

## $(\alpha, \beta)$ -Metric on Cartan Space

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### Article History:

**Received:** 10-09-2024

**Revised:** 15-11-2024

**Accepted:** 25-11-2024

### Abstract:

In Finsler geometry, a Cartan space or Cartan manifold refers to a special type of geometrical structure where there is a preferred connection or curvature, often associated with a homogeneous space that carries additional symmetries. In the present research paper, we have deduced necessary and sufficient conditions under which an  $(\alpha, \beta)$ -metric,  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - k^2/3 \frac{\beta^4(x, \omega)}{\alpha^3(x, \omega)}$ , on a Cartan space admitting  $h$ -metrical  $d$ -connection becomes a locally Minkowski and conformally flat space.

**Keyword:** Cartan Space,  $(\alpha, \beta)$ -metric, Minkowski space, Conformally flat space.

**AMS Subject Classification:** 53B40, 53C60

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## 1 Introduction

The term "Cartan space" can refer to spaces that have a particular kind of connection (typically the Cartan connection) or spaces that have been generalized to accommodate more complex geometrical structures. This space was founded by E. Cartan a French mathematician and geometer [2]. Cartan space is the dual of a Finsler space [6] and this dual space was defined using a linear functional named as Legendre transformation. The relation between Cartan space and Finsler space has been studied by F. Brickell [1], H. Rund [10] and others. R. Miron ([6], [7]) introduced the theory of Hamiltonian space, He proved that Cartan space is a particular case of Hamilton space. The notion of  $(\alpha, \beta)$ -metric in Cartan space was introduced by T. Igrashi ([4], [3]). He obtained the metric tensors and some invariants which characterize the special class of Cartan spaces with  $(\alpha, \beta)$ -metric. H.G. Nagaraja [8], G. Shanker [11], M. Rafee ([9], [15]) and Tripathi [13] have also made significant development in the theory of Cartan spaces with  $(\alpha, \beta)$ -metric. The paper is organized as follows:

In Section 2, we give basic definitions and results required for subsequent sections. In Section 3, we deal with Cartan space with an  $(\alpha, \beta)$  -metric,  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2 \beta^4(x, \omega)}{3 \alpha^3(x, \omega)}$ , admitting  $h$ -metrical  $d$ -connection. In Section 4, we study the conformal change of Cartan space and find some important results.

## 2 Preliminaries

We recall some important definitions like Finsler metric in cotangent bundle, Cartan space etc:

### Definition 2.1 (Finsler metric of cotangent bundle)

Let  $M$  be a smooth manifold and  $T^*M$  be its cotangent bundle. A  $C^\infty$  function  $K: T^*M \setminus \{0\} \rightarrow R$  is

called Finsler metric or Finsler fundamental function on the cotangent bundle  $T^*M$  if it satisfies the following properties:

- (1) Positivity:  $K(x, \omega) \geq 0$  for all  $\omega \in T_p^*M$ .
- (2) Positive Homogeneity:  $K(x, \omega)$  is +ve 1-homogeneous on the fibers of the cotangent bundle  $T^*M$ , i.e.,  $K(x, \lambda\omega) = \lambda K(x, \omega)$ ,  $\forall \lambda > 0$ ; for any  $x \in M$ ,  $\omega \in T_x^*M$ .
- (3) Strict Convexity of  $K(x, \omega)$ : The hessian matrix defined by  $g^{ij}(x, \omega) = \frac{1}{2} \frac{\partial^2 K^2}{\partial \omega^i \partial \omega^j}(x, \omega)$  is positive definite for all  $(x, \omega) \in T^*M \setminus \{0\}$ .

**Definition 2.2 (Cartan Space)**

A differentiable manifold  $M$  equipped with a Finsler metric  $K(x, \omega)$  defined on the cotangent bundle  $T^*M$  is called a Cartan space..

Cartan space is denoted by  $C = (M, K(x, \omega))$ , where  $K(x, \omega)$  represents norm of the differential one form  $\omega \in T_x^*M$  based at any point  $x \in M$ . The function  $K(x, \omega)$  is called the fundamental function and  $g^{ij}(x, \omega) = \frac{1}{2} \frac{\partial^2 K^2}{\partial \omega^i \partial \omega^j}(x, \omega)$  is called the fundamental metric tensor of the Cartan space  $C$ . In Cartan space the metric  $K: T^*M \rightarrow [0, \infty)$  is defined from cotangent bundle  $T^*M$  to non-negative real numbers, so at a point  $x \in M$ ,  $K(x, -)$  eats one-form  $\omega \in T_p^*M$  and spits non-negative reals, amounts to saying that Cartan space is constructed on the cotangent bundle  $T^*M$  in the same way a Finsler space  $(M, F(x, y))$ , where  $F: TM \rightarrow [0, \infty)$ , is constructed on the tangent bundle  $TM$ .

Next we define the norm of a differential one form  $\omega \in T_p^*M$  in local coordinates or in terms of fundamental metric tensor  $g^{ij}$  of the corresponding Cartan space  $(M, K(x, \omega))$ .

**Definition 2.3 (Finsler norm of a differential one form)**

Let  $(M, K(x, \omega))$  be a Cartan space, where  $K(x, \omega)$  is a Finsler metric on the cotangent bundle  $T^*M$ . Then the norm of a differential one form  $\omega \in T_p^*M$  at any fixed point  $x \in M$  is denoted by  $K_x(\omega)$  and defined by

$$K_x^2(\omega) = \frac{1}{2} \frac{\partial^2 K^2}{\partial \omega^i \partial \omega^j}(x, \omega) \omega_i \omega_j$$

$$= g^{ij}(x, \omega) \omega_i \omega_j,$$

where  $g^{ij}(x, \omega) = \frac{1}{2} \frac{\partial^2 K^2}{\partial \omega^i \partial \omega^j}(x, \omega)$  is the fundamental metric tensor of the Finsler metric  $K(x, \omega)$  of cotangent bundle.

Moreover, when referring to a Cartan space with an  $(\alpha, \beta)$ -metric, this could mean the Cartan space is endowed with a metric that is parametrized by two parameters, typically denoted as  $\alpha$  and  $\beta$ . This type of metric is used in various contexts, including in the study of spacetimes or in certain models of differential geometry and gravity, such as generalized theories of relativity. Let us precisely define what is meant by a Cartan space with  $(\alpha, \beta)$ -metric.

**Definition 2.4** If the fundamental function  $K(x, \omega)$  of a Cartan space  $C = (M, K(x, \omega))$  is a function of variables  $\beta(x, \omega) = \omega_i b^i(x)$ , where  $a^{ij}(x)$  is a Riemannian metric and  $b^i(x)$  is a vector field depending only on  $x$ , then  $C$  is called Cartan space with  $(\alpha, \beta)$ -metric. Here it is to be remarked that  $K(x, \omega)$  must satisfy all the conditions imposed on the fundamental function of a Cartan space.

Let us consider a Cartan space  $C = (M, K(x, \omega))$  with an  $(\alpha, \beta)$ -metric,  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2}{3} \frac{\beta^4(x, \omega)}{\alpha^3(x, \omega)}$ , where  $\alpha = (a^{ij}(x, \omega)\omega_i\omega_j)^{\frac{1}{2}}$  and  $\beta = \omega_i b^i(x)$ .

The fundamental tensor  $g^{ij}(x, \omega)$  and its reciprocal tensor  $g_{ij}(x, \omega)$  of the Cartan space  $C = (M, K(\alpha, \beta))$  are given by [3]

$$g^{ij} = \rho a^{ij} + \rho_0 b^i b^j + \rho_{-1}(b^i \omega_j + b^j \omega_i) + \rho_{-2} \omega_i \omega_j, \tag{1}$$

where  $\rho, \rho_0, \rho_{-1}$  and  $\rho_{-2}$  are invariants which are defined and calculated as follows:

$$\begin{aligned} \rho &= \frac{1}{2\alpha} K_\alpha \\ &= \frac{\alpha^5 - 2k\alpha^2\beta^2 + k^2\beta^4}{2\alpha^5} \\ \rho_0 &= \frac{1}{2} K_{\beta\beta} \\ &= \frac{2k\alpha^2 - 2k^2\beta^2}{\alpha^3} \\ \rho_{-1} &= \frac{1}{2\alpha} K_{\alpha\beta} \\ &= -\frac{2k^2\beta^3 - 2k\alpha^2\beta}{\alpha^5} \\ \rho_{-2} &= \frac{1}{2\alpha^2} \left( K_{\alpha\alpha} - \frac{1}{\alpha} K_\alpha \right) \\ &= \frac{6k\alpha^2\beta^2 - 5k^2\beta^4 - \alpha^4}{2\alpha^7} \end{aligned}$$

and

$$g_{ij} = \sigma a_{ij} - \sigma_0 b_i b_j + \sigma_{-1}(b_i \omega_j + b_j \omega_i) + \sigma_{-2} \omega_i \omega_j, \tag{2}$$

where

$$\begin{aligned} \sigma &= \frac{1}{\rho} \\ &= \frac{2\alpha^5}{\alpha^5 - 2k\alpha^2\beta^2 + k^2\beta^4} \\ \sigma_0 &= \frac{\rho_0}{\rho\tau} \\ \tau &= \sigma + \sigma_0 B^2 + \rho_{-1}\beta \\ \sigma_{-1} &= \frac{\rho_{-1}}{\rho\tau} \end{aligned}$$

$$\sigma_{-2} = \frac{\rho_{-2}}{\rho\tau},$$

where  $B^2 = b^i b_j$  and  $B$  represents the norm of the differential form  $\beta(x, \omega) = \omega_i b^i(x) \in T_p^*M$ .

Moreover, the Cartan connection, which is a generalized connection often used to describe homogeneous spaces or spaces with symmetries, the torsion tensor is defined in a similar way, but it incorporates the structure of the Cartan connection itself. A Cartan connection provides a framework to study affine connections on a space that is often non-Riemannian. The Cartan connection is particularly useful for describing curved spaces with torsion and other structural properties, especially in the study of Lie groups, affine manifolds, and non-metric connections.

The Cartan torsion tensor can be thought of as a measure of the failure of the Cartan connection to satisfy the torsion-free condition in a way similar to the standard torsion tensor in Riemannian geometry.

The Cartan torsion tensor  $C^{ijk}$  [5] in the Cartan space with an  $(\alpha, \beta)$ -metric is given by

$$C^{ijk} = -\frac{1}{2} [r_{-1} b^i b^j b^k + \{\rho_{-1} a^{ij} b^k + \rho_{-2} a^{ij} \omega^k + r_{-2} b^i b^j \omega^k + r_{-3} b^i \omega^j \omega^k + i|j|k\} + r_{-4} \omega^i \omega^j \omega^k], \tag{3}$$

where its coefficients  $r_{-1}$ ,  $r_{-2}$ ,  $r_{-3}$  and  $r_{-4}$  are defined and calculated as follows:

$$\begin{aligned} r_{-1} &= \frac{1}{2} K_{\beta\beta\beta} \\ &= \frac{-4k^2\beta^2}{\alpha^3} \\ r_{-2} &= \frac{1}{2\alpha} K_{\alpha\beta\beta} \\ &= \frac{6k^2\beta^2 - 2k\alpha^2}{\alpha^5} \\ r_{-3} &= \frac{1}{2\alpha^2} \left( K_{\alpha\alpha\beta} - \frac{1}{\alpha} K_{\alpha\beta} \right) \\ &= \frac{6k\alpha^2\beta - 10k^2\beta^3}{\alpha^7} \\ r_{-4} &= \frac{1}{2\alpha^3} \left( K_{\alpha\alpha\alpha} - \frac{3}{\alpha} K_{\alpha\alpha} + \frac{3}{\alpha^2} K_{\alpha} \right) \\ &= \frac{35k^2\beta^4 - 30k\alpha^2\beta^2 + 3\alpha^4}{2\alpha^9}. \end{aligned}$$

We use the symole ‘:’ to denote the covariant differentiation with respect to Christoffel symbols  $\gamma_{jk}^i$  constructed from  $a_{ij}$ . Whenever we talk about Christoffel symbols  $\gamma_{jk}^i$  constructed from  $a_{ij}$ , we mean  $\gamma_{jk}^i = \frac{1}{2} a^{li} \left( \frac{\partial a_{kl}}{\partial x^j} + \frac{\partial a^{lj}}{\partial x^k} - \frac{\partial a^{jk}}{\partial x^l} \right)$ . Since  $\omega_{i:k} = 0$  and from Ricci’s theorem of tensor calculus [14] we have  $a_{:k}^{ij} = 0$ , if  $b_{:k}^i = 0$ , then  $g_{:k}^{ij} = 0$ . Also, let  $\Gamma_{jk}^i(p) = \frac{1}{2} g^{ir} (\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$  be the Christoffel symbols constructed from fundamental metric tensor  $g_{ij}(x, \omega)$  of the Cartan space  $(M, K(x, \omega))$ . Now, for the Cartan space  $(M, K(x, \omega))$ , we state canonical  $d$ -connection is a triplet

given by

$$D\Gamma = (N_{jk}, H_{jk}^i, C_i^{jk}),$$

where

$$N_{ij} = \Gamma_{ij}^k \omega_k - \frac{1}{2} \Gamma_{hr}^k \omega_k \omega^r \partial^h g_{ij} \quad (4)$$

$$H_{jk}^i = \frac{1}{2} g^{ir} (\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}) \quad (5)$$

$$C_i^{jk}(x, \omega) = -\frac{1}{2} g_{ir}(x, \omega) \frac{\partial g^{jk}(x, \omega)}{\partial \omega^r} = g_{ir}(x, \omega) C^{rjk}(x, \omega). \quad (6)$$

are respectively called canonical  $N$ -connection, Christoffel symbols and  $d$ -tensor field of type (2,1).

Let  $h$ -covariant derivative with respect to  $D\Gamma$  be denoted by the symbol  $'|_h$ . Then, we have the following definition for later use.

**Definition 2.5 [9]** *An  $h$ -metrical  $d$ -connection on a Cartan space  $C = (M, K(\alpha(x, \omega), \beta(\omega)))$  with  $(\alpha, \beta)$ -metric is a  $d$ -connection,  $D\Gamma$  on  $C$ , satisfying the following properties:*

- (1)  $h$ -deflection tensor  $D_{ij}(= \omega_{i|j}) = 0$
- (2)  $\alpha_{|h}^{ij} = 0$
- (3)  $g_{|h}^{ij} = 0$ .

### 3 Cartan spaces with an $(\alpha, \beta)$ -metric with $h$ -metrical $d$ -connection

In this section we impose the condition of  $d$ -connection  $D\Gamma$  on the Cartan space with an  $(\alpha, \beta)$ -metric to be  $h$ -metrical and in consequence we analyse what shapes the corresponding Cartan space assumes.

First we take the  $h$ -covariant derivative of the given  $(\alpha, \beta)$ -metric as follows:

$$\begin{aligned} K(x, \omega) &= \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2 \beta^4(x, \omega)}{3 \alpha^3(x, \omega)} \\ g^{ij}(\omega_i \omega_{j|h} + \omega_j \omega_{i|h}) + \omega_i \omega_j g_{|h}^{ij} \\ &= \alpha_{|h} + \epsilon\beta_{|h} + 2k \left( \frac{2\alpha\beta\beta_{|h} - \beta^2\alpha_{|h}}{\alpha^2} \right) - \frac{k^3}{3} \left( \frac{4\alpha^3\beta^3\beta_{|h} - 3\alpha^2\beta^4\alpha_{|h}}{\alpha^6} \right) \end{aligned}$$

As we have stipulated the  $d$ -connection  $D\Gamma$  of the Cartan space is  $h$ -metrical, therefore by definition 2.5, we have

$$\omega_{j|h} = 0, \omega_{i|h} = 0, \alpha_{|h} = 0, g_{|h}^{ij} = 0$$

Using these values in above expression, we get

$$\begin{aligned} g^{ij}(\omega_i \times 0 + \omega_j \times 0) + \omega_i \omega_j \times 0 &= 0 + \epsilon\beta_{|h} + 2k \left( \frac{2\alpha\beta\beta_{|h} - \beta^2 \times 0}{\alpha^2} \right) - \\ \frac{k^3}{3} \left( \frac{4\alpha^3\beta^3\beta_{|h} - 3\alpha^2\beta^4 \times 0}{\alpha^6} \right) \end{aligned}$$

$$\beta_{|h} = 0 \quad (\because \alpha \neq 0, \beta \neq 0) \tag{7}$$

$$(\omega_i b^i(x))_{|h} = 0 \quad (\because \beta(x, \omega) = \omega_i b^i(x))$$

$$\omega_i b^i(x)_{|h} + b^i(x) \omega_{i|h} = 0$$

As we have stipulated the  $d$ -connection  $D\Gamma$  of the Cartan space is  $h$ -metrical, therefore by definition 2.5, we have

$$\omega_{i|h} = 0,$$

Using these values in above expression, we get

$$\omega_i b^i(x)_{|h} + b^i(x) \times 0 = 0$$

$$\omega_i b^i(x)_{|h} = 0$$

$$b^i(x)_{|h} = 0 \tag{8}$$

Now, we find  $h$ -covariant derivatives of the coefficients of metric tensor  $g^{ij}$  and then use conditions of  $h$ -metrical  $d$ -connection  $D\Gamma$  of Cartan space as follows, we get

$$\because \rho = \frac{\alpha^5 - 2k\alpha^2\beta^2 + k^2\beta^4}{2\alpha^5}$$

$$\because \rho_{|h} = 0. \tag{9}$$

$$\because \rho_0 = \frac{2k\alpha^2 - 2k^2\beta^2}{\alpha^3}$$

$$\because \rho_{0|h} = 0. \tag{10}$$

$$\because \rho_{-1} = \frac{-4k^2\beta^2}{\alpha^3}$$

$$\because \rho_{-1|h} = 0. \tag{11}$$

$$\because \rho_{-2} = \frac{6k^2\beta^2 - 2k\alpha^2}{\alpha^5}$$

$$\because \rho_{-2|h} = 0. \tag{12}$$

The  $h$ -covariant differentiation of the equation (1) gives

$$g_{|h}^{ij} = \rho a_{|h}^{ij} + a^{ij} \rho_{|h} + \rho_0 (b^i b^j)_{|h} + b^i b^j \rho_0 + \rho_{-1} (b^i \omega^j + b^j \omega^i)_{|h} +$$

$$(b^i \omega^j + b^j \omega^i) \rho_{-1|h} + \rho_{-2} (\omega^i \omega^j)_{|h} + \omega^i \omega^j \rho_{-2|h}$$

$$g_{|h}^{ij} = \rho a_{|h}^{ij} + a^{ij} \rho_{|h} + \rho_0 (b^i b_{|h}^j + b^j b_{|h}^i) + b^i b^j \rho_{0|h} + \rho_{-1} (b^i \omega_{|h}^j + \omega^i b_{|h}^j + b^j \omega_{|h}^i + \omega^j b_{|h}^i) \rho_{-1|h} (b^i \omega^j + b^j \omega^i) + \rho_{-2|h} (\omega^i \omega_{|h}^j + \omega^j \omega_{|h}^i) + \omega^i \omega^j \rho_{-2|h}.$$

Using the conditions of  $h$ -metrical  $d$ -connection  $D\Gamma$  of Cartan space and equations (8), (9), (10), (11) and (12), above equation reduces to

$$g_{|h}^{ij} = 0.$$

Thus, allowing  $d$ -connection  $D\Gamma$  of the Cartan space to be  $h$ -metrical, it gives two important quantities namely  $a_{|h}^{ij} = 0$  (by definition of  $h$ -metrical  $d$ -connection) and  $g_{|h}^{ij} = 0$ , i.e.,  $h$ -covariant derivatives of fundamental metric tensors of associated Riemannian space and Cartan space vanishes.

Now, since  $a_{|h}^{ij} = 0$  and  $g_{|h}^{ij} = 0$ , therefore there corresponding Christoffel symbols will also be same, i.e.,  $H_{jh}^i = \gamma_{jh}^i$  and its equivalent condition is given by

$$b_{:k}^i = 0 \tag{13}$$

Now, since  $H_{jh}^i = \gamma_{jh}^i$  therefore the curvature tensor  $D_{hjk}^i$  of  $D\Gamma$  coincides with the curvature tensor  $R_{hjk}^i$  of Riemannian connection  $R\Gamma = (\gamma_{jk}^i, \gamma_{jk}^i \gamma_i, 0)$ , i.e.,

$$D_{hjk}^i = R_{hjk}^i$$

If the Riemannian curvature tensor vanishes, i.e.,  $R_{hjk}^i = 0$ , the curvature tensor of  $d$ -connection also vanishes, i.e.,  $D_{hjk}^i = 0$ . This discussion can be summarized as follows:

**Proposition 3.1** *A Cartan space  $C$  with the  $(\alpha, \beta)$ -metric  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2 \beta^4(x, \omega)}{3 \alpha^3(x, \omega)}$  admitting a  $h$ -metrical  $d$ -connection is locally flat if and only if the associated Riemannian space is locally flat.*

Now, we find  $h$ -covariant derivatives of the coefficients of Cartan torsion tensor  $C^{ijk}$  and then use conditions of  $h$ -metrical  $d$ -connection  $D\Gamma$  of Cartan space and equation (7) as follows:

$$\begin{aligned} \therefore r_{-1} &= \frac{-4k^2\beta^2}{\alpha^3} \\ \therefore r_{-1|h} &= 0 \end{aligned} \tag{14}$$

$$\begin{aligned} \therefore r_{-2} &= \frac{6k^2\beta^2 - 2k\alpha^2}{\alpha^5} \\ \therefore r_{-2|h} &= 0 \end{aligned} \tag{15}$$

$$\begin{aligned} \therefore r_{-3} &= \frac{6k\alpha^2\beta - 10k^2\beta^3}{\alpha^7} \\ \therefore r_{-3|h} &= 0 \end{aligned} \tag{16}$$

$$\begin{aligned} \therefore r_{-4} &= \frac{35k^2\beta^4 - 30k\alpha^2\beta^2 + 3\alpha^4}{2\alpha^9} \\ \therefore r_{-4|h} &= 0 \end{aligned} \tag{17}$$

Now we calculate the value of  $h$ -covariant derivative of  $d$ -tensor field  $C_i^{jk}$  of type (2,1) under the assumption of  $h$ -metrical  $d$ -connection as follows:

$$\therefore C_k^{ij} = g_{kr} C^{rij}$$

$$\begin{aligned}
 &\therefore C_{k|h}^{ij} = (g_{kr}C^{rij})_{|h} \\
 &= g_{kr} \times C_{|h}^{rij} + C^{rij} \times 0g_{kr|h} \\
 &= g_{kr}C_{|h}^{rij} \\
 &= -g_{kr} \frac{1}{2} [r_{-1}b^r b^i b^j + r_{-2}b^r b^i \omega^j + r_{-3}b^r \omega^i \omega^j + r_{-4}\omega^r \omega^i \omega^j + \rho_{-1}a^{ri} b^j + \\
 &\rho_{-2}a^{ri} \omega^j + r|i|j]_{|h} \\
 &= -g_{kr} \frac{1}{2} [r_{-1} \times (b^r b^i b^j)_{|h} + b^r b^i b^j \times 0r_{-1|h} + r_{-2} \times (b^r b^i \omega^j)_{|h} + b^r b^i \omega^j \times \\
 &0r_{-2|h} + r_{-3} \times (b^r \omega^i \omega^j)_{|h} + b^r \omega^i \omega^j \times 0r_{-3|h} + r_{-4} \times (\omega^r \omega^i \omega^j)_{|h} + \omega^r \omega^i \omega^j \times \\
 &0r_{-4|h} + \rho_{-1} \times (a^{ri} b^j)_{|h} + a^{ri} b^j \times 0\rho_{-1|h} + \rho_{-2} \times (a^{ri} \omega^j)_{|h} + a^{ri} \omega^j \times \\
 &0\rho_{-2|h} + (r|i|j)_{|h}] \\
 &= -g_{kr} \frac{1}{2} [r_{-1}(b^r b^i b^j)_{|h} + r_{-2}(b^r b^i \omega^j)_{|h} + r_{-3}(b^r \omega^i \omega^j)_{|h} + r_{-4}(\omega^r \omega^i \omega^j)_{|h} + \\
 &\rho_{-1}(a^{ri} b^j)_{|h} + \rho_{-2}(a^{ri} \omega^j)_{|h} + (r|i|j)_{|h}] \\
 &= -g_{kr} \frac{1}{2} [r_{-1}(b^r b^i 0b_{|h}^j + b^r b^j 0b_{|h}^i + b^i b^j 0b_{|h}^r) + r_{-2}(b^r b^i 0\omega_{|h}^j + b^r \omega^j 0b_{|h}^i + \\
 &b^i \omega^j 0b_{|h}^r) + \\
 &r_{-3}(b^r \omega^i 0\omega_{|h}^j + b^r \omega^j 0\omega_{|h}^i + \omega^i \omega^j 0b_{|h}^r) + r_{-4}(\omega^r \omega^i 0\omega_{|h}^j + \omega^r \omega^j 0\omega_{|h}^i + \\
 &\omega^i \omega^j 0\omega_{|h}^r) + \\
 &\rho_{-1}(a^{ri} 0b_{|h}^j + b^j 0a_{|h}^{ri}) + \rho_{-2}(a^{ri} 0\omega_{|h}^j + \omega^j 0a_{|h}^{ri}) + 0(r|i|j)_{|h}] \\
 &C_{k|h}^{ij} = 0 \tag{18}
 \end{aligned}$$

Thus we shown that Cartan torsion tensor vanishes under the assumption of  $h$ -metrical  $d$ -connection.

One knows that a Cartan space  $C$  is Berwald space if and only if  $C_{k|h}^{ij} = 0$  [12]. Hence from equation (18), we have the following proposition:

**Proposition 3.2** *A Cartan space  $C$  with the  $(\alpha, \beta)$ -metric  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2 \beta^4(x, \omega)}{3 \alpha^3(x, \omega)}$  admitting  $h$ -metrical  $d$ -connection is a Berwald space*

In [12], it is deduced that a locally Minkowski space is a Berwald space in which curvature tensor vanishes. Hence, from the Propositions 3.1 and 3.2, we have following theorem:

**Theorem 3.3** *A Cartan space  $C$  with the  $(\alpha, \beta)$ -metric  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2 \beta^4(x, \omega)}{3 \alpha^3(x, \omega)}$  admitting  $h$ -metrical  $d$ -connection is locally Minkowski space if and only if the associated Riemannian space is locally flat.*

#### 4 Conformal change of Cartan space with an $(\alpha, \beta)$ -metric

In this section our aim is to conformally transform a Cartan space  $(M, K(x, \omega))$  to another Cartan space  $(M, \tilde{K}(x, \omega))$  and then to determine the nature of curvature tensor  $\tilde{D}_{hjk}^i$  in the conformally transformed space  $(M, \tilde{K}(x, \omega))$  under the influence of  $h$ -metrical  $d$ -connection on the original Cartan space  $(M, K(x, \omega))$ . That is, we are going to determine the shape of conformally transformed space  $(M, \tilde{K}(x, \omega))$  under the stipulation of  $h$ -metrical  $d$ -connection on  $(M, K(x, \omega))$ .

For that, consider an  $n$ -dimensional Cartan space  $C = (M, K(x, \omega))$  equipped with a real smooth  $n$ -manifold  $M$  and the  $(\alpha, \beta)$ -metric  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2}{3} \frac{\beta^4(x, \omega)}{\alpha^3(x, \omega)}$ , where  $\alpha = (a^{ij}(x, \omega)\omega_i\omega_j)^{\frac{1}{2}}$  and  $\beta = \omega_i b^i(x)$ . By a conformal change  $\sigma: K \rightarrow \tilde{K}$  such that  $\tilde{K}(\tilde{\alpha}, \tilde{\beta}) = e^\sigma K(\alpha, \beta)$ , we have the another Cartan space  $\tilde{C}^n = (M, \tilde{K}(\tilde{\alpha}, \tilde{\beta}))$ , where  $\tilde{\alpha} = e^\sigma \alpha$  and  $\tilde{\beta} = e^\sigma \beta$ .

Putting  $\alpha = (a^{ij}(x, \omega)\omega_i\omega_j)^{\frac{1}{2}}$  and  $\beta = \omega_i b^i(x)$  in the above relations, we get

$$\begin{aligned}\tilde{\alpha} &= e^\sigma \alpha \\ \tilde{\alpha} &= e^\sigma (a^{ij}(x, \omega)\omega_i\omega_j)^{\frac{1}{2}} \\ \tilde{\alpha} &= (e^{2\sigma} a^{ij}(x, \omega)\omega_i\omega_j)^{\frac{1}{2}} \\ \tilde{\alpha} &= (\underline{\tilde{a}^{ij}} \omega_i \omega_j)^{\frac{1}{2}} \\ \tilde{a}^{ij} &= e^{2\sigma} a^{ij}(x, \omega)\end{aligned}$$

and

$$\begin{aligned}\tilde{\beta} &= e^\sigma \beta \\ \tilde{\beta} &= e^\sigma \omega_i b^i(x) \\ \tilde{\beta} &= \omega_i \underline{e^\sigma b^i(x)} \\ \tilde{\beta} &= \omega_i \underline{b^i} \\ \tilde{b}^i &= e^\sigma b^i(x)\end{aligned}$$

Now we calculate the Christoffel symbols  $\tilde{\gamma}_{rk}^p$  in conformally transformed space  $(M, \tilde{K}(x, \omega))$

as follows:

We know from Riemannian geometry Christoffel symbols of second kind  $\gamma_{rk}^p$  from fundamental metric tensor  $a^{pq}(x, \omega)$  can be defined as

$$\gamma_{qk}^p = \frac{1}{2} a^{lp} \left( \frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l} \right)$$

Similarly, we can also define the Christoffel symbols  $\tilde{\gamma}_{rk}^p$  in conformally transformed space  $(M, \tilde{K}(x, \omega))$  as

$$\tilde{\gamma}_{qk}^p = \frac{1}{2} \tilde{a}^{lp} \left( \frac{\partial \tilde{a}_{kl}}{\partial x^q} + \frac{\partial \tilde{a}_{lq}}{\partial x^k} - \frac{\partial \tilde{a}_{qk}}{\partial x^l} \right)$$

$$\begin{aligned}
 &= \frac{1}{2} e^{2\sigma} a^{lp} (x, \omega) \left( \frac{\partial e^{2\sigma} a_{kl}(x, \omega)}{\partial x^q} + \frac{\partial e^{2\sigma} a_{lq}(x, \omega)}{\partial x^k} - \frac{\partial e^{2\sigma} a_{qk}(x, \omega)}{\partial x^l} \right) \\
 &= \frac{1}{2} e^{2\sigma} a^{lp} \left[ \left( e^{2\sigma} \frac{\partial a_{kl}}{\partial x^q} + a_{kl} \frac{\partial e^{2\sigma}}{\partial x^q} \right) + \left( e^{2\sigma} \frac{\partial a_{lq}}{\partial x^k} + a_{lq} \frac{\partial e^{2\sigma}}{\partial x^k} \right) - \left( e^{2\sigma} \frac{\partial a_{jq}}{\partial x^l} + a_{jq} \frac{\partial e^{2\sigma}}{\partial x^l} \right) \right] \\
 &= \frac{1}{2} e^{2\sigma} a^{lp} \left[ \left( e^{2\sigma} \frac{\partial a_{kl}}{\partial x^q} + 2e^{2\sigma} a_{kl} \frac{\partial \sigma}{\partial x^q} \right) + \left( e^{2\sigma} \frac{\partial a_{lq}}{\partial x^k} + 2e^{2\sigma} a_{lq} \frac{\partial \sigma}{\partial x^k} \right) - \right. \\
 &\quad \left. \left( e^{2\sigma} \frac{\partial a_{qk}}{\partial x^l} + 2e^{2\sigma} a_{qk} \frac{\partial \sigma}{\partial x^l} \right) \right] \\
 &= \frac{1}{2} e^{4\sigma} a^{lp} \left[ \left( \frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l} \right) + \left( 2a_{kl} \frac{\partial \sigma}{\partial x^q} + 2a_{lq} \frac{\partial \sigma}{\partial x^k} - 2a_{qk} \frac{\partial \sigma}{\partial x^l} \right) \right] \\
 &= e^{4\sigma} \left[ \frac{1}{2} a^{lp} \left( \frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l} \right) + \left( a^{lp} a_{kl} \sigma_q + a^{lp} a_{lq} \sigma_k - a^{lp} a_{qk} \sigma_l \right) \right] \\
 &= e^{4\sigma} \left[ \gamma_{qk}^p + \left( \delta_k^p \sigma_q + \delta_q^p \sigma_k - a_{qk} \sigma^l \right) \right]
 \end{aligned}$$

Hence, the components of Christoffel symbols  $\tilde{\gamma}_{qk}^p$ , constructed from  $\tilde{a}^{pq}$ , in conformally transformed space are given by

$$\tilde{\gamma}_{qk}^p = \gamma_{qk}^p + B_{qk}^p, \quad (19)$$

where  $B_{qk}^p = \sigma_k \delta_q^p + \sigma_q \delta_k^p - a_{kq} \sigma^p$ ,  $\sigma^p = \sigma_q a^{pq}$ .

The covariant derivative of  $\tilde{b}^p$  with respect to  $\tilde{\gamma}_{rk}^p$ , yields

$$\tilde{b}_{:k}^p = e^\sigma (b_{:k}^p + 2\sigma_k b^p + b^r \sigma_r \delta_k^p - \sigma_p b^r a_{rk}). \quad (20)$$

Transvecting the equation (20) by  $\tilde{b}^k$ , and putting

$$M^p = \frac{1}{B^2} \left( b^k b_{:k}^p - \frac{b_{:r}^r b^p}{n+4} \right), \quad (21)$$

we have  $\sigma^p = \tilde{M}^p - M^p$ , from which we get  $\sigma_p = \tilde{M}_p - M_p$ . Substituting the values of  $\sigma_p$  and  $\sigma^p$  in equation (19) and using  $D_{hq}^p = \gamma_{hq}^p + \delta_h^p M_q + \delta_h^p M_q + \delta_q^p M_h - M^p a_{hq}$ , we find

$$\tilde{D}_{hq}^p = D_{hq}^p. \quad (22)$$

Here  $D_{hq}^p$  is a symmetric and conformally invariant linear connection on  $M$ .

The whole discussion can be summarized in the following proposition.

**Proposition 4.1** *Let  $C = (M, K(x, \omega))$  be a Cartan space the  $(\alpha, \beta)$ -metric  $K(x, \omega) = \alpha(x, \omega) + \epsilon \beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2}{3} \frac{\beta^4(x, \omega)}{\alpha^3(x, \omega)}$ . Then, there exists a conformally invariant symmetric linear connection  $D_{qk}^p$  on  $M$ .*

Next, if we denote the curvature tensor of  $D_{qk}^p$  by  $D_{hqk}^p$ , then from the equation (22), we get

$$\tilde{D}_{hqk}^p = D_{hqk}^p. \quad (23)$$

Since  $b_{:k}^p = 0$ , from equation (21), we get  $M^i = 0$ . Hence, we deduce that  $D_{qk}^p = \gamma_{qk}^p$  and  $D_{hqk}^p =$

$R_{hqk}^p$ .

Thus we have the following proposition.

**Proposition 4.2** *Let  $C = (M, K)$  be a Cartan space the  $(\alpha, \beta)$ -metric  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2 \beta^4(x, \omega)}{3 \alpha^3(x, \omega)}$  admitting  $h$ -metrical  $d$ -connection. Then, there exists a conformally invariant symmetric linear connection  $D_{qh}^p$  such that  $D_{qk}^p = \gamma_{qk}^p$  and it's curvature tensor  $D_{hqk}^p = R_{hqk}^p$ .*

Next, if the associated Riemannian space  $(M, \alpha)$  is locally flat, that is,  $R_{hqk}^p = 0$ , then from Proposition 4.2 and equation (23), we deduce that  $\tilde{D}_{hqk}^p = 0$ , that is, the space  $C$  is conformally flat. Thus we have the following theorem.

**Theorem 4.3** *Let  $C = (M, K)$  be a Cartan space the  $(\alpha, \beta)$ -metric  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2 \beta^4(x, \omega)}{3 \alpha^3(x, \omega)}$  admitting  $h$ -metrical  $d$ -connection. Then the space  $C$  is conformally flat if and only if the associated Riemannian space is locally flat.*

**Conclusion:** The conditions derived clarify how the  $(\alpha, \beta)$ -metric  $K(x, \omega) = \alpha(x, \omega) + \epsilon\beta(x, \omega) + 2k \frac{\beta^2(x, \omega)}{\alpha(x, \omega)} - \frac{k^2 \beta^4(x, \omega)}{3 \alpha^3(x, \omega)}$  influences the geometric structure of the Cartan space, particularly regarding its transformation into a locally Minkowski space. Moreover, it identifies the criteria under which the space becomes conformally flat, suggesting that the curvature can be transformed to zero by a conformal transformation under  $h$ -metrical  $d$ -connection. These results may have implications for theoretical physics, particularly in contexts like general relativity, where understanding the structure of spacetime is crucial.

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