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Endomorphisms and Their Role in Novel Derivations of Hilbert Algebras

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Abstract:

This study introduces and explores left-f-derivations, right-f-derivations, and f-derivations of type I and type II within the framework of Hilbert algebras, focusing on their fundamental properties and structural significance. We show that the kernel of an f-derivation, $\mathrm{Ker}_d(A)$, forms a near filter and operates as a subalgebra of the Hilbert algebra A. These findings enhance the understanding of Hilbert algebras and offer new perspectives on the role of derivations in algebraic logic.

Keywords: Hilbert algebra, near filter, subalgebra, left-f-derivation, right-f-derivation.

1. Introduction

The concept of Hilbert algebras, introduced by Henkin in the early 1950s, emerged as a groundbreaking algebraic framework aimed at formalizing implications within intuitionistic and other non-classical logics [9]. Unlike classical logic, which operates on binary truth values, non-classical logic, including intuitionistic logic, requires more nuanced structures to capture their subtleties. Hilbert algebras provided a sophisticated tool for this purpose, offering an algebraic lens through which to analyze and comprehend the behaviour of implication in these alternative logical systems. By the 1960s, the fundamental importance of Hilbert algebras was further solidified through Diego's seminal work, demonstrating that Hilbert algebras constitute a locally finite variety [7]. This revelation not only underscored the structural richness of Hilbert algebras but also integrated them firmly within the broader domain of algebraic logic. Diego's results provided essential insights, laying a rigorous foundation for further explorations into their properties and applications. As a result, Hilbert algebras have become an essential structure in the study of non-classical logics, facilitating deeper inquiries into the interplay between algebra and logic.

Building upon these foundational developments, subsequent research delved deeper into both the algebraic and logical aspects of Hilbert algebras, further expanding their theoretical and practical relevance. Notably, Busneag [4,5] and Jun [15] explored the critical role of filters within Hilbert algebras, establishing that these filters act as deductive systems fundamental to the underlying logical structure. Their work unveiled profound connections between the algebraic properties of Hilbert algebras and their logical interpretations, offering a refined understanding of how these systems model non-classical logic. Additionally, Dudek's contributions [8] extended the versatility of Hilbert algebras through the concept of fuzzification, where the algebraic framework was adapted to accommodate fuzzy logic, characterized by truth values that exist on a continuum rather than binary distinctions. This innovation enriched the study of Hilbert algebras, enabling their application in domains that address uncertainty, partial truth, and vagueness—critical aspects of real-world reasoning systems. The inclusion of fuzzy subalgebras and deductive systems demonstrated the flexibility of Hilbert algebras and further reinforced their importance in contemporary logical and algebraic research.

The study of derivations has seen remarkable progress in recent years, particularly in their application to diverse algebraic structures. In 2021, Muangkarn et al. [18] examined f_q-derivations, while Bantaojai et al. [3] explored derivations induced by endomorphisms within B-algebras. This investigation revealed new insights into the interplay between derivations and algebraic morphisms, broadening the understanding of structural transformations in these systems. Continuing this trajectory, Bantaojai et al. [1,2] expanded their research in 2022 to encompass derivations on d-algebras and B-algebras, further enriching the theoretical framework and offering deeper perspectives on the behaviour of derivations in more complex algebraic systems. Simultaneously, Muangkarn et al. [17,19] focused on the structural implications of derivations induced by endomorphisms in BG-algebras and d-algebras, shedding light on the intricate relationships between these operations and the underlying algebraic properties. Additionally, Iampan et al. [10,20,21] made substantial contributions by studying derivations in UP-algebras, emphasizing the versatility and broad applicability of derivation theory across various algebraic frameworks. Building on this rich foundation, Iampan et al. [11,13] introduced and rigorously developed the concepts of (l,r)-derivations, (r,l)-derivations, and general derivations within the context of Hilbert algebras. Their work not only advanced the understanding of derivations in these specific algebraic structures but also illuminated their logical implications. Subsequently, their investigation into the relationship between derivations and endomorphisms opened up new pathways for exploring the structural properties of Hilbert algebras. This research has significantly enriched the algebraic theory surrounding Hilbert algebras, contributing valuable insights into the role of derivations in both classical and non-classical logical systems. Collectively, these studies have deepened our comprehension of derivations across a wide array of algebraic structures, laying a robust foundation for future research in this dynamic area of algebraic theory.

This paper introduces and investigates left-f-derivations, right-f-derivations, and f-derivations of type I and type II within the framework of Hilbert algebras. We explore the fundamental properties of these derivations, highlighting their structural roles and interactions. Notably, we establish that the kernel of an f-derivation, $\mathrm{Ker}_d(A)$, forms a near filter and functions as a subalgebra of the Hilbert algebra A. These results not only deepen our understanding of the algebraic structure of Hilbert algebras but also pave the way for future research into the broader implications of derivations in algebraic logic and their potential applications.

2. Preliminaries

Before proceeding, it's important to revisit the concept of Hilbert algebras, introduced by Diego in 1966 [7]. As a key structure in algebraic logic, they capture the behaviour of implication in non-classical systems, making them central to further studies of derivations and algebraic operations.

Definition 2.1. [7] A *Hilbert algebra* is a triplet with the formula $A=(A,\cdot,1)$, where A is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of A that is true according to the axioms stated below:

$$(\forall x,y\in A)(x\cdot(y\cdot x)=1) \tag{2.1}$$

$$(\forall x,y,z\in A)((x\cdot(y\cdot z))\cdot((x\cdot y)\cdot(x\cdot z))=1) \tag{2.2}$$

$$(\forall x,y\in A)(x\cdot y=1 \text{ and } y\cdot x=1\Rightarrow x=y) \tag{2.3}$$

In [8], the following conclusion was established.

Lemma 2.2. Let $A=(A,\cdot,1)$ be a Hilbert algebra. Then

(1) $(\forall x\in A)(x\cdot x=1)$,

(2) $(\forall x\in A)(1\cdot x=x)$,

(3) $(\forall x\in A)(x\cdot 1=1)$,

(4) $(\forall x,y,z\in A)(x\cdot(y\cdot z)=y\cdot(x\cdot z))$,

(5) $(\forall x,y,z\in A)((x\cdot z)\cdot((z\cdot y)\cdot(x\cdot y))=1)$.

In a Hilbert algebra $A=(A,\cdot,1)$, the binary relation $\leq$ is defined by

$$(\forall x,y\in A)(x\leq y\Leftrightarrow x\cdot y=1),$$

which is a partial order on A with 1 as the largest element.

Definition 2.3. [16] A nonempty subset D of a Hilbert algebra $A=(A,\cdot,1)$ is called a *subalgebra* of A if $x\cdot y\in D$ for all $x,y\in D$.

Definition 2.4. [6] A nonempty subset D of a Hilbert algebra $A=(A,\cdot,1)$ is called an *ideal* of A if the following conditions hold:

(1) $1\in D$,

(2) $(\forall x,y\in A)(y\in D\Rightarrow x\cdot y\in D)$,

(3) $(\forall x,y_1,y_2\in A)(y_1,y_2\in D\Rightarrow (y_1\cdot(y_2\cdot x))\cdot x\in D)$.

Definition 2.5. [12] A nonempty subset D of a Hilbert algebra $A=(A,\cdot,1)$ is called a *near filter* of A if the following conditions hold:

(1) $1\in D$,

(2) $(\forall x, y \in A)(y \in D \Rightarrow x \cdot y \in D)$.

Definition 2.6. [12] A nonempty subset D of a Hilbert algebra $A = (A, \cdot, 1)$ is called a *filter* of A if the following conditions hold:

(1) $1 \in D$,

(2) $(\forall x, y \in A)(x \cdot y, x \in D \Rightarrow y \in D)$.

Definition 2.7. Let $A = (A, \cdot, 1_A)$ and $B = (B, \text{å}, 1_B)$ be Hilbert algebras. A function $f : A \to B$ is called a *homomorphism* if

$$f(x \cdot y) = f(x) \,\text{å}\, f(y) \text{ for all } x, y \in A.$$

Now, $f(1_A) = f(1_A \cdot 1_A) = f(1_A) \,\text{å}\, f(1_A) = 1_B$. A homomorphism $f : A \to A$ is said to be an *endomorphism*.

For any x, y in a Hilbert algebra $A = (A, \cdot, 1)$, we define $x \vee y$ by $(y \cdot x) \cdot x$. By Lemma 2.2 (4), we can prove that $x \vee y$ is an upper bound of x and y. That is,

$$(\forall x, y \in X)(x \cdot (x \vee y) = 1), \tag{2.4}$$

$$(\forall x, y \in X)(y \cdot (x \vee y) = 1). \tag{2.5}$$

A Hilbert algebra $A = (A, \cdot, 1)$ is said to be $\vee$*-commutative* [14] if for all $x, y \in A, (y \cdot x) \cdot x = (x \cdot y) \cdot y$, that is, $x \vee y = y \vee x$. From [14], we know that

$$(\forall x \in X)(x \vee x = x), \tag{2.6}$$

$$(\forall x \in X)(x \vee 1 = 1 \vee x = 1). \tag{2.7}$$

3. Left and right-f-derivations of type I

In this section, we introduce the concepts of left-f-derivations, right-f-derivations, and f-derivations of type I in Hilbert algebras, examining their fundamental properties. We then focus on analyzing the subset $\mathrm{Ker}_d(A)$ associated with a left (or right) f-derivation of type I, highlighting its structural significance within the algebra.

Definition 3.1. Let $A = (A, \cdot, 1)$ be a Hilbert algebra and f be an endomorphism of A. A self-map $d : A \to A$ is called a *left-f-derivation of type* I of A if it satisfies the identity

$$d(x \cdot y) = (d(x) \cdot f(y)) \vee (x \cdot y) \text{ for all } x, y \in A.$$

Similarly, a self-map $d : A \to A$ is called a *right-f-derivation of type* I of A if it satisfies the identity

$$d(x \cdot y) = (f(x) \cdot d(y)) \vee (x \cdot y) \text{ for all } x, y \in A.$$

Moreover, if d is a left-f-derivation of type I and a right-f-derivation of type I of A, it is called an *f-derivation of type* I of A.

Example 3.2. Let $A=\{1,2,3,4\}$ be a Hilbert algebra with a fixed element 1 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	1	1	4
4	1	1	3	1

Then $(A,\cdot,1)$ is a Hilbert algebra. We define an endomorphism f on A as follows:

$$f=\begin{pmatrix}1&2&3&4\\1&1&3&4\end{pmatrix}$$

Define a self-map $d_1 : A \to A$ as follows:

$$d_1=\begin{pmatrix}1&2&3&4\\1&1&3&4\end{pmatrix}$$

Hence, d_1 is a left- f -derivation of type I of A. Define a self-map $d_2 : A \to A$ as follows:

$$d_2=\begin{pmatrix}1&2&3&4\\1&2&1&1\end{pmatrix}$$

Hence, d_1 is a right- f -derivation of type I of A.

Definition 3.3. A self-map d of a Hilbert algebra $A=(A,\cdot,1)$ is called *regular* if $d(1)=1$.

Theorem 3.4. In a Hilbert algebra $A=(A,\cdot,1)$, the following statements hold:

(1) every left- f -derivation of type I of A is regular,

(2) every right- f -derivation of type I of A is regular.

Proof. (1) Assume that d is a left- f -derivation of type I of A. Then

$d(1)=d(1\cdot 1)$ [Lemma 2.2 (1)]

$=(d(1)\cdot f(1))\vee(1\cdot 1)$

$=(d(1)\cdot 1)\vee 1$ [Lemma 2.2 (1)]

$=1$. [(2.7)]

Hence, d is regular.

(2) Assume that d is a right- f -derivation of type I of A. Then

$d(1)=d(1\cdot 1)$ [Lemma 2.2 (1)]

$= (f(1)\cdot d(1)) \vee (1\cdot 1)$

$= (1\cdot d(1)) \vee 1$ [Lemma 2.2 (1)]

$=1$. [(2.7)]

Hence, d is regular.

Corollary 3.5. Every f -derivation of type I of a Hilbert algebra A is regular.

Theorem 3.6. In a Hilbert algebra $A = (A,\cdot,1)$, the following statements hold:

(1) if d is a left- f -derivation of type I of A, then $d(x) = f(x) \vee x$ for all $x \in A$,

(2) if d is a right- f -derivation of type I of A, then $d(x) = d(x) \vee x$ for all $x \in A$.

Proof. (1) Assume that d is a left- f -derivation of type I of A. Then, for all $x \in A$,

$d(x) = d(1\cdot x)$ [Lemma 2.2 (2)]

$= (d(1)\cdot f(x)) \vee (1\cdot x)$

$= (1\cdot f(x)) \vee (1\cdot x)$ [regular]

$= f(x) \vee x$. [Lemma 2.2 (2)]

(2) Assume that d is a right- f -derivation of type I of A. Then, for all $x \in A$,

$d(x) = d(1\cdot x)$ [Lemma 2.2 (2)]

$= (f(1)\cdot d(x)) \vee (1\cdot x)$

$= (1\cdot d(x)) \vee (1\cdot x)$

$= d(x) \vee x$. [Lemma 2.2 (2)]

Corollary 3.7. If d is an f -derivation of type I of A, then

$$d(x) = f(x) \vee x = d(x) \vee x \text{ for all } x \in A.$$

Proposition 3.8. Let d be a left- f -derivation of type I of a Hilbert algebra $A = (A,\cdot,1)$. Then the following properties hold: for any $x, y \in A$,

(1) $f(x) \le d(x)$,

(2) $d(x)\cdot f(y) \le d(x\cdot y)$,

(3) $d(x\cdot f(x))\cdot f(f(x)) \le d(d(x))$,

(4) $d(y\cdot x)\cdot f(x) \le d(x \vee y)$,

(5) $d(x) = d(x) \vee f(x)$.

Proof. (1) For all $x \in A$,

$f(x)\cdot d(x)=f(x)\cdot(f(x)\vee x)$ [Theorem 3.6 (1)]

$=1$. [(2.4)]

Hence, $f(x)\leq d(x)$ for all $x\in A$.

(2) For all $x,y\in A$,

$(d(x)\cdot f(y))\cdot d(x\cdot y)=(d(x)\cdot f(y))\cdot((d(x)\cdot f(y))\vee(x\cdot y))$

$=1$. [(2.4)]

Hence, $d(x)\cdot f(y)\leq d(x\cdot y)$ for all $x,y\in A$.

(3) For all $x\in A$,

$(d(x\cdot f(x))\cdot f(f(x)))\cdot d(d(x))=(d(x\cdot f(x))\cdot f(f(x)))\cdot d(f(x)\vee x)$ [Theorem 3.6 (1)]

$=(d(x\cdot f(x))\cdot f(f(x)))\cdot d((x\cdot f(x))\cdot f(x))$

$=1$. [(2)]

Hence, $d(x\cdot f(x))\cdot f(f(x))\leq d(d(x))$ for all $x\in A$.

(4) For all $x,y\in A$,

$(d(y\cdot x)\cdot f(x))\cdot d(x\vee y)=(d(y\cdot x)\cdot f(x))\cdot d((y\cdot x)\cdot x)$

$=1$. [(2)]

Hence, $d(y\cdot x)\cdot f(x)\leq d(x\vee y)$ for all $x,y\in A$.

(5) For all $x\in A$,

$d(x)=1\cdot d(x)$ [Lemma 2.2 (2)]

$=(f(x)\cdot d(x))\cdot d(x)$ [(1)]

$=d(x)\vee f(x)$.

Proposition 3.9. Let d be a right-f-derivation of type I of a Hilbert algebra $A=(A,\cdot,1)$. Then the following properties hold: for any $x,y\in A$,

(1) $x\leq d(x)$,

(2) $f(x)\cdot d(y)\leq d(x\cdot y)$,

(3) $f(x\cdot d(x))\cdot d(d(x))\leq d(d(x))$,

(4) $f(y\cdot x)\cdot d(x)\leq d(x\vee y)$.

Proof. (1) For all $x\in A$,

$x\cdot d(x)=x\cdot(d(x)\vee x)$ [Theorem 3.6 (2)]

$=1$. [(2.5)]

Hence, $x \leq d(x)$ for all $x \in A$.

(2) For all $x, y \in A$,

$(f(x) \cdot d(y)) \cdot d(x \cdot y) = (f(x) \cdot d(y)) \cdot ((f(x) \cdot d(y)) \vee (x \cdot y))$

$=1$. [(2.4)]

Hence, $f(x) \cdot d(y) \leq d(x \cdot y)$ for all $x, y \in A$.

(3) For all $x \in A$,

$(f(x \cdot d(x)) \cdot d(d(x))) \cdot d(d(x)) = (f(x \cdot d(x)) \cdot d(d(x))) \cdot d(d(x) \vee x)$ [Theorem 3.6 (2)]

$= (f(x \cdot d(x)) \cdot d(d(x))) \cdot d((x \cdot d(x)) \cdot d(x))$

$=1$. [(2)]

Hence, $f(x \cdot d(x)) \cdot d(d(x)) \leq d(d(x))$ for all $x \in A$.

(4) For all $x, y \in A$,

$(f(y \cdot x) \cdot d(x)) \cdot d(x \vee y) = (f(y \cdot x) \cdot d(x)) \cdot d((y \cdot x) \cdot x)$

$=1$. [(2)]

Hence, $f(y \cdot x) \cdot d(x) \leq d(x \vee y)$ for all $x, y \in A$.

Definition 3.10. Let d be a self-map of a Hilbert algebra $A = (A, \cdot, 1)$. We define the *kernel* $\mathrm{Ker}_d(A)$ of A as follows:

$$\mathrm{Ker}_d(A) = \{x \in A : d(x) = 1\}$$

Theorem 3.11. If d is a right- f -derivation of type I of a Hilbert algebra $A = (A, \cdot, 1)$, then $y \vee x \in \mathrm{Ker}_d(A)$ for all $y \in \mathrm{Ker}_d(A)$ and $x \in A$.

Proof. Assume that d is a right- f -derivation of A. Let $y \in \mathrm{Ker}_d(A)$ and $x \in A$. Then $d(y) = 1$. Thus,

$d(y \vee x) = d((x \cdot y) \cdot y)$

$= (f(x \cdot y) \cdot d(y)) \vee ((x \cdot y) \cdot y)$

$= (f(x \cdot y) \cdot 1) \vee ((x \cdot y) \cdot y)$

$= 1 \vee ((x \cdot y) \cdot y)$ [Lemma 2.2 (3)]

$=1$. [(2.7)]

Hence, $y \vee x \in \mathrm{Ker}_d(A)$.

Theorem 3.12. If d is a right- f -derivation of type I of a commutative Hilbert algebra $A=(A,\cdot,1)$ and for any $x,y\in A$ is such that $y\leq x$ and $y\in \mathrm{Ker}_d(A)$, then $x\in \mathrm{Ker}_d(A)$.

Proof. Assume that d is a right- f -derivation of type I of A. Let $x,y\in A$ be such that $y\leq x$ and $y\in \mathrm{Ker}_d(A)$. Then $y\cdot x=1$ and $d(y)=1$. Thus,

$d(x)=d(1\cdot x)$ [Lemma 2.2 (2)]

$=d((y\cdot x)\cdot x)$

$=d((x\cdot y)\cdot y)$ [commutative]

$=(f(x\cdot y)\cdot d(y))\vee((x\cdot y)\cdot y)$

$=(f(x\cdot y)\cdot 1)\vee((x\cdot y)\cdot y)$

$=1\vee((x\cdot y)\cdot y)$ [Lemma 2.2 (3)]

$=1$. [(2.7)]

Hence, $x\in \mathrm{Ker}_d(A)$.

Theorem 3.13. If d is a right- f -derivation of type I of a Hilbert algebra $A=(A,\cdot,1)$, then $\mathrm{Ker}_d(A)$ is a near filter (subalgebra) of A.

Proof. Assume that d is a right- f -derivation of type I of A. By Theorem 3.4 (2), we have $d(1)=1$ and so $1\in \mathrm{Ker}_d(A)\neq\varnothing$. Let $x\in A$ and $y\in \mathrm{Ker}_d(A)$. Then $d(y)=1$. Thus,

$d(x\cdot y)=(f(x)\cdot d(y))\vee(x\cdot y)$

$=(f(x)\cdot 1)\vee(x\cdot y)$

$=1\vee(x\cdot y)$ [Lemma 2.2 (3)]

$=1$. [(2.7)]

Hence, $x\cdot y\in \mathrm{Ker}_d(A)$, so $\mathrm{Ker}_d(A)$ is a near filter of A.

4. Left and right- f -derivations of type II

Building on the concepts discussed earlier, in this section, we redefine left- f -derivations, right- f -derivations, and f -derivations of type II in Hilbert algebras, examining their fundamental properties. We then analyze the subset $\mathrm{Ker}_d(A)$ associated with these derivations, emphasizing its structural importance within the algebra.

Definition 4.1. Let $A=(A,\cdot,1)$ be a Hilbert algebra and f be an endomorphism of A. A self-map $d:A\rightarrow A$ is called a *left- f -derivation of type* II of A if it satisfies the identity

$$d(x\cdot y)=(x\cdot y)\vee(d(x)\cdot f(y)) \text{ for all } x,y\in A.$$

Similarly, a self-map $d: A \to A$ is called a *right-* f *-derivation of type* II of A if it satisfies the identity

$$d(x \cdot y) = (x \cdot y) \vee (f(x) \cdot d(y)) \text{ for all } x, y \in A.$$

Moreover, if d is a left- f -derivation of type II and a right- f -derivation of type II of A, it is called an f *-derivation of type* II of A.

Example 4.2. Let $A = \{1, 2, 3, 4\}$ be a Hilbert algebra with a fixed element 1 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	2	1	4
4	1	1	1	1

Then $(A, \cdot, 1)$ is a Hilbert algebra. We define an endomorphism f on A as follows:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

Define a self-map $d_3 : A \to A$ as follows:

$$d_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 4 \end{pmatrix}$$

Hence, d_3 is a left- f -derivation of type II and a right- f -derivation of type II of A, so it is an f -derivation of type II of A.

Theorem 4.3. In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:

(1) every left- f -derivation of type II of A is regular,

(2) every right- f -derivation of type II of A is regular.

Proof. (1) Assume that d is a left- f -derivation of type II of A. Then

$d(1) = d(1 \cdot 1)$ [Lemma 2.2 (1)]

$= (1 \cdot 1) \vee (d(1) \cdot f(1))$

$= 1 \vee (d(1) \cdot 1)$ [Lemma 2.2 (1)]

$= 1$. [(2.7)]

Hence, d is regular.

(2) Assume that d is a right- f -derivation of type II of A. Then

$d(1) = d(1 \cdot 1)$ [Lemma 2.2 (1)]

$= (1\cdot 1) \vee (f(1)\cdot d(1))$

$= 1 \vee (1\cdot d(1))$ [Lemma 2.2 (1)]

$= 1$. [(2.7)]

Hence, d is regular.

Corollary 4.4. Every f-derivation of type II of a Hilbert algebra A is regular.

Theorem 4.5. In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:

(1) if d is a left-f-derivation of type II of A, then $d(x) = x \vee f(x)$ for all $x \in A$,

(2) if d is a right-f-derivation of type II of A, then $d(x) = x \vee d(x)$ for all $x \in A$.

Proof. (1) Assume that d is a left-f-derivation of type II of A. Then, for all $x \in A$,

$d(x) = d(1\cdot x)$ [Lemma 2.2 (2)]

$= (1\cdot x) \vee (d(1)\cdot f(x))$

$= (1\cdot x) \vee (1\cdot f(x))$ [regular]

$= x \vee f(x)$. [Lemma 2.2 (2)]

(2) Assume that d is a right-f-derivation of type II of A. Then, for all $x \in A$,

$d(x) = d(1\cdot x)$ [Lemma 2.2 (2)]

$= (1\cdot x) \vee (f(1)\cdot d(x))$

$= (1\cdot x) \vee (1\cdot d(x))$

$= x \vee d(x)$. [Lemma 2.2 (2)]

Corollary 4.6. If d is an f-derivation of type II of A, then

$$d(x) = x \vee f(x) = x \vee d(x) \text{ for all } x \in A.$$

Proposition 4.7. Let d be a left-f-derivation of type II of a Hilbert algebra $A = (A, \cdot, 1)$. Then the following properties hold: for any $x \in A$,

(1) $x \le d(x)$,

(2) $d(x) = d(x) \vee x$.

Proof. (1) For all $x \in A$,

$x \cdot d(x) = x \cdot (x \vee f(x))$ [Theorem 4.5 (1)]

$= 1$. [(2.4)]

Hence, $x \le d(x)$ for all $x \in A$.

(2) For all $x \in A$,

$d(x) = 1 \cdot d(x)$ [Lemma 2.2 (2)]

$= (x \cdot d(x)) \cdot d(x)$ [(1)]

$= d(x) \vee x$.

Proposition 4.8. Let d be a right-f-derivation of type II of a Hilbert algebra $A = (A, \cdot, 1)$. Then the following properties hold: for any $x \in A$,

(1) $x \leq d(x)$,

(2) $d(x) = d(x) \vee x$.

Proof. (1) For all $x \in A$,

$x \cdot d(x) = x \cdot (x \vee d(x))$ [Theorem 4.5 (2)]

$= 1$. [(2.4)]

Hence, $x \leq d(x)$ for all $x \in A$.

(2) For all $x \in A$,

$d(x) = 1 \cdot d(x)$ [Lemma 2.2 (2)]

$= (x \cdot d(x)) \cdot d(x)$ [(1)]

$= d(x) \vee x$.

Theorem 4.9. If d is a right-f-derivation of type II of a Hilbert algebra $A = (A, \cdot, 1)$, then $y \vee x \in \mathrm{Ker}_d(A)$ for all $y \in \mathrm{Ker}_d(A)$ and $x \in A$.

Proof. Let $y \in \mathrm{Ker}_d(A)$ and $x \in A$. Then $d(y) = 1$. Thus,

$d(y \vee x) = d((x \cdot y) \cdot y)$

$= ((x \cdot y) \cdot y) \vee (f(x \cdot y) \cdot d(y))$

$= (y \vee x) \vee (f(x \cdot y) \cdot 1)$

$= (y \vee x) \vee 1$ [Lemma 2.2 (3)]

$= 1$. [(2.7)]

Hence, $y \vee x \in \mathrm{Ker}_d(A)$.

Theorem 4.10. If d is a right-f-derivation of type II of a commutative Hilbert algebra $A = (A, \cdot, 1)$ and for any $x, y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$, then $x \in \mathrm{Ker}_d(A)$.

Proof. Let $x, y \in A$ be such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$. Then $y \cdot x = 1$ and $d(y) = 1$. Thus,

$d(x) = d(1 \cdot x)$ [Lemma 2.2 (2)]

$= d((y \cdot x) \cdot x)$

$= d((x \cdot y) \cdot y)$ [commutative]

$= ((x \cdot y) \cdot y) \vee (f(x \cdot y) \cdot d(y))$

$= (y \vee x) \vee (f(x \cdot y) \cdot 1)$

$= (y \vee x) \vee 1$ [Lemma 2.2 (3)]

$=1$. [(2.7)]

Hence, $x \in \mathrm{Ker}_d(A)$.

Theorem 4.11. If d is a right-f-derivation of type II of a Hilbert algebra $A = (A, \cdot, 1)$, then $\mathrm{Ker}_d(A)$ is a near filter (subalgebra) of A.

Proof. Assume that d is a right-f-derivation of type II of A. By Theorem 4.3 (2), we have $d(1) = 1$ and so $1 \in \mathrm{Ker}_d(A) \neq \varnothing$. Let $x \in A$ and $y \in \mathrm{Ker}_d(A)$. Then $d(y) = 1$. Thus,

$d(x \cdot y) = (x \cdot y) \vee (f(x) \cdot d(y))$

$= (x \cdot y) \vee (f(x) \cdot 1)$

$= (x \cdot y) \vee 1$ [Lemma 2.2 (3)]

$=1$. [(2.7)]

Hence, $x \cdot y \in \mathrm{Ker}_d(A)$, so $\mathrm{Ker}_d(A)$ is a near filter of A.

5. Conclusion

This study introduced and investigated left-f-derivations, right-f-derivations, and f-derivations of type I and type II within the framework of Hilbert algebras, focusing on their fundamental properties and structural roles. We demonstrated that the kernel of an f-derivation, $\mathrm{Ker}_d(A)$, forms a near filter and functions as a subalgebra of the Hilbert algebra A. These results deepened the understanding of the algebraic structure of Hilbert algebras and provided new insights into the broader implications of derivations in algebraic logic. Future research could extend these concepts to other non-classical algebraic systems and explore their applications in logical frameworks and computational logic models.

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