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M-open Maps and its Applications in Pythagorean Fuzzy Topological Spaces

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Abstract:

In this paper, we introduce and investigate Pythagorean fuzzy M-open and closed maps in Pythagorean fuzzy topological spaces and also discuss about some properties and characterization of Pythagorean fuzzy maps. Also one real life applications, we applied entropy measure for decision making problem of diet selection based on the performance.

Keywords Pythagorean fuzzy M-open maps, Pythagorean fuzzy M-closed maps, Pythagorean Fuzzy Entropy.

1. Introduction

Considering the imprecision in decision-making, Zadeh [35] introduced the idea of fuzzy set which has a membership function, μ that assigns to each element of the universe of discourse, a number from the unit nterval $[0,1]$ to indicate the degree of belongingness to the set under consideration. The notion of fuzzy sets generalizes classical sets theory by allowing intermediate situations between the whole and nothing. In a fuzzy set, a membership function is defined to describe the degree of membership of an element to a class. The membership value ranges from 0 to 1, where 0 shows that the element does not belong to a class, 1 means belongs, and other values indicate the degree of membership to a class. For fuzzy sets, the membership function replaced the characteristic function in crisp sets. The concept of fuzzy set theory seems to be inconclusive because of the exclusion of nonmembership function and the disregard for the possibility of hesitation margin.

Atanassov critically studied these shortcomings and proposed a concept called intuitionistic fuzzy sets (IFSs) [1, 2, 4, 5]. The construct (that is, IFS's) incorporates both membership function, μ and nonmembership function, ν with hesitation margin, π (that is, neither membership nor non-membership functions), such that $\mu+\nu\leq 1$ and $\mu+\nu+\pi=1$. Atanassov [3] introduced intuitionistic fuzzy sets of second type ($IFSST$) with the property that the sum of the square of the membership and non-membership degrees is less than or equal to one. This concept generalizes IFS's in a way. The notion of IFS's provides a flexible framework to elaborate uncertainty and vagueness.

The idea of IFS seems to be resourceful in modelling many real-life situations like medical diagnosis [7, 8, 12, 28, 29], career determination [10], selection process [11], and multi-criteria decision-making [15, 16, 17], among others.

There are situations where $\mu + \nu \geq 1$ unlike the cases capture in IFS's. This limitation in IFS naturally led to a construct, called Pythagorean fuzzy sets (pfs's). Pythagorean fuzzy set (pfs) proposed in [32, 33, 34] is a new tool to deal with vagueness considering the membership grade, μ and non-membership grade, ν satisfying the conditions $\mu + \nu \leq 1$ or $\mu + \nu \geq 1$, and also, it follows that $\mu^2 + \nu^2 + \pi^2 = 1$, where π is the Pythagorean fuzzy set index. In fact, the origin of Pythagorean fuzzy sets emanated from $IFSST$ earlier studied in the literature. As a generalized set, PFS has close relationship with IFS. The construct of PFS's can be used to characterize uncertain information more sufficiently and accurately than IFS. Garg [14] presented an improved score function for the ranking order of interval-valued Pythagorean fuzzy sets ($IVPFS$s). Based on it, a Pythagorean fuzzy technique for order of preference by similarity to ideal solution ($TOPSIS$) method by taking the preferences of the experts in the form of interval-valued Pythagorean fuzzy decision matrices was discussed. Other explorations of the theory of PFS's can be found in [6, 9, 13, 18, 19, 25, 26]. Saha [27] defined δ-open sets in topological spaces. Vadivel et al. [31] introduced δ-open sets in a neutrosophic topological space. The notion of M-open sets in topological spaces were introduced by El-Maghrabi and Al-Juhani [23] in 2011 and studied some of their properties. The class of sets namely, M-open sets are playing more important role in topological spaces, because of their applications in various fields of Mathematics and other real fields. Recently, Jeeva et al. [20, 21, 22] introduced neutrosophic soft M-open sets in neutrosophic topological spaces and developed the concepts of neutrosophic soft M-Continuity and M-Irresolute maps.

Entropy can be viewed as a gauge of the degree of uncertainty present in a set, regardless of how fuzzy, intuitionistic, ambiguous, etc. the set may be. Since the pfs in this case can also handle uncertain data, it follows naturally that we are also interested in determining the entropy of an pfs. In 1965, Zadeh [35] made the first reference to entropy as a fuzziness metric. More recently, De Luca-Termini [8] axiomatized the entropy that is not probabilistic.

The remainder of this paper is organized as follows. In section 2, some basic definitions of fs's, IFS's and pfs's are briefly reviewed. In sections 3 and 4, We develop the concept of some Pythagorean fuzzy open and closed maps in Pythagorean fuzzy topological space and also specialized some of their basic properties with examples. Finally, we presented an entropy measure for pfs's and one real-world scenarios where this entropy measure can be used are mentioned in section 4. The paper is concluded in section 5.

2 Preliminaries

We recall some basic notions of fuzzy sets, IFS's and pfs's .

Definition 2.1 *[35] Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0,1]$. That is:*

$$\mu_A(x) = \begin{cases} 1, & \text{if} \quad x \in X \\ 0, & \text{if} \quad x \notin X \\ (0,1) & \text{if } x \text{ ispartlyin } X. \end{cases}$$

Alternatively, a fuzzy set A in X is an object having the form $A = \{< x, \mu_A(x) > | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x)}{x} \right\rangle | x \in X\right\}$, where the function $\mu_A(x): X \to [0,1]$ defines the degree of membership of the element, $x \in X$.

The closer the membership value $\mu_A(x)$ to 1, the more x belongs to A, where the grades 1 and 0 represent full membership and full nonmembership. Fuzzy set is a collection of objects with graded membership, that is, having degree of membership. Fuzzy set is an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in a binary terms according to a bivalent condition; an element either belongs or does not belong to the set. Classical bivalent sets are in fuzzy set theory called crisp sets. Fuzzy sets are generalized classical sets, since the indicator function of classical sets is special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Fuzzy sets theory permits the gradual assessment of the membership of element in a set; this is described with the aid of a membership function valued in the real unit interval [0,1].

Let us consider two examples:

(i) all employees of XYZ who are over $1.8m$ in height; (ii) all employees of XYZ who are tall. The first example is a classical set with a universe (all XYZ employees) and a membership rule that divides the universe into members (those over $1.8m$) and nonmembers. The second example is a fuzzy set, because some employees are definitely in the set and some are definitely not in the set, but some are borderline.

This distinction between the ins, the outs, and the borderline is made more exact by the membership function, μ. If we return to our second example and let A represent the fuzzy set of all tall employees and x represent a member of the universe X (i.e. all employees), then $\mu_A(x)$ would be $\mu_A(x) = 1$ if x is definitely tall or $\mu_A(x) = 0$ if x is definitely not tall or $0 < \mu_A(x) < 1$ for borderline cases.

Definition 2.2 *[1, 2, 4, 5] Let a nonempty set X be fixed. An IFS A in X is an object having the form:* $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ *or* $A = \left\{\left\langle \frac{\mu_A(x), \nu_A(x)}{x} \right\rangle | x \in X\right\}$*, where the functions* $\mu_A(x): X \to [0,1]$ *and* $\nu_A(x): X \to [0,1]$ *define the degree of membership and the degree of nonmembership, respectively, of the element* $x \in X$ *to A, which is a subset of X, and for every* $x \in X$*:* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$*. For each A in X:* $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ *is the intuitionistic fuzzy set index or hesitation margin of* x *in* X*. The hesitation margin* $\pi_A(x)$ *is the degree of nondeterminacy of* $x \in X$ *to the set A and* $\pi_A(x) \in [0,1]$*. The hesitation margin is the function that expresses lack of knowledge of whether* $x \in X$ *or* $x \notin X$*. Thus:* $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$*.*

Example 2.1 *Let* $X = \{x, y, z\}$ *be a fixed universe of discourse and* $A = \left\{\left\langle \frac{0.6, 0.1}{x} \right\rangle, \left\langle \frac{0.8, 0.1}{y} \right\rangle, \left\langle \frac{0.5, 0.3}{z} \right\rangle\right\}$*, be the intuitionistic fuzzy set in* X*. The hesitation margins of the elements* x, y, z *to* A *are as follows:* $\pi_A(x) = 0.3, \pi_A(y) = 0.1$ *and* $\pi_A(z) = 0.2$*.*

Definition 2.3 *[32, 33, 34] Let X be a universal set. Then, a Pythagorean fuzzy set A, which is a set of ordered pairs over X, is defined by the following:* $A = \{< x, \mu_A(x), \nu_A(x) | x \in X\}$ *or* $A =$

$\left\{\left\langle \frac{\mu_A(x),\nu_A(x)}{x} \right\rangle | x \in X\right\}$*, where the functions* $\mu_A(x): X \to [0,1]$ *and* $\nu_A(x): X \to [0,1]$ *define the degree of membership and the degree of nonmembership, respectively, of the element* $x \in X$ *to* A*, which is a subset of* X*, and for every* $x \in X$*,* $0 \leq (\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$*. Supposing* $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$*, then there is a degree of indeterminacy of* $x \in X$ *to* A *defined by* $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\nu_A(x))^2]}$ *and* $\pi_A(x) \in [0,1]$*. In what follows,* $(\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 = 1$*. Otherwise,* $\pi_A(x) = 0$ *whenever* $(\mu_A(x))^2 + (\nu_A(x))^2 = 1$*. We denote the set of all PFS's over X by* $pfs(X)$*.*

Definition 2.4 *[34] Let* A *and* B *be* pfs*'s of the forms* $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ *and* $B = \{< a, \lambda_B(a), \mu_B(a) > | a \in X\}$*. Then [(i)]*

1. $A \subseteq B$ if and only if $\lambda_A(a) \leq \lambda_B(a)$ and $\mu_A(a) \geq \mu_B(a)$ for all $a \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $\bar{A} = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$.
4. $A \cap B = \{< a, \lambda_A(a) \wedge \lambda_B(a), \mu_A(a) \vee \mu_B(a) > | a \in X\}$.
5. $A \cup B = \{< a, \lambda_A(a) \vee \lambda_B(a), \mu_A(a) \wedge \mu_B(a) > | a \in X\}$.
6. $0_X = \{< a, 0,1 > | a \in X\}$ and $1_X = \{< a, 1,0 > | a \in X\}$.
7. $\overline{1} = 0$ and $\overline{0} = 1$.

Definition 2.5 *[24] An Pythagorean fuzzy topology by subsets of a non-empty set* X *is a family* τ *of* pfs*'s satisfying the following axioms. [(i)]*

1. $\phi, X \in \tau$.
2. $G_1 \cap G_2 \in \tau$ for every $G_1, G_2 \in \tau$ and
3. $\bigcup \quad G_i \in \tau$ for any arbitrary family $\{G_i | i \in j\} \subseteq \tau$. The pair (X, τ) is called an Pythagorean fuzzy topological space ($pfts$ in short) and any pfs G in τ is called an Pythagorean fuzzy open set ($pfos$ in short) in X. The complement $\bar{A}$ of an Pythagorean fuzzy open set A in an $pfts(X, \tau)$ is called an Pythagorean fuzzy closed set ($pfcs$ in short).

Definition 2.6 *[24] Let* (X, τ) *be an* $pfts$ *and* $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ *be an* pfs *in* X*. Then the interior and the closure of* A *are denoted by* $pfint(A)$ *and* $pfcl(A)$ *and are defined as follows:* $pfcl(A) = \cap \{K | K \text{ isan } pfcs \text{ and } A \subseteq K\}$ *and* $pfint(A) = \cup \{G | G \text{ isan } pfos \text{ and } G \subseteq A\}$*. Also, it can be established that* $pfcl(A)$ *is an* $pfcs$ *and* $pfint(A)$ *is an* $pfos$*,* A *is an* $pfcs$ *if and only if* $pfcl(A) = A$ *and* A *is an* $pfos$ *if and only if* $pfint(A) = A$*. We say that* A *is* pf*-dense if* $pfcl(A) = X$*.*

Lemma 2.1 *[30] For any Pythagorean fuzzy set* A *in* (X, τ)*, we have* $X - pfint(A) = pfcl(X - A)$ *and* $X - pfcl(A) = pfint(X - A)$*.*

Definition 2.7 *[30] Let* (X, τ) *be an* $pfts$ *and* A *be an* pfs*. Then* A *is said to be an Pythagorean fuzzy (i) regular open set (*$pfros$ *in short) if* $A = pfint(pfcl(A))$*. (ii) regular closed set (*$pfrcs$ *in short) if* $A = pfcl(pfint(A))$*. By Lemma 2.1, it follows that* A *is an* $pfros$ *iff* $\bar{A}$ *is an* $pfrcs$*.*

3 Pythagorean fuzzy M-open mappings

Definition 3.1 *Let* (X_I, Γ_P) *(or* X_I*) be an* $pfts$ *and* $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_I\}$ *be an* pfs *in* X_I*. Then the (i)* $pf\delta$*-interior of* A *are denoted by* $pf\delta int(A)$ *and are defined as follows.* $pf\delta int(A) = \cup \{G | G \text{ is an } pfros \text{ and } G \subseteq A\}$*. (ii)* $pf\delta$*-closure of* A *are denoted by* $pf\delta cl(A)$ *and are defined as follows.* $pf\delta cl(A) = \cap \{K | K \text{ is an } pfrcs \text{ and } A \subseteq K\}$.

Definition 3.2 *Let* (X_I, Γ_P) *be an* $pfts$ *and* $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_I\}$ *be an* pfs *in* X_I*. A set* A *is said to be* pf *[(i)]*

1. δ-open set (briefly, $pf\delta os$) if $A = pf\delta int(A)$,
2. δ-pre open set (briefly, $pf\delta\mathcal{P}os$) if $A \subseteq pfint(pf\delta cl(A))$,
3. δ-semi open set (briefly, $pf\delta\mathcal{S}os$) if $A \subseteq pfcl(pf\delta int(A))$,
4. e open set (briefly, $pfeos$) if $A \subseteq pfcl(pf\delta int(A)) \cup pfint(pf\delta cl(A))$,
5. δ (resp. δ-pre, δ-semi and e) dense if $pf\delta cl(A)$ (resp. $pf\delta\mathcal{P}cl(A), pf\delta\mathcal{S}cl(A)$ and $pfecl(A)) = X_1$.

The complement of an $pf\delta os$ (resp. $pf\delta\mathcal{P}os, pf\delta\mathcal{S}os$ and $pfeos$) is called an $pf\delta$ (resp. $pf\delta\mathcal{P}, pf\delta\mathcal{S}$ and pfe) closed set (briefly, $pf\delta cs$ (resp. $pf\delta\mathcal{P}cs, pf\delta\mathcal{S}cs$ and $pfecs$)) in X_1.

The family of all $pf\delta os$ (resp. $pf\delta cs, pf\delta\mathcal{P}os, pf\delta\mathcal{P}cs, pf\delta\mathcal{S}os, pf\delta\mathcal{S}cs, pfeos$ and $pfecs$) of X_1 is denoted by $pf\delta OS(X_1)$, (resp. $pf\delta CS(X_1), pf\delta\mathcal{P}OS(X_1), pf\delta\mathcal{P}CS(X_1), pf\delta\mathcal{S}OS(X_1), pf\delta\mathcal{S}CS(X_1)$, $pfeOS(X_1)$ and $pfeCS(X_1)$).

Definition 3.3 *Let* (X, τ) *be an* $pfts$ *and* $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_I\}$ *be an* pfs *in* X_I*. Then the (i)* $pf\delta$*-pre (resp.* $pf\delta$*-semi and* pfe*)-interior of* A *are denoted by* $pf\delta\mathcal{P}int(A)$ *(resp.* $pf\delta\mathcal{S}int(A)$ *and* $pfeint(A)$*) and are defined as follows:* $pf\delta\mathcal{P}int(A)$ *(resp.* $pf\delta\mathcal{S}int(A)$ *and* $pfeint(A)$ $= \cup \{G | G$ *in a* $pf\delta\mathcal{P}os$ *(resp.* $pf\delta\mathcal{S}os$ *and* $pfeos$*) and* $G \subseteq A\}$*, (ii)* $pf\delta$*-pre (resp.* $pf\delta$*-semi and* pfe*)-closure of* A *are denoted by* $pf\delta\mathcal{P}cl(A)$ *(resp.* $pf\delta\mathcal{S}cl(A)$ *and* $pfecl(A)$*) and are defined as follows:* $pf\delta\mathcal{P}cl(A)$ *(resp.* $pf\delta\mathcal{S}cl(A)$ *and* $pfecl(A)) = \cap \{K | K$ *is an* $pf\delta\mathcal{P}cs$ *(resp.* $pf\delta\mathcal{S}cs, pfecs$*) and* $A \subseteq K\}$.

Definition 3.4 *Let* (X_I, Γ_P) *be an* $pfts$ *and* $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_I\}$ *be an* pfs *in* X_I*. A set* A *is said to be* pf

1. θ-interior of A (briefly, $pf\theta int(A)$) is defined by $pf\theta int(A) = \cup \{pfint(B) : B \subseteq A \,\&\, B$ isa $pfcs$ in $X_1\}$.
2. θ-open set (briefly, $pf\theta os$) if $A = pf\theta int(A)$.
3. θ -semi open set (briefly, $pf\theta\mathcal{S}os$) if $A \subseteq pfcl(pf\theta int(A))$.
4. M-open set (briefly, $pfMos$) if $A \subseteq pfcl(pf\theta int(A)) \cup pfint(pf\delta cl(A))$.

The complement of a $pfMos$ (resp. $pf\theta os$ & $pf\theta\mathcal{S}os$) is called an pfM (resp. $pf\theta$ & $pf\theta\mathcal{S}$) closed set (briefly, $pfMcs$ (resp. $pf\theta cs$ & $pf\theta\mathcal{S}cs$)) in X_1.

The family of all $pf\theta os$ (resp. $pf\theta cs, pf\theta\mathcal{S}os, pf\theta\mathcal{S}cs, pfMos$ and $pfMcs$) of X_1 is denoted by $pf\theta OS(X_1)$, (resp. $pf\theta CS(X_1), pf\theta\mathcal{S}OS(X_1), pf\theta\mathcal{S}CS(X_1), pfMOS(X_1)$ and $pfMCS(X_1)$).

Definition 3.5 *Let (X_1, Γ_P) be an $pfts$ and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_1\}$ be an pfs in X_1. Then the pf*

1. M (resp. $pf\theta$-semi)-interior of A (briefly, $pfMint(A)$ (resp. $pf\theta\mathcal{S}int(A)$) is defined by $pfMint(A)$ (resp. $pf\theta int(A)$ and $pf\theta\mathcal{S}int(A)$) $= \cup \{B: B \subseteq A$ and B is a $pfMos$ (resp. $pf\theta\mathcal{S}os$) in $X_1\}$.
2. M (resp. θ-semi)-closure of A (briefly, $pfMcl(A)$ (resp. $pf\theta\mathcal{S}cl(A)$) is defined by $pfMcl(A)$ (resp. $pf\theta\mathcal{S}cl(A)$) $= \cap \{B: A \subseteq B$ and A is a $pfMcs$ (resp. $pf\theta\mathcal{S}cs$) in $X_1\}$.

Definition 3.6 *Let (X_1, Γ_P) and (X_2, Ψ_P) be any two $pfts$'s. A mapping $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ is said to be a Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, e, θ, $\theta\mathcal{S}$ and M)-continuous (briefly, $pfCts$ (resp. $pf\delta Cts$, $pf\delta\mathcal{P}Cts$, $pf\delta\mathcal{S}Cts$, $pfeCts$, $pf\theta Cts$, $pf\theta\mathcal{S}Cts$ and $pfMCts$)) if the inverse image of every $pfos$ in (X_2, Ψ_P) is a $pfos$ (resp. $pf\delta os$, $pf\delta\mathcal{P}os$, $pf\delta\mathcal{S}os$, $pfeos$, $pf\theta os$, $pf\theta\mathcal{S}os$ and $pfMos$) in (X_1, Γ_P).*

Definition 3.7 *Let (X_1, Γ_P) and (X_2, Ψ_P) be any two $pfts$'s. A mapping $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ is said to be a Pythagorean fuzzy (resp. θ, $\theta\mathcal{S}$, δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, M and e)-open (briefly, pfO (resp. $pf\theta O$, $pf\theta\mathcal{S}O$, $pf\delta O$, $pf\delta\mathcal{P}O$, $pf\delta\mathcal{S}O$, $pfMO$ and $pfeO$)) mapping if the image of every $pfos$ in (X_1, Γ_P) is a $pfos$ (resp. $pf\theta os$, $pf\theta\mathcal{S}os$, $pf\delta os$, $pf\delta\mathcal{P}os$, $pf\delta\mathcal{S}os$, $pfMos$ and $pfeos$) in (X_2, Ψ_P).*

Proposition 3.1 *Let (X_1, Γ_P) & (X_2, Ψ_P) be a $pfts$'s. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be a mapping. Then the following statements are hold for $pfts$, but not conversely.*

1. Every $pf\theta O$ is a pfO.
2. Every $pf\theta O$ is a $pf\theta\mathcal{S}O$.
3. Every $pf\theta\mathcal{S}O$ is a $pfMO$.
4. Every $pf\delta O$ is a $pf\delta\mathcal{S}O$.
5. Every $pf\delta O$ is a $pf\delta\mathcal{P}O$.
6. Every $pf\delta\mathcal{S}O$ is a $pfeO$.
7. Every $pf\delta\mathcal{P}O$ is a $pfMO$.
8. Every $pfMO$ is a $pfeO$.
9. Every $pf\delta O$ is a pfO.

Proof.

1. Let B be a $pfos$ in (X_1, Γ_P). Since h_P is $pf\theta O$, $h_P(B)$ is $pf\theta os$ in (X_2, Ψ_P). Since every $pf\theta os$ is a $pfos$, $h_P(B)$ is a $pfos$ in (X_2, Ψ_P). Hence, h_P is a pfO.
2. Let B be a $pfos$ in (X_1, Γ_P). Since h_P is $pf\theta O$, $h_P(B)$ is $pf\theta os$ in (X_2, Ψ_P). Since every $pf\theta os$ is a $pf\theta\mathcal{S}os$, $h_P(B)$ is a $pf\theta\mathcal{S}os$ in (X_2, Ψ_P). Hence, h_P is a $pf\theta\mathcal{S}O$.

3. Let B be a $pfos$ in (X_1,Γ_P). Since h_P is $pf\theta\mathcal{S}O$, $h_P(B)$ is $pf\theta\mathcal{S}os$ in (X_2,Ψ_P). Since every $pf\theta\mathcal{S}os$ is a $pfMos$, $h_P(B)$ is a $pfMos$ in (X_2,Ψ_P). Hence, h_P is a $pfMO$.

4. Let B be a $pfos$ in (X_1,Γ_P). Since h_P is $pf\delta O$, $h_P(B)$ is $pf\delta os$ in (X_2,Ψ_P). Since every $pf\delta os$ is a $pf\delta\mathcal{S}os$, $h_P(B)$ is a $pf\delta\mathcal{S}os$ in (X_2,Ψ_P). Hence, h_P is a $pf\delta\mathcal{S}O$.

5. Let B be a $pfos$ in (X_1,Γ_P). Since h_P is $pf\delta O$, $h_P(B)$ is $pf\delta os$ in (X_2,Ψ_P). Since every $pf\delta os$ is a $pf\delta\mathcal{P}os$, $h_P(B)$ is a $pf\delta\mathcal{P}os$ in (X_2,Ψ_P). Hence, h_P is a $pf\delta\mathcal{P}O$.

6. Let B be a $pfos$ in (X_1,Γ_P). Since h_P is $pf\delta\mathcal{S}O$, $h_P(B)$ is $pf\delta\mathcal{S}os$ in (X_2,Ψ_P). Since every $pf\delta\mathcal{S}os$ is a $pfeos$, $h_P(B)$ is a $pfeos$ in (X_2,Ψ_P). Hence, h_P is a $pfeO$.

7. Let B be a $pfos$ in (X_1,Γ_P). Since h_P is $pf\delta\mathcal{P}O$, $h_P(B)$ is $pf\delta\mathcal{P}os$ in (X_2,Ψ_P). Since every $pf\delta\mathcal{P}os$ is a $pfMos$, $h_P(B)$ is a $pfMos$ in (X_2,Ψ_P). Hence, h_P is a $pfMO$.

8. Let B be a $pfos$ in (X_1,Γ_P). Since h_P is $pfMO$, $h_P(B)$ is $pfMos$ in (X_2,Ψ_P). Since every $pfMos$ is a $pfeos$, $h_P(B)$ is a $pfeos$ in (X_2,Ψ_P). Hence, h_P is a $pfeO$.

9. Let B be a $pfos$ in (X_1,Γ_P). Since h_P is $pf\delta O$, $h_P(B)$ is $pf\delta os$ in (X_2,Ψ_P). Since every $pf\delta os$ is a $pfos$, $h_P(B)$ is a $pfos$ in (X_2,Ψ_P). Hence, h_P is a pfO

Remark 3.1 *We obtain the following diagram from the results are discussed above.*

$$\begin{tikzcd}
pf\theta O \arrow[rr] \arrow[d] & & pfO \\
pf\theta\mathcal{S}O \arrow[ddd] & & pf\delta O \arrow[u] \arrow[dl] \arrow[ddd] \\
& pf\delta\mathcal{P}O \arrow[ddl] & \\
& & \\
pfMO \arrow[dr] & & pf\delta\mathcal{S}O \arrow[dl] \\
& pfeO &
\end{tikzcd}$$

Note: $A \rightarrow B$ denotes A implies B, but not conversely.

Example 3.1 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3 *&* A_4 *in* X_1 *are defined as,*

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

Now, we have $\Gamma_P = \Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \rightarrow (X_2, \Psi_P)$ be an identity mapping. Then, h_P is pfO but not $pf\theta O$, because the set A_1 is $pfos$ in X_1 but $h_P(A_1) = A_1$ is not $pf\theta os$ in X_2.

Example 3.2 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

$$B_1 = \{< x_1, 0.80, 0.20 >, < x_2, 0.60, 0.40 >\}$$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \rightarrow (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pf\theta\mathcal{S}O$ (resp. $pf\delta\mathcal{S}O$) but not $pf\theta O$ (resp. $pf\delta O$), because the set B_1 is $pfos$ in X_1 but $h_P(B_1) = B_1$ is not $pf\theta os$ (resp. $pf\delta os$) in X_2.

Example 3.3 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$$A_1 = B_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \rightarrow (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfMO$ but not $pf\theta\mathcal{S}O$, because the set B_1 is $pfos$ in X_1 but $h_P(B_1) = B_1$ is not $pf\theta\mathcal{S}os$ in X_2.

Example 3.4 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

$$B_1 = \{< x_1, 0.40, 0.20 >, < x_2, 0.40, 0.40 >\}$$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \rightarrow (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfeO$ but not $pfMO$, because the set B_1 is $pfos$ in X_1 but $h_P(B_1) = B_1$ is not $pfMos$ in X_2.

Example 3.5 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$B_1 = A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is pfO (resp. $pfeO$ and $pf\delta\mathcal{P}O$) but not $pf\delta O$ (resp. $pf\delta\mathcal{S}O$ and $pf\delta O$), because the set B_1 is $pfos$ in X_1 but $h_P(B_1) = B_1$ is not $pf\delta os$ (resp. $pf\delta\mathcal{S}os$ and $pf\delta os$) in X_2.

Example 3.6 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

$$B_1 = \{< x_1, 0.80, 0.20 >, < x_2, 0.60, 0.30 >\}$$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfMO$ but not $pf\delta\mathcal{P}O$, because the set B_1 is $pfos$ in X_1 but $h_P(B_1) = B_1$ is not $pf\delta\mathcal{P}os$ in X_2.

Theorem 3.1 *A mapping* $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ *is* $pfMO$ *iff for every* pfs K *of* (X_1, Γ_P), $h_P(pfint(K)) \subseteq pfMint(h_P(K))$.

Proof. Necessity: Let h_P be a $pfMO$ mapping and K be a $pfos$ in (X_1, Γ_P). Now, $pfint(K) \subseteq K$ implies $h_P(pfint(K)) \subseteq h_P(K)$. Since h_P is a $pfMO$ mapping, $h_P(pfint(K))$ is $pfMos$ in (X_2, Ψ_P) such that $h_P(pfint(K)) \subseteq h_P(K)$. Therefore $h_P(pfint(K)) \subseteq pfMint(h_P(K))$.

Sufficiency: Assume K is a $pfos$ of (X_1, Γ_P). Then $h_P(K) = h_P(pfint(K)) \subseteq pfMint(h_P(K))$. But $pfMinth_P(K) \subseteq h_P(K)$. So $h_P(K) = pfMint(K)$ which implies $h_P(K)$ is a $pfMos$ of (X_2, Ψ_P) and hence h_P is a $pfMO$

Theorem 3.2 *If* $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ *is a* $pfMO$ *mapping, then* $pfint(h_P^{-1}(K)) \subseteq h_P^{-1}(pfMint(K))$ *for every* pfs K *of* (X_2, Ψ_P).

Proof. Let K be a pfs of (X_2, Ψ_P). Then $pfint(h_P^{-1}(K))$ is a $pfos$ in (X_1, Γ_P). Since h_P is $pfMO$, $h_P(pfint(h_P^{-1}(K)))$ is $pfMos$ in (X_2, Ψ_P) and hence $h_P(pfint(h_P^{-1}(K))) \subseteq pfMint(h_P(h_P^{-1}(K))) \subseteq pfMint(K)$. Thus $pfint(h_P^{-1}(K)) \subseteq h_P^{-1}(pfMint(K))$.

Theorem 3.3 *A mapping* $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ *is* $pfMO$ *iff for each* pfs G *of* (X_2, Ψ_P) *and for each* $pfcs$ K *of* (X_1, Γ_P) *containing* $h_P^{-1}(G)$*, there is a* $pfMcs$ H *of* (X_2, Ψ_P) *such that* $G \subseteq H$ *and* $h_P^{-1}(H) \subseteq K$.

Proof. Necessity: Assume h_P is a $pfMO$ mapping. Let G be the $pfcs$ of (X_2, Ψ_P) and K is a $pfcs$ of (X_1, Γ_P) such that $h_P^{-1}(G) \subseteq K$. Then $H = (h_P^{-1}(K^c))^c$ is $pfMcs$ of (X_2, Ψ_P) such that $h_P^{-1}(H) \subseteq K$.

Sufficiency: Assume K is a $pfos$ of (X_1,Γ_P). Then $h_P^{-1}((h_P(K))^c) \subseteq K^c$ and K^c is $pfcs$ in (X_1,Γ_P). By hypothesis, there is a $pfMcs\ H$ of (X_2,Ψ_P) such that $(h_P(K))^c \subseteq H$ and $h_P^{-1}(H) \subseteq K^c$. Therefore $K \subseteq (h_P^{-1}(H))^c$. Hence $H^c \subseteq h_P(K) \subseteq h_P((h_P^{-1}(H))^c) \subseteq H^c$ which implies $h_P(K) = H^c$. Since H^c is $pfMos$ of (X_2,Ψ_P), $h_P(K)$ is $pfMO$ in (X_2,Ψ_P) and thus h_P is $pfMO$ mapping.

Theorem 3.4 *A mapping $h_P\colon (X_1,\Gamma_P) \to (X_2,\Psi_P)$ is $pfMO$ iff $h_P^{-1}(pfMcl(G)) \subseteq pfcl(h_P^{-1}(G))$ for every pfs G of (X_2,Ψ_P).*

Proof. Necessity: Assume h_P is a $pfMO$ mapping. For any $pfs\ G$ of (X_2,Ψ_P), $h_P^{-1}(G) \subseteq pfcl(h_P^{-1}(G))$. Therefore by Theorem 3.3, there exists a $pfMcs\ K$ in (X_2,Ψ_P) such that $G \subseteq K$ and $h_P^{-1}(K) \subseteq pfcl(h_P^{-1}(G))$. Therefore we obtain that $h_P^{-1}(pfMcl(G)) \subseteq h_P^{-1}(K) \subseteq pfcl(h_P^{-1}(G))$.

Sufficiency: Assume G is a pfs of (X_2,Ψ_P) and K is a $pfcs$ of (X_1,Γ_P) containing $h_P^{-1}(G)$. Put $H = pfcl(G)$, then $G \subseteq H$ and H is $pfMcs$ and $h_P^{-1}(H) \subsetneq pfcl(h_P^{-1}(G)) \subseteq K$. Then by Theorem 3.3, h_P is $pfMO$ mapping.

Theorem 3.5 *If $h_P\colon (X_1,\Gamma_P) \to (X_2,\Psi_P)$ and $g_P\colon (X_2,\Psi_P) \to (X_3,\Phi_P)$ be two pf mappings and $g_P \circ h_P\colon (X_1,\Gamma_P) \to (X_3,\Phi_P)$ is $pfMO$. If $g_P\colon (X_2,\Psi_P) \to (X_3,\Phi_P)$ is $pfMIrr$, then $h_P\colon (X_1,\Gamma_P) \to (X_2,\Psi_P)$ is $pfMO$ mapping.*

Proof. Let K be a $pfos$ in (X_1,Γ_P). Then $(g_P \circ h_P)(K)$ is $pfMos$ of (X_3,Φ_P) because $g_p \circ h_P$ is $pfMO$ mapping. Since g_P is $pfMIrr$ and $(g_P \circ h_P)(K)$ is $pfMos$ of (X_3,Φ_P), $g_P^{-1}((g_P \circ h_P)(K)) = h_P(K)$ is $pfMos$ in (X_2,Ψ_P). Hence h_P is $pfMO$ mapping

Theorem 3.6 *If $h_P\colon (X_1,\Gamma_P) \to (X_2,\Psi_P)$ is pfO and $g_P\colon (X_2,\Psi_P) \to (X_3,\Phi_P)$ is $pfMO$ mappings, then $g_P \circ h_P\colon (X_1,\Gamma_P) \to (X_3,\Phi_P)$ is $pfMO$.*

Proof. Let K be a $pfos$ in (X_1,Γ_P). Then $h_P(K)$ is a $pfos$ of (X_2,Ψ_P) because h_P is a pfO mapping. Since g_P is $pfMO$, $g_P(h_P(K)) = (g_P \circ h_P)(K)$ is a $pfMos$ of (X_3,Φ_P). Hence $g_P \circ h_P$ is $pfMO$ mapping.

4 Pythagorean fuzzy M-closed mapping

Definition 4.1 *Let (X_1,Γ_P) and (X_2,Ψ_P) be any two $pfts$'s. A mapping $h_P\colon (X_1,\Gamma_P) \to (X_2,\Psi_P)$ is said to be a Pythagorean fuzzy (resp. θ, $\theta\mathcal{S}$, δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, M and e)-closed (briefly, pfC (resp. $pf\theta C$, $pf\theta\mathcal{S}C$, $pf\delta C$, $pf\delta\mathcal{P}C$, $pf\delta\mathcal{S}C$, $pfMC$ and $pfeC$)) mapping if the image of every $pfcs$ in (X_1,Γ_P) is a $pfcs$ (resp. $pf\theta cs$, $pf\theta\mathcal{S}cs$, $pf\delta cs$, $pf\delta\mathcal{P}cs$, $pf\delta\mathcal{S}cs$, $pfMcs$ and $pfecs$) in (X_2,Ψ_P).*

Proposition 4.1 *Let (X_1,Γ_P) & (X_2,Ψ_P) be a $pfts$'s. Let $h_P\colon (X_1,\Gamma_P) \to (X_2,\Psi_P)$ be a mapping. Then the following statements are hold for $pfts$, but not conversely.*

1. Every $pf\theta C$ is a pfC.
2. Every $pf\theta C$ is a $pf\theta\mathcal{S}C$.
3. Every $pf\theta\mathcal{S}C$ is a $pfMC$.
4. Every $pf\delta C$ is a $pf\delta\mathcal{S}C$.
5. Every $pf\delta C$ is a $pf\delta\mathcal{P}C$.

6. Every $pf\delta\mathcal{S}C$ is a $pfeC$.
7. Every $pf\delta\mathcal{P}C$ is a $pfMC$.
8. Every $pfMC$ is a $pfeC$.
9. Every $pf\delta C$ is a pfC.

Proof.

1. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pf\theta C$, $h_P(B)$ is $pf\theta cs$ in (X_2, Ψ_P). Since every $pf\theta cs$ is a $pfcs$, $h_P(B)$ is a $pfcs$ in (X_2, Ψ_P). Hence, h_P is a pfC.

2. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pf\theta C$, $h_P(B)$ is $pf\theta cs$ in (X_2, Ψ_P). Since every $pf\theta cs$ is a $pf\theta\mathcal{S}cs$, $h_P(B)$ is a $pf\theta\mathcal{S}cs$ in (X_2, Ψ_P). Hence, h_P is a $pf\theta\mathcal{S}C$.

3. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pf\theta\mathcal{S}C$, $h_P(B)$ is $pf\theta\mathcal{S}cs$ in (X_2, Ψ_P). Since every $pf\theta\mathcal{S}cs$ is a $pfMcs$, $h_P(B)$ is a $pfMcs$ in (X_2, Ψ_P). Hence, h_P is a $pfMC$.

4. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pf\delta C$, $h_P(B)$ is $pf\delta cs$ in (X_2, Ψ_P). Since every $pf\delta cs$ is a $pf\delta\mathcal{S}cs$, $h_P(B)$ is a $pf\delta\mathcal{S}cs$ in (X_2, Ψ_P). Hence, h_P is a $pf\delta\mathcal{S}C$.

5. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pf\delta C$, $h_P(B)$ is $pf\delta cs$ in (X_2, Ψ_P). Since every $pf\delta cs$ is a $pf\delta\mathcal{P}cs$, $h_P(B)$ is a $pf\delta\mathcal{P}cs$ in (X_2, Ψ_P). Hence, h_P is a $pf\delta\mathcal{P}C$.

6. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pf\delta\mathcal{S}C$, $h_P(B)$ is $pf\delta\mathcal{S}cs$ in (X_2, Ψ_P). Since every $pf\delta\mathcal{S}cs$ is a $pfecs$, $h_P(B)$ is a $pfecs$ in (X_2, Ψ_P). Hence, h_P is a $pfeC$.

7. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pf\delta\mathcal{P}C$, $h_P(B)$ is $pf\delta\mathcal{P}cs$ in (X_2, Ψ_P). Since every $pf\delta\mathcal{P}cs$ is a $pfMcs$, $h_P(B)$ is a $pfMcs$ in (X_2, Ψ_P). Hence, h_P is a $pfMC$.

8. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pfMC$, $h_P(B)$ is $pfMcs$ in (X_2, Ψ_P). Since every $pfMcs$ is a $pfecs$, $h_P(B)$ is a $pfecs$ in (X_2, Ψ_P). Hence, h_P is a $pfeC$.

9. Let B be a $pfcs$ in (X_1, Γ_P). Since h_P is $pf\delta C$, $h_P(B)$ is $pf\delta cs$ in (X_2, Ψ_P). Since every $pf\delta cs$ is a $pfcs$, $h_P(B)$ is a $pfcs$ in (X_2, Ψ_P). Hence, h_P is a pfC

Remark 4.1 *We obtain the following diagram from the results are discussed above.*

$$\begin{tikzcd}
pf\theta C \arrow[rr] \arrow[d] & & pfC \\
pf\theta\mathcal{S}C \arrow[dd] & & pf\delta C \arrow[u] \arrow[dl] \arrow[dd] \\
& pf\delta\mathcal{P}C \arrow[dl] & \\
pfMC \arrow[dr] & & pf\delta\mathcal{S}C \arrow[dl] \\
& pfeC &
\end{tikzcd}$$

Note: $A \to B$ denotes A implies B, but not conversely.

Example 4.1 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3 *&* A_4 *in* X_1 *are defined as,*

$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$

$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$

$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$

$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$

Now, we have $\Gamma_P = \Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is pfC but not $pf\theta C$, because the set A_1^c is $pfcs$ in X_1 but $h_P(A_1^c) = A_1^c$ is not $pf\theta cs$ in X_2.

Example 4.2 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$

$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$

$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$

$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$

$B_1 = \{< x_1, 0.80, 0.20 >, < x_2, 0.60, 0.40 >\}$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pf\theta\mathcal{S}C$ (resp. $pf\delta\mathcal{S}C$) but not $pf\theta C$ (resp. $pf\delta C$), because the set B_1^c is $pfcs$ in X_1 but $h_P(B_1^c) = B_1^c$ is not $pf\theta cs$ (resp. $pf\delta cs$) in X_2.

Example 4.3 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$A_1 = B_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$

$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$

$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$

$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfMC$ but not $pf\theta\mathcal{S}C$, because the set B_1^c is $pfcs$ in X_1 but $h_P(B_1^c) = B_1^c$ is not $pf\theta\mathcal{S}cs$ in X_2.

Example 4.4 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$

$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$

$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$

$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$

$B_1 = \{< x_1, 0.40, 0.20 >, < x_2, 0.40, 0.40 >\}$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfeC$ but not $pfMC$, because the set B_1^c is $pfcs$ in X_1 but $h_P(B_1^c) = B_1^c$ is not $pfMcs$ in X_2.

Example 4.5 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$B_1 = A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is pfC (resp. $pfeC$ and $pf\delta\mathcal{P}C$) but not $pf\delta C$ (resp. $pf\delta\mathcal{S}C$ and $pf\delta C$), because the set B_1^c is $pfcs$ in X_1 but $h_P(B_1^c) = B_1^c$ is not $pf\delta cs$ (resp. $pf\delta\mathcal{S}cs$ and $pf\delta cs$) in X_2.

Example 4.6 *Let* $X_1 = X_2 = \{x_1, x_2\}$ *and* pfs*'s* A_1, A_2, A_3, A_4 *in* X_2 *&* B_1 *in* X_1 *are defined as,*

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

$$B_1 = \{< x_1, 0.80, 0.20 >, < x_2, 0.60, 0.30 >\}$$

Now, we have $\Gamma_P = \{0_X, 1_X, B_1\}$ and $\Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfMC$ but not $pf\delta\mathcal{P}C$, because the set B_1^c is $pfcs$ in X_1 but $h_P(B_1^c) = B_1^c$ is not $pf\delta\mathcal{P}cs$ in X_2.

Theorem 4.1 *A mapping* $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ *is* $pfMC$ *iff for each* pfs G *of* (X_2, Ψ_P) *and for each* $pfos$ K *of* (X_1, Γ_P) *containing* $h_P^{-1}(G)$*, there is a* $pfMos$ L *of* (X_2, Ψ_P) *such that* $G \subseteq L$ *and* $h_P^{-1}(L) \subseteq K$.

Proof. Necessity: Assume h_P is a $pfMC$ mapping. Let G be the $pfcs$ of (X_2, Ψ_P) and K is a $pfos$ of (X_1, Γ_P) such that $h_P^{-1}(G) \subseteq K$. Then $L = 1_X - h_P^{-1}(K^c)$ is $pfMos$ of (X_2, Ψ_P) such that $h_P^{-1}(L) \subseteq K$.

Sufficiency: Assume K is a $pfcs$ of (X_1, Γ_P). Then $(h_P(K))^c$ is a pfs of (X_2, Ψ_P) and K^c is $pfos$ in (X_1, Γ_P) such that $h_P^{-1}((h_P(K))^c) \subseteq K^c$. By hypothesis, there is a $pfMos$ L of (X_2, Ψ_P) such that $(h_P(K))^c \subseteq L$ and $h_P^{-1}(L) \subseteq K^c$. Therefore $K \subseteq (h_P^{-1}(L))^c$. Hence $L^c \subseteq h_P(L) \subseteq h_P((h_P^{-1}(L))^c) \subseteq L^c$ which implies $h_P(K) = L^c$. Since L^c is $pfMcs$ of (X_2, Ψ_P), $h_P(K)$ is $pfMcs$ in (X_2, Ψ_P) and thus h_P is $pfMC$ mapping.

Theorem 4.2 *If* $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ *is* pfC *and* $g_P: (X_2, \Psi_P) \to (X_3, \Phi_P)$ *is* $pfMC$*. Then* $g_P \circ h_P: (X_1, \Gamma_P) \to (X_3, \Phi_P)$ *is* $pfMC$.

Proof. Let K be a $pfcs$ in (X_1, Γ_P). Then $h_P(K)$ is $pfcs$ of (X_2, Ψ_P) because h_P is pfC mapping. Now $(g_P \circ h_P)(K) = g_P(h_P(K))$ is $pfMcs$ in (X_3, Φ_P) because g_P is $pfMC$ mapping. Thus $g_P \circ h_P$ is $pfMC$ mapping

Theorem 4.3 *If* $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ *is* $pfMC$ *map, then* $pfMcl(h_P(K)) \subseteq h_P(pfcl(K))$.

Proof. Obvious.

Theorem 4.4 *Let* $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ *and* $g_P: (X_2, \Psi_P) \to (X_3, \Phi_P)$ *are* $pfMC$ *mappings. If every* $pfMcs$ *of* (X_2, Ψ_P) *is* $pfcs$*, then* $g_P \circ h_P: (X_1, \Gamma_P) \to (X_3, \Phi_P)$ *is* $pfMC$.

Proof. Let K be a $pfcs$ in (X_1, Γ_P). Then $h_P(K)$ is $pfMcs$ of (X_2, Ψ_P) because h_P is $pfMC$ mapping. By hypothesis, $h_P(K)$ is $pfcs$ of (X_2, Ψ_P). Now $g_P(h_P(K)) = (g_P \circ h_P)(K)$ is $pfMcs$ in (X_3, Φ_P) because g_P is $pfMC$ mapping. Thus $g_P \circ h_P$ is $pfMC$ mapping

Theorem 4.5 *Let* $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ *be a bijective mapping. Then the following statements are equivalent: [(i)]*

1. h_P is a $pfMO$ mapping.
2. h_P is a $pfMC$ mapping.
3. h_P^{-1} is $pfMCts$ mapping.

Proof. (i) ⇒ (ii): Let us assume that h_P is a $pfMO$ mapping. By definition, K is a $pfos$ in (X_1, Γ_P), then $h_P(K)$ is a $pfMos$ in (X_2, Ψ_P). Here, K is $pfcs$ in (X_1, Γ_P). Then $1_X - K$ is a $pfos$ in (X_1, Γ_P). By assumption, $h_P(1_X - K)$ is a $pfMos$ in (X_2, Ψ_P). Hence, $1_Y - h_P(1_X - K)$ is a $pfMcs$ in (X_2, Ψ_P). Therefore, h_P is a $pfMC$ mapping.

(ii) ⇒ (iii): Let K be a $pfcs$ in (X_1, Γ_P) By (ii), $h_P(K)$ is a $pfMcs$ in (X_2, Ψ_P). Hence, $h_P(K) = (h_P^{-1})^{-1}(K)$. So h_P^{-1} is a $pfMcs$ in (X_2, Ψ_P). Hence, h_P^{-1} is $pfMCts$.

(iii) ⇒ (i): Let K be a $pfos$ in (X_1, Γ_P). By (iii), $(h_P^{-1})^{-1}(K) = h_P(K)$ is a $pfMO$ mapping.

5 Application

Entropy as a measure of fuzziness was first proposed by Zadeh [35]. Later many mathematicians defined several entropy measures. In this section, we focus on defining an entropy measure for pfs that connects the degree of membership and non-membership. As an example, we have applied the proposed entropy measure in the field of seasons.

Definition 5.1 *Let* $A = \{< x, \mu_A(x), \nu_A(x) | x \in X\}$ *be a* pfs *in* X*. The new entropy measure for* A *denoted by* $\varepsilon_{pfs}(A)$*, is a function,* $\varepsilon_{pfs}: \tau_{pfs}(X) \to [0,1]$ *and is defined as* $\varepsilon_{pfs}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n} (\alpha_A - \gamma_A)^2;\ forevery`x_i \in A$*, where* $\tau_{pfs}(X)$ *denote the family of all* pfs*'s on* X*.*

Example 5.1 *The development of the each nation is measured by the economical development, Technical development, Educational growth, AI implementation, Import and Export, Defence status, Sports and etc. In Olympics and commonwealth games the developed countries are the toppers of the list, so the developing countries are also making lot of efforts and allot funds for sports. Even-though* *practice and coaching plays the major role, the diet supplement also plays a vital role in athlete's performance. Each sports academy, there are well trained nutritions to plan for the diet of the athlete.*

In a private sports academy named as X has three diet plans for athlete who is on the training of the national games. Each diet was scheduled for a week, implemented at the gap of one week difference, the performance was rated by four trainers, for each day on the diet week. The rating of the trainers was converted as pfs and entropy measure is used to select the best diet for the athlete . The following table displays pfs of performance of each diet week. Here we use the notations for Diet 1, Diet 2 and Diet 3 are Di_1, Di_2 and Di_3.

Table 1. Reviews of the Hotels based on the Criteria

	Day 1 (D_1)	Day 2 (D_2)	Day 3 (D_3)	Day 4 (D_4)
Di_1	$< Di_1, D_1; 0.8,0.0 >$	$< Di_1, D_2; 0.6,0.6 >$	$< Di_1, D_3; 0.4,0.6 >$	$< Di_1, D_4; 0.3,0.3 >$
Di_2	$< Di_2, D_1; 0.7,0.3 >$	$< Di_2, D_2; 0.8,0.6 >$	$< Di_2, D_3; 0.7,0.3 >$	$< Di_2, D_4; 0.8,0.2 >$
Di_3	$< Di_3, D_1; 0.7,0.1 >$	$< Di_3, D_2; 0.8,0.0 >$	$< Di_3, D_3; 0.7,0.4 >$	$< Di_3, D_4; 0.6,0.3 >$
	Day 5 (D_5)	Day 6 (D_6)	Day 7 (D_7)	
Di_1	$< Di_1, D_5; 0.8,0.2 >$	$< Di_1, D_6; 0.7,0.7 >$	$< Di_1, D_7; 0.7,0.5 >$	
Di_2	$< Di_2, D_5; 0.2,0.4 >$	$< Di_2, D_6; 0.6,0.2 >$	$< Di_2, D_7; 0.7,0.5 >$	
Di_3	$< Di_3, D_5; 0.7,0.5 >$	$< Di_3, D_6; 0.7,0.2 >$	$< Di_3, D_7; 0.8,0.5 >$	

Clearly, all values in the Table 1 are pfs's. Now we calculate the ε_{pfs} of each value.

Table 2. Entropy measure of each Diet.

	entropy measure
Di_1	0.73
Di_2	0.76
Di_3	**0.61**

From Table 2, Clearly that $\varepsilon_{pfs}(Di_3) < \varepsilon_{pfs}(Di_1) < \varepsilon_{pfs}(Di_2)$.

Hence we conclude that Di_3 is the best for the athlete.

6 Conclusion

In this paper, we have studied a new class of maps called Pythagorean fuzzy M open and Pythagorean fuzzy M closed and their properties are discussed. Also we applied entropy measure for decision making problem of calculation of diet selection based on the performance. In future, we decide to apply entropy measure for decision making in various fields.

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