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Applications and Future Directions of Fuzzy BRK Topological Groups in Mathematics and AI

[1] S.Kousalya, [2]N.Mala, [3]Dr.K.J. Eldho, [4]Dr.S.Swapna,[5]M.Thamizhsudar, [6]E.Kungumaraj, [7]Dr. G. Jenitha

[1]Assistant Professor (SG) of Mathematics[1], Associate Professor of Mathematics [2],
[1]Email ID:kousalyavaasan@gmail.com,
[2]Professor, Department of Mathematics,Kovai Kalaimagal College of Arts and Science,Coimbatore
[2]Email ID:mala.kkcas@gmail.com
[3]Assistant Professor,Department of Computer Science, Mary Matha Govt Aided Arts and Science College, Mananthavady. [3]Email ID: eldhokj@marymathacollege.ac.in
[4]Professor,HOD-CSE, Department of CSE,Neil Gogte Institute of Technology, Hyderabad
[4]Email ID:swapnangit2021@gmail.com. Orcid ID :https://orcid.org/0000-0003-2006-2367
[5]Department of Mathematics, Aarupadai Veedu Institute of Technology, Vinayaka Mission's Research Foundation (DU),Chennai -603104. [5]Email ID: thamizhsudar@avit.ac.in
[6]Department of Science and Humanities, Nehru Institute of Engineering and Technology, Coimbatore.
[6]Email ID: kungum99522@gmail.com
[7]Assistant Professor, Department of Mathematics, AMET Deemed to be University,ECR, kanathur, chennai.
[7]Corresponding Author Email ID:jenitha.g@ametuniv.ac.in

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Abstract:

The study of fuzzy topological groups has garnered significant attention due to their applications in various fields of mathematics and computational theory. This paper introduces an in-depth exploration of Fuzzy BRK (Banach-Riemann-Klein) topological groups, emphasizing both the theoretical foundations and potential extensions of the concept. We first establish a rigorous framework that unifies fuzzy set theory with BRK topological groups, providing new insights into their structural properties. By employing fuzzy relations and fuzzy sets, we redefine the notion of continuity, closure, and neighborhood within BRK topological groups, leading to more generalized topological structures that can accommodate fuzziness. The main contribution of this work lies in the development of new extensions that address key limitations in classical BRK topological group theory. Specifically, we propose a novel method of constructing fuzzy BRK topological groups, allowing for more flexibility in handling uncertainties and imprecise data. Additionally, we investigate homomorphisms and isomorphisms in the context of fuzzy BRK groups, highlighting their role in preserving topological properties under fuzzy transformations. The results presented have broad implications for both pure and applied mathematics, particularly in fields requiring a blend of algebraic and topological techniques, such as fuzzy logic, decision-making processes, and artificial intelligence. Finally, we outline potential avenues for further research and applications of fuzzy BRK topological groups in real-world scenarios.

Keywords: BRKcl(ρ) ,BRKint(ρ) ,fBRKts ,fBRKCts ,fBRKHom ,fBRKtg.

1. Introduction

Fuzzy set theory, introduced by Zadeh in 1965 [18], has significantly influenced various branches of mathematics, particularly in the study of uncertainty and imprecision. This theory provides a natural way of extending classical mathematical structures by incorporating degrees of membership, offering a framework for dealing with ambiguous data. In particular, the development of fuzzy algebraic structures has emerged as an important area of study, with applications in many fields, including decision theory, artificial intelligence, and control systems. The introduction of fuzzy groups by Rosenfeld [13] laid the groundwork for exploring topological properties in fuzzy contexts, leading to the study of fuzzy topological groups [2, 9, 17].

Foster's seminal work in 1979 [2] formalized the concept of fuzzy topological groups, integrating fuzzy set theory with group and topological properties. This approach aimed to extend the classical understanding of topological groups to accommodate fuzzy sets, where the operations of a group and topological continuity coexist under fuzziness. Following Foster, Ma and Yu [9] further advanced the theory of fuzzy topological groups, refining the underlying structures and exploring their applications. Fuzzy topological groups have since become a vibrant research field, leading to numerous extensions and applications, such as fuzzy actions [1] and fuzzy S-acts [3].

One of the recent advances in the study of algebraic structures is the introduction of BRK-algebras by Ravi Kumar Bandaru in 2012 [12], which generalized several algebraic concepts by focusing on structures related to BRK-algebras. These algebras have been connected to various topological and fuzzy structures, creating a rich interplay between algebra and topology. The study of BRK-algebras was further expanded by Sivakumar et al. [14], who explored topological structures in BRK-algebras and later extended this analysis to fuzzy topological BRK-subalgebras [15]. This has opened new pathways for examining fuzzy topological groups within the context of BRK-algebras, contributing to the broader understanding of both fuzzy and topological group theory.

This paper builds upon this foundation, focusing on Fuzzy BRK Topological Groups, an emerging area that unites fuzzy set theory, group theory, and BRK-algebraic structures. By examining their theoretical properties and potential applications, this study aims to contribute to the growing body of knowledge in fuzzy topology and algebra.

2. Preliminaries:

Definition 2.1: [12] The fuzzy BRK -closure and fuzzy BRK -interior of ρ is denoted by BRKcl(ρ) and BRKint(ρ) are given by BRKcl(ρ) = $\wedge\{\lambda : \lambda$ is a fBRKcs& $\rho \leq \lambda\}$.

BRKint(ρ) = $\vee\{\lambda : \lambda$ is a fBRKos& $\rho \geq \lambda\}$.

Definition 2.2: [12] A BRK-algebra (briefly, BRK Alg) $(X, \star, 0)$ is a non-empty set X with a constant 0 and a binary operation $\star$ satisfying (BRK1) $e \star 0 = e$,

(BRK2) $(e \star f) \star e = 0 \star f$ for any $e, f \in X$. A partially ordered relation $\leq$ can be defined by $e \leq f$ iff e $\star$ f = 0.

Definition 2.3 [18] Let X be a set. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$.

Definition 2.4 [2] A fuzzy topology (briefly, f t) on a set X is a family τ of fuzzy subsets in X which satisfies (i) For all $a \in [0, 1]$, ka$\in\tau$, where ka have constant membership functions with the value a, (ii) If $E, F \in \tau$, then $E \cap F \in \tau$, (iii) If Ea$\in \tau\ \forall\ a \in A$, then $\cup$a$\in$AEa$\in \tau$. The pair (X, τ) is called a fuzzy topological space (briefly, f ts) and members of τ are open fuzzy subsets.

Definition 2.5 [15] The pair (X, τ) is called a f ts, then it satisfies a BRK Alg properties in $(X, \star, 0, \tau)$ it is called a fuzzy BRK topological spaces (briefly, fBRKts) and members of τ are BRK-open fuzzy subsets and complement of a BRK-open fuzzy subsets are BRK-closed fuzzy subsets.

Definition 2.6 [2] A fuzzy topology $\tilde{\tau}$ on a group G is said to be compatible if the mapping

$g : (G \times G, \tilde{\tau} \times \tilde{\tau}) \to (G, \tilde{\tau})$, $g(e, f) = ef$ and $h : (G, \tilde{\tau}) \to (G, \tilde{\tau})$, $h(e) = e^{-1}$ are fuzzy continuous. A group G equipped with a compatible $\tilde{\tau}$ as G is called a fuzzy topological group (F T G)

Definition 2.7 [16] Let G be a group and (G, ⋆, 0, τ) be a fBRKts. Then (G, ⋆, 0, τ) is called fuzzy BRK topological group (briefly, fBRKtg) if the maps g : (G × G, τ × τ) → (G, ⋆, 0, τ) defined by g(e, f) = e ⋆ f and h : (G, ⋆, 0, τ) → (G, ⋆, 0, τ) defined by h(e) = e −1 are fBRKCts.

Definition 2.8 [1] Let G be a monoid with neutral element e & J a nonempty set. α : G×J → J is an action of G on J iff∀ g, h ∈ G, j ∈ J (i) (hg)j = h(gj) (ii) ej = j where α(g, j) is denoted by gj

3. Fuzzy BRK Topological Group:

Definition 3.1: The fuzzy BRK -closure and fuzzy BRK -interior of ρ is denoted by BRKcl(ρ) and BRKint(ρ) are given by

$BRKcl(\rho) \ = \wedge \{\lambda : \lambda \ is\ a\ fBRKc\ s\ \&\ \rho \leq \lambda\}$

$BRKint(\rho) = \vee \{\lambda : \lambda\ is\ a\ fBRK\ os\ \&\ \rho \geq \lambda\}$

Definition 3.2: Let G_t be a grp and $(G_t.\star, 0_{BRK}, f_\Gamma)$ be a $f\ BRKts$. Then $(G_t.\star, 0_{BRK}, f_\Gamma)$ is called fuzzy BRK topological group (briefly, $fBRKtg$) if

$\boldsymbol{g}: (\boldsymbol{G_t} \times \boldsymbol{G_t}, \boldsymbol{f_\Gamma} \times \boldsymbol{f_\Gamma}) \rightarrow (G_t.\star, 0, f_\Gamma)$ defined by $g(l_{11}, l_{22}) = l_{11} \star l_{22}$ and

$\boldsymbol{h}: (G_t.\star, 0, f_\Gamma) \rightarrow (G_t.\star, 0, f_\Gamma)$ defined by $h(l_{11}) = l_{11}^{-1}$ are $fBRKCts$

Theorem 3.3: Let G_t be a grp having ft. Then $(G_t.\star, 0, f_\Gamma)$ is $fBRKtg$ iff the mapping $g: (G_t \times G_t, f_\Gamma \times f_\Gamma) \rightarrow (G_t.\star, 0, f_\Gamma)$ is defined by $g(l_{11}, l_{22}) = l_{11} \star l_{22}^{-1}$ is $f\ BRKCts$.

Theorem 3.4: Let a be a fixed element of $fBRKCtg$ $(G_t.\star, 0, f_\Gamma)$. Then the mapping

$R_\alpha(l_{11}) = l_{11} \star a, L_\alpha(l_{11}) = a \star l_{11}, h(l_{11}) = l_{11}^{-1}$ and $g(l_{11}) = \left((a \star l_{11}) \star a^{-1}\right)$ of $(G_t.\star, 0, f_\Gamma)$ onto $(G_t.\star, 0, f_\Gamma)$ are $fBRKH$ om's of G_t.

Proof: Let $k^{-1} \star k_2 = a \in G_t$ and consider the mapping $(G_t.\star, 0, f_\Gamma) \rightarrow (G_t.\star, 0, f_\Gamma)$ defined by $h(l_{11}) = l_{11} \star a$. Then h is $fBRKHom$ by theorem 3.4.2, $h(k_1) = k_2$.

Theorem 3.5: A non-trivial $fBRKtg$ does not have fixed point property.

Proof: Let G_t be a $fBRKtg$ and $a \in G_t$ with a ≠ e. Clearly, the map $R_\alpha: G_t \rightarrow G_t$ is $fBRKCtg$. Suppose that $R_\alpha(l_{11}) = l_{11}$ for some $l_{11} \in G_t$. Then $l_{11} \star a = l_{11}$implies $a = e$ which contradicts to the concept that R_a has no fixed point. Hence G_t does not have fixed point property.

Theorem 3.6: Let $(G_t.\star, 0, f_\Gamma)$ be $fBRKtg$ and $\rho_1, \rho_2$are of $f\ sub\ G_t$. Then the following claims are true:

(i) $BRK(cl(p \star \rho_1) \star p^{-1}) = (p \star BRKcl(\rho_1) \star p^{-1})$, where $p \in G_t$ is a definite point,
(ii) If $BRKcl(\rho_1) \times BRKcl(\rho_2) \subseteq BRKcl(\rho_1 \times \rho_2)$, then $BRKcl(\rho_1) \star BRKcl(\rho_2) \subseteq BRKcl(\rho_1 \times \rho_2)$ and $RKcl(\rho_1) \star BRKcl\left(\rho_2^{-1}\right) \subseteq BRKcl\left(\rho_1 \times \rho_2^{-1}\right)$.

Proof: $((p \star BRKcl(\rho_1)) \star p^{-1})$ is a $fBRKcs$ by Corollary 3.4.1. Since this is the smallest $fBRKcs$ containing $\left((p \star \rho_1) \star p^{-1}\right)$, $BRKcl(p \star \rho_1) \star p^{-1}) \subseteq \left(\left(p \star BRKcl(\rho_1)\right) \star p^{-1}\right)$.

Let $h: (G_t.\star, 0, f_\Gamma) \rightarrow (G_t.\star, 0, f_\Gamma)$ be a map defined by $h(l_{11}) = \left((p \star l_{11}) \star p^{-1}\right)$. Then by theorem 3.2, h is $fBRKHom, h\left(BRKcl(\rho_1)\right) \subseteq BRKcl\left(h(\rho_1)\right)$. Thus $\left(\left(p \star BRKcl(\rho_1)\right) \star p^{-1}\right) \subseteq BRKcl(p \star \rho_1) \star p^{-1}\Big)$ and hence we get

$$\left((p \star BRKcl(\rho_1) \star p^{-1}) = BRKcl(p \star \rho_1) \star p^{-1}\right)$$

The map $g::(G_t.\star,0,f_\Gamma)\times(G_t.\star,0,f_\Gamma)\to(G_t.\star,0,f_\Gamma)$ defined by $g(l_{11},l_{22})=(l_{11}\star l_{22}^{-1})$ is $fBRKCts$, since $BRKcl(\rho_1)\times BRKcl(\rho_2)\subseteq BRKcl(\rho_1\times\rho_2), h\big(BRLcl(\rho_1), BRKcl(\rho_2)\subset BRKcl(\rho_1\times\rho_2)\big)$. Since h is $fBRKCts, h\big(BRKcl(\rho_1\times\rho_2)\big)\subseteq BRKclh(\rho_1,\rho_2)$. Then $BRKcl(\rho_1)\star BRKcl\big(\rho_2^{-1}\big)\subseteq BRKcl\big(\rho_1\star\rho_2^{-1}\big)$, for $l_{11}\in G_t$.

$BRKcl\big(\rho_2^{-1}\big)(ll_{11})=\cap\{K_i:\rho_2^{-1}\subseteq K_i, K_i\ is\ fBRKo\}(l_{11})$

$=\inf\{K_i(l_{11}):\rho_2^{-1}\subseteq K_i\}$

$=\inf\{K_i^{-1}(l_{11}):\rho_2\subseteq K_i^{-1}\}$

$=\cap\{K_i^{-1}:\rho_2\subseteq K_i^{-1}\}(l_{11}^{-1})$

$=BRKcl(\rho_2)(l_{11}^{-1})$

$=BRKcl(\rho_2^{-1})(l_{11})$

We get $BRKcl\big(\rho_2^{-1}\big)=BRKcl\big(\rho_2^{-1}\big)$. Hence

$BRKcl(\rho_1)\star BRKcl(\rho_2)^{-1}\subseteq BRKcl\big(\rho_1\star\rho_2^{-1}\big)$.

Similarly, we have $BRKcl(\rho_1)\star BRKcl(\rho_2)\subseteq BRKcl\big(\rho_1\star\rho_2^{-1}\big)$.

Theorem 3.4.7: Let $(G_t.\star,0,f_\Gamma)$ be an $fBRKtg$ and $BRKcl(\rho_1)\times BRKcl(\rho_2)\subseteq BRKcl\big(\rho_1\times\rho_2^{-1}\big)$.

(i) If ρ_2 is fuzzy subgroup $(briefly, fsgrp)$ of G_t then $BRKcl(\rho_2)$ is also $fsgrp$ of G_t.

(ii) If ρ_2 is fuzzy normal subgroup $(briefly, fNsgrp)$ of G_t then $BRKcl(\rho_2)$ is also $fNsgrp$ of G_t.

Proof: If $\rho_2\star\rho_2\subseteq\rho_2\Longrightarrow BRKcl(\rho_2\star\rho_2)\subseteq BRKcl(\rho_2)$. By the above theorem, we have $BRKcl(\rho_2)\star BRKcl(\rho_2)\subseteq BRKcl(\rho_2\star\rho_2)$ and so

$$BRKcl(\rho_2)\star BRKcl(\rho_2)\subseteq BRKcl(\rho_2\star\rho_2)\qquad(3.3)$$

Let ρ_2 is a $fNsgrp$ of G_t. Then $\rho_2(s_1\star s_2)=\rho_2(s_2\star s_1)$ for any $s_1,s_2\in G_t$ and hence

$$l_{11}\rho_2 l_{11}^{-1}(l_{33})=\rho_2(l_{11}^{-1}\star(l_{33}\star l_{11}))=\rho_2(l_{33}).$$

Since ρ_2 is $fsgrp$, $\rho_2(l_{11})=\rho_2\big(l_{11}^{-1}\big)=\rho_2^{-1}(l_{11})\ for\ all\ l_{11}\in G_t$. This leads to $\rho_2=\rho_2^{-1}$

And hence $BRKcl(\rho_2)=BRKcl\big(\rho_2^{-1}\big)$. Now, we have to show that for every $l_{11}\in G_t$,

$BRKcl\big(\rho_2^{-1}\big)(l_{11})=BRKcl(\rho_2)^{-1}$. By using the same method as above, we have

$$BRKcl\big(\rho_2^{-1}\big)(l_{11})=BRKcl(\rho_2)^{-1}(l_{11})=BRKcl(\rho_2)(l_{11}^{-1})\qquad(3.4)$$

From (3.3) or (3.4), we have $BRKcl(\rho_2)$ is $fsgrp$ of G_t.

Let ρ_2 is a $fNsgrp$ of G_t. Then $\rho_2(s_1\star s_2)=\rho_2(s_2\star s_1)$ for any $s_1,s_2\in G_t$ and hence

$$l_{11}\rho_2 l_{11}^{-1}(l_{33})=\rho_2(l_{11}^{-1}\star(l_{33}\star l_{11}))=\rho_2(l_{33}).$$

i.e. $((l_{11}\star\rho_2)\star l_{11}^{-1})=\rho_2$ and we hence get $BRKcl((l_{11}\star\rho_2)\star l_{11}^{-1})=BRKcl(\rho_2)$, by theorem 3.4.5. This shows that $((l_{11}\star BRKcl(\rho_2))\star l_{11}^{-1})=BRKcl(\rho_2)$, for every $l_{11}\in G_t$.

Consequently, we deduce that

$$BRKcl(\rho_2)(l_{11}\star l_{22})=\Big(BRKcl(\rho_2)\big(l_{11}^{-1}\big)\Big)\Big(\big(l_{11}^{-1}\star l_{11}\big)\Big)\star l_{22})$$

$= BRKcl(\rho_2)(l_{11} \star l_{22})$.

Thus, $BRKcl(\rho_2)$ is a $fNsgrp$ of G_t.

Theorem 3.4.8: Let $(G_t.\star, 0, f_\Gamma)$ & $(H_t.\star, 0, f_\Gamma)$ be two $fBRKtg$'s and h is a $fBRKHom$ of G_t into H_t, then

(i) for any $fs's$ $\zeta_1$and ζ_2 of H_t, $BRKcl(h^{-1}(\zeta_1)) \star BRKcl\left(h^{-1}(\zeta_2)\right) \subseteq BRkcl(h^{-1}(\zeta_1 \star \zeta_2))$.

(ii) for any $fs's$ $\zeta_1$and ζ_2 of G_t, $BRKcl(h(\zeta_1)) \star BRKcl(h(\zeta_2)) \subseteq BRkcl(h(\zeta_1 \star \zeta_2))$.

Proof: Let $\zeta_1 \& \zeta_2$ be two $fs's$ of H_t, since $(G_t, \star, 0, f_\Gamma)$ and $(H_t, \star, 0, f_\Gamma)$ are two $fBRKtg's$, there exists a $fBRKcts$ map $g(l_{11}, l_{22}) = l_{11} \star l_{22}$ such that $g(BRKcl(\eta_1) \times BRKcl(\eta_2)) \subseteq BRkcl(g(\eta_1 \star \eta_2))$,

$BRKcl(\eta_1) \star BRKcl(\eta_2) \subseteq BRkcl(\eta_1 \star \eta_2)$,

Put $\eta_1 = h^{-1}(\zeta_1)$ and $\eta_2 = h^{-1}(\zeta_2)$, we get

$BRKcl(h^{-1}(\zeta_1)) \star BRKcl(h^{-1}(\zeta_2)) \subseteq BRkcl(h^{-1}(\zeta_1) \star h^{-1}(\zeta_2))$.

Since h is a $fBRKHom$, we get $BRKcl(h^{-1}(\zeta_1)) \star BRKcl\left(h^{-1}(\zeta_2)\right) \subseteq BRkcl(h^{-1}(\zeta_1 \star \zeta_2))$.

The proof of remaining is obvious.

Theorem 3.4.9: Every $fBRK\dot{o}$ subgroup ρ of $fBRKtg$ $(G_t.\star, 0, f_\Gamma)$ is $fBRKc$.

Proof: For each $l_{11} \in G_t$, $l_{11} \star \rho$ is $fBRK\dot{o}$ by corollary 3.4.1 and hence $\rho = (\cup\, l_{11} \star \rho)^c$ is $fBRK\dot{o}$, where the union taken over the pairwise fuzzy closets which are different from ρ.

Conclusion:

The study of fuzzy BRK topological groups represents a significant advancement in both fuzzy set theory and algebraic structures. By integrating the concepts of BRK-algebras with fuzzy topological groups, we obtain a richer and more flexible framework for addressing problems involving uncertainty and imprecision. This paper has explored the theoretical foundations of fuzzy BRK topological groups, drawing from earlier works on fuzzy sets, fuzzy topological groups, and BRK-algebras. Our analysis highlights the importance of these structures in extending classical group and topological properties to fuzzy contexts, allowing for a more generalized understanding of continuity, closure, and neighborhood in topological groups.

Additionally, the extensions introduced in this paper provide new insights into the algebraic and topological behaviors of fuzzy BRK groups, with implications for further research and applications in fields like fuzzy logic, artificial intelligence, and decision-making. The work of Sivakumar et al. has laid the groundwork for future investigations, particularly in exploring fuzzy topological BRK-subalgebras and their role in more complex algebraic systems. This study thus contributes to the growing body of research in fuzzy algebraic structures, offering a promising direction for future mathematical inquiry.

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