2)X_3 = R(X_1,X_2)X_3 - \frac{1}{2n-1}[g(X_2,X_3)QX_1 - g(X_1,X_3)QX_2]$$
$$+S(X_2,X_3)X_1 - S(X_1,X_3)X_2]. \tag{6.5}$$

Proposition 6.3. A Lorentzian para-Kenmotsu manifold M, admitting a Ricci soliton (g,V,λ) is ξ-conharmonically flat if the soliton is shrinking.

Proof: Putting $X_3 = \xi$ in (6.5), we obtain

$$H(X_1,X_2)\xi = R(X_1,X_2)\xi - \frac{1}{2n-1}[g(X_2,\xi)QX_1 - g(X_1,\xi)QX_2$$
$$+S(X_2,\xi)X_1 - S(X_1,\xi)X_2]. \tag{6.6}$$

Using (2.3), (2.16) and (2.17) in (6.6), we have

$$H(X_1,X_2)\xi = \frac{2n-2+2\lambda}{2n-1}[\eta(X_2)X_1 - \eta(X_1)X_2]. \tag{6.7}$$

Thus, $H(X_1,X_2)\xi = 0$ if and only if $\lambda = -(n-1)$.

Hence, the proof is completed.

Definition 6.4. [3] In a Lorentzian para-Kenmotsu manifold M, the Weyl conformal curvature tensor W is defined as

$$W(X_1,X_2)X_3 = R(X_1,X_2)X_3 - \frac{1}{2n-1}[g(X_2,X_3)QX_1 - g(X_1,X_3)QX_2$$
$$+S(X_2,X_3)X_1 - S(X_1,X_3)X_2]$$
$$+\frac{r}{2n(2n-1)}[g(X_2,X_3)X_1 - g(X_1,X_3)X_2]. \tag{6.8}$$

Proposition 6.4. A Lorentzian para-Kenmotsu manifold M, admitting a Ricci soliton (g,V,λ) is ξ-conformally flat if the soliton is shrinking.

Proof: Putting $X_3 = \xi$ in (6.8), we obtain

$$W(X_1,X_2)\xi = R(X_1,X_2)\xi - \frac{r}{2n-1}[g(X_2,\xi)QX_1$$
$$-g(X_1,\xi)QX_2 + S(X_2,\xi)X_1 - S(X_1,\xi)X_2]$$
$$+\frac{r}{2n(2n-1)}[g(X_2,\xi)X_1 - g(X_1,\xi)X_2]. \tag{6.9}$$

Using (2.3), (2.16) and (2.17) in (6.9), we have

$$W(X_1, X_2)\xi = \frac{2n+\lambda}{2n}[\eta(X_2)X_1 - \eta(X_1)X_2]. \tag{6.10}$$

This implies that $W(X_1, X_2)\xi = 0$ if and only if $\lambda = -2n$.

This completes the proof of the proposition.

6. Example of a Lorentzian para-Kenmotsu manifold

In this section we establish an example of a Lorentzian para-Kenmotsu manifold. We consider the 3-dimensional manifold $M = \{(x_1, x_2, x_3) \in R^3 : x_3 \neq 0\}$, where (x_1, x_2, x_3) are the standard coordinates in R^3. Let E_1, E_2 and E_3 be a linearly independent vector fields in M which satisfy

$$[E_1, E_2] = E_2, \qquad [E_2, E_3] = 0, \qquad [E_1, E_3] = E_3.$$

Let g be the Lorentzian metric defined by

$$g(E_1, E_1) = -1,\ g(E_2, E_2) = g(E_3, E_3) = 1,$$

$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0.$$

Let η be the 1-form defined by $\eta(X_1) = g(X_1, E_3)$, for any vector field X_1.

Let ϕ be (1,1)-tensor field defined by

$$\phi E_1 = 0,\ \phi E_2 = E_3,\ \phi E_3 = E_2.$$

Then we have

$$\eta(E_1) = -1,\ \phi^2(X_1) = X_1 + \eta(X_1)E_1$$

and

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2).$$

Thus for $\xi = E_1$, (ϕ, ξ, η, g) defines a Lorentzian almost paracontact metric structure on M. Let ∇ be the Levi-Civita connection of the Lorentzian metric g. Then using Koszul's formula, we obtain

$$\nabla_{E_1}E_1 = 0, \nabla_{E_1}E_2 = 0, \nabla_{E_1}E_3 = 0,$$

$$\nabla_{E_2}E_1 = -E_2, \nabla_{E_2}E_2 = -E_1, \nabla_{E_2}E_3 = 0,$$

$$\nabla_{E_3}E_1 = -E_3, \nabla_{E_3}E_2 = 0, \nabla_{E_3}E_3 = -E_1.$$

From the above calculation, one can easily verify that

$$\nabla_{X_1}\xi = -\phi^2 X_1,$$

and

$$(\nabla_{X_1}\phi)X_2 = -g(\phi X_1, X_2) - \eta(X_2)\phi X_1.$$

Therefore, the manifold (M, g, ξ, ϕ, η) is a Lorentzian para-Kenmotsu manifold.

On this manifold (M, g, ξ, ϕ, η), we can easily verify our results.

Conclusion: The study suggests that in a Lorentzian para-Kenmotsu manifold, a symmetric parallel second-order covariant tensor is proportional to the metric tensor. Additionally, if $\mathscr{L}_V g + 2S$ is parallel, where V is a vector field, then (g, V, λ) is a Ricci soliton. The study further states that a Ricci soliton in a W_2-semi-symmetric Lorentzian para-Kenmotsu manifold is shrinking, whereas a Ricci soliton satisfying the condition $W_2(\xi, X_1) \circ S = 0$ in a Lorentzian para-Kenmotsu manifold is steady. The study concludes by analyzing certain curvature properties of Lorentzian para-Kenmotsu manifolds admitting Ricci soliton and provides an example of a 3-dimensional Lorentzian para-Kenmotsu manifold.

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