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Degree of Approximation of the Conjugate of Functions Belonging to LIP (α, r) −Class by $(C, 1)(E, q)(E, q)$ Means of Conjugate Fourier Series

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Abstract:
This research paper is related to the degree of approximation of the conjugate of 2π −periodic function belonging to the $\text{Lip}(\alpha, r)(0 < \alpha \leq 1, r \geq 1)$- class by using $(C, 1)(E, q)(E, q)$ means of the conjugate Fourier series. Our result may be useful for the coming researchers in the future.

Keywords: $\text{Lip}(\alpha, r)$ − class, conjugate Fourier series, $(C, 1)(E, q)(E, q)$ means.

1. Introduction

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series and the sequence $\{s_n\}$ its nth partial sum.The sequence -to-sequence transform

$$C_n^1 = \frac{1}{n+1} \sum_{k=0}^{n} s_k, \quad n = 0,1,2, \ldots \tag{1}$$

defines the Cesàro means of order one of $\{s_n\}$. If $lim_{n\to\infty} C_n^1 = s$, the series $\sum_{n=0}^{\infty} u_n$ is said to be $(C, 1)$ summable to s.

The sequence-to-sequence transform

$$E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k, q > 0, n = 0,1,2, \ldots \tag{2}$$

defines the Euler mean of order $q > 0$ of $\{s_n\}$.

$$C_n^1 E_n^q E_n^q = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} s_v \tag{3}$$

The series $\sum_{n=0}^{\infty} u_n$ is said to be $(C, 1)(E, q)(E, q)$ summable to s, if im $_{n\to\infty} C_n^1 E_n^q E_n^q = s$.

For a 2π periodic signal which is integrable in the sense of Lebesgue over $(-\pi, \pi)$.

The conjugate of Fourier series is defined by

$$\sum_{k=1}^{\infty} (b_k coskx - a_k coskx) \tag{4}$$

and nth partial sum is defined by

$$\tilde{s}_n(f; x) = \sum_{k=1}^{\infty} (b_k coskx - a_k coskx) \tag{5}$$

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The conjugate of f denoted by $\tilde{f}$ is defined by

$$\tilde{f}(x) = -\frac{1}{2\pi} lim_{\xi\to 0}\int_{\xi}^{\pi} \psi(t)cos\left(\frac{t}{2}\right) dt,$$

where $\psi(t) = f(x+t) - f(x-t)$

A function $f \in$ Lipα, if

$$|f(x+t) - f(x+t)| = O(|t|^{\alpha}) \quad for \quad 0 < \alpha \leq 1.$$

and $f \in Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi} |f(x)|^r\right)^{\frac{1}{r}} = O(t^{\alpha}),\ \ 0 < \alpha \leq 1, r \geq 1.$$

L_p- norm is defined by

$$f_p = \left(\int_0^{2\pi} |f(x)|^p\right)^{\frac{1}{p}},\ p \geq 1.$$

L∞-norm of a function $f: R \to R$ is defined by f_{∞}

$$f_{\infty} = sup\{|f(x)|/f: R \to R\}$$

The degree of approximation of function $f: R \to R$ by a trigonometric polynomial t_n[1] is defined by

$$t_n - f_{\infty} = sup\{|t_n - f|: x \in R\} or t_n - f_p = min t_n - f.$$

This method of approximation is called trigonometric Fourier approximation.

We also write

$$C_n^1 E_n^q E_n^q = \frac{1}{n+1}\sum_{k=0}^{n} \frac{1}{(1+q)^k}\sum_{u=0}^{k}\binom{k}{u}\frac{q^{k-u}}{(1+q)^u}\sum_{v=0}^{u}\binom{u}{v} q^{u-v}\frac{cos\left(v+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}$$

and $\tau = \left[\frac{1}{t}\right]$, the integral part of $\frac{1}{t}$.

2. Known theorem

Various investigators such as Dhakal[2], Lal and Singh[8], Mittal et al. [6,7], Qureshi[4,5] Sonker and Singh[9] have studied the degree of approximation in various function spaces such as Lip α , Lip(α, r), Lip$(\xi(t), r)$ and weighted $(L_r, \xi(t))$ by using triangular matrix summability and product summability (C,1)(E,1), (N,p_n)(E,1). Sonker and Singh[9] have determined the degree of approximation of the conjugate of signals (functions) belonging to Lip(α, r)-class by$(C,1)(E,q)$ means of conjugate trigonometric Fourier series. Sonker and Singh have proved the following:

Theorem 1

[9] Let $f(x)$ be a 2π-periodic, Lebesgue integrable function and belonging to the Lip(α, r)- class with $r \geq 1$ and $\alpha r \geq 1$. Then the degree of approximation of $\tilde{f}(x)$, the conjugate of $f(x)$ by $(C,1)(E,q)$ means of its conjugate Fourier series is given by

$$C_n^1 E_n^q - \tilde{f}_r = O\left(n^{\frac{1}{r}-\alpha}\right), n = 0,1,2, \ldots ., \tag{6}$$

Main theorem

The objective of this paper is to establish the following theorem.

Theorem 2

Let $f(x)$ be a 2π-periodic, Lebesgue integrable function and belonging to the Lip(α, r)- class with $r \geq 1$ and $\alpha r \geq 1$. Then the degree of approximation of $\tilde{f}(x)$, the conjugate of $f(x)$ by $(C, 1)(E, q)(E, q)$ means of its conjugate Fourier series is given by

$$C_n^1 E_n^q E_n^q - \tilde{f}_r = O\left(n^{\frac{1}{r}-\alpha}\right), n = 0,1,2, \ldots ., \tag{7}$$

provided

$$\left(\int_0^{\frac{\pi}{n+1}} (|\psi(t)|/t^\alpha)^r dt\right)^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right), \tag{8}$$

$$\left(\int_{\frac{\pi}{n+1}}^{\pi} \left(t^{-\delta}|\psi(t)|/t^\alpha\right)^r dt\right)^{\frac{1}{r}} = O\left((n+1)^\delta\right), \tag{9}$$

Where δ is an arbitrary number such that $(\alpha + \delta)s < -1$ and $1/s = 1 - 1/r$ for $r > 1$.

4. Lemmas

We need the following lemmas for the proof of our theorem.

4.1 Lemma

$|K_n(t)| = O\left(\frac{1}{t}\right) + O\left((n+1)t\right)$ for $0 \leq t \leq \frac{\pi}{n+1} \leq \frac{\pi}{v+1}$

Proof.

$$|K_n(t)| = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} \frac{cos\left(v+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}$$

$$= \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} \frac{cos\left(v+1-\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}$$

$$\leq \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} \frac{cos(v+1)t cos\left(\frac{t}{2}\right) + sin(v+1)t sin\left(\frac{t}{2}\right)}{sin\left(\frac{t}{2}\right)}$$

$$= \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} \left[O\left(\frac{1}{t}\right) + O(sin(v+1)t)\right]$$

$$= \left[\frac{1}{(n+1)t} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v}\right]$$

$$+ \left[\frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} (v+1)t\right]$$

$$= O\left[\frac{1}{(n+1)t}(n+1)\right] + O\left[\frac{1}{(n+1)}(n+1)(n+1)t\right]$$

$$= O\left(\frac{1}{t}\right) + O\big((n+1)t\big),$$

In view of $\sin(v+1)t \le (v+1)t$ for $0 \le t \le \frac{\pi}{v+1}$ and $\left(sin\left(\frac{t}{2}\right)\right)^{-1} < \frac{\pi}{t}$ for $0<t\le \pi$ [3, p.247].

4.2 Lemma

$|K_n(t)| = O\left(\frac{1}{t}\right) + O(1)$ for $\frac{\pi}{v+1} \le t \le \pi$.

Proof.

$$|K_n(t)| \le \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} \frac{cos\left(v+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}$$

$$= \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} \frac{cos\left(v+1-\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}$$

$$\le \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} \frac{cos(v+1)tcos\left(\frac{t}{2}\right)+sin(v+1)tsin\left(\frac{t}{2}\right)}{sin\left(\frac{t}{2}\right)}$$

$$= \frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} \left[O\left(\frac{1}{t}\right) + O(1)\right]$$

$$= \left[\frac{1}{(n+1)t} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v}\right]$$

$$+ \left[\frac{1}{(n+1)} \sum_{k=0}^{n} \frac{1}{(1+q)^k} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v}\right]$$

$$= O\left[\frac{1}{(n+1)t}(n+1)\right] + \left[\frac{1}{(n+1)}(n+1)\right]$$

$$= O\left(\frac{1}{t}\right) + O(1),$$

In view of $|sin(v+1)t| \le 1$ and $\left(sin\left(\frac{t}{2}\right)\right)^{-1} \le \frac{\pi}{t}$ for $0 < t \le \pi$ [3, p.247]

5. Proof of main Theorem

The integral representation of $\tilde{s}_n(f;x)$ is given by

$$\tilde{s}_n(f;x) = -\frac{1}{\pi}\int_0^{\pi} \psi(t) \frac{cos\left(\frac{t}{2}\right) - cos\left(n+\frac{1}{2}\right)t}{2sin\left(\frac{t}{2}\right)} dt.$$

Therefore, we have

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(t) \frac{cos\left(n+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)} dt.$$

Now, denoting $(C,1)(E,q)(E,q)$ transform of $\tilde{s}_n(f;x)$ by $C_n^1 E_n^q E_n^q$, we write

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$$C_n^1 E_n^q E_n^q - \tilde{f} = \frac{1}{2\pi(n+1)}\left[\sum_{k=0}^{n} \frac{1}{(1+q)^k} \int_0^{\pi} \frac{\psi(t)}{sin\frac{t}{2}} \sum_{u=0}^{k} \binom{k}{u} \frac{q^{k-u}}{(1+q)^u} \sum_{v=0}^{u} \binom{u}{v} q^{u-v} cos\left(v + \frac{1}{2}\right) t\right] \quad (10)$$

$$= \left[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}\right] \psi(t) K_n(t) dt = I_1 + I_2, say. \quad (11)$$

Using Lemma 4.1, Hölder's inequality, condition (8) and Minkwiski's inequality, we have

$$|I_1| = \int_0^{\frac{\pi}{n+1}} |\psi(t)||K_n(t)| dt$$

$$\leq \left[\int_0^{\frac{\pi}{n+1}} (|\psi(t)/t^{\alpha})^r\right]^{\frac{1}{r}} \left[lim_{\epsilon\to 0} \int_{\epsilon}^{\frac{\pi}{n+1}} (t^{\alpha}|K_n(t)|)^s dt\right]^{\frac{1}{s}}$$

$$= O((n+1)^{-1}) \left[lim_{\epsilon\to 0} \int_{\epsilon}^{\frac{\pi}{n+1}} (t^{\alpha-1} + (n+1)t^{\alpha+1})^s dt\right]^{\frac{1}{s}}$$

$$= O((n+1)^{-1}) \left[\left(lim_{\epsilon\to 0} \int_{\epsilon}^{\frac{\pi}{n+1}} t^{(\alpha-1)s} dt\right)^{\frac{1}{s}} + \left(lim_{\epsilon\to 0} \int_{\epsilon}^{\frac{\pi}{n+1}} (n+1) t^{(\alpha+1)s} dt\right)^{\frac{1}{s}}\right]$$

$$= O((n+1)^{-1})\left[(n+1)^{-\alpha+1-1/s} + (n+1)(n+1)^{-\alpha-1-1/s}\right]$$

$$= O((n+1)^{-1})\left[(n+1)^{-\alpha+1/r} + (n+1)(n+1)^{-\alpha-1-1+1/r}\right]$$

$$= O\left[(n+1)^{-\alpha+\frac{1}{r}-1} + (n+1)^{-\alpha-2+\frac{1}{r}}\right]$$

$$= O\left((n+1)^{-\alpha-1+\frac{1}{r}}\right) \quad (12)$$

Now, we consider

$$|I_2| = \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n+1}} |\psi(t)||K_n(t)| dt.$$

Using Lemma 4.2, condition (9) and Minkowiski's inequality, we have

$$|I_2| \leq \left[\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{t^{\alpha}}\right)^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{\alpha}|K_n(t)|}{t^{-\delta}}\right)^s\right]^{\frac{1}{s}}$$

$$= O\left((n+1)^{\delta}\right) \left[\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{\alpha}}{t^{-\delta}}\left(O\left(\frac{1}{t}\right) + O(1)\right)\right)^s dt\right]^{\frac{1}{s}}$$

$$= O\left((n+1)^{\delta}\right) \left[\int_{\frac{\pi}{n+1}}^{\pi} \left(t^{\alpha+\delta-1} + t^{\alpha+\delta}\right)^s dt\right]^{\frac{1}{s}}$$

$$= O\left((n+1)^{\delta}\right) \left[\left(\int_{\frac{\pi}{n+1}}^{\pi} t^{(\alpha+\delta-1)s} dt\right)^{\frac{1}{s}} + \left(\int_{\frac{\pi}{n+1}}^{\pi} t^{(\alpha+\delta)s} dt\right)^{\frac{1}{s}}\right]$$

$$= O\big((n+1)^{\delta}\big)\left[(n+1)^{(-\alpha-\delta+1)-\frac{1}{s}} + (n+1)^{(-\alpha-\delta)-\frac{1}{s}}\right] \quad (1+(\alpha+\delta)s \le 0)$$

$$= O\left[(n+1)^{-\alpha+1-\frac{1}{s}} + (n+1)^{-\alpha-\frac{1}{s}}\right]$$

$$= O\left[(n+1)^{-\alpha+\frac{1}{r}} + (n+1)^{-\alpha-1+\frac{1}{r}}\right]$$

$$= O\left[(n+1)^{-\alpha+\frac{1}{r}}(1+(n+1)^{-1})\right]$$

$$= O\left((n+1)^{-\alpha+\frac{1}{r}}\right) \qquad (13)$$

Combining (12) and (13), we have

$$|C_n^1 E_n^q E_n^q - \tilde{f}| = O\left((n+1)^{-\alpha+\frac{1}{r}}\right).$$

Hence,

$$C_n^1 E_n^q E_n^q - \tilde{f}_r = \left(\int_0^{2\pi} |C_n^1 E_n^q E_n^q - \tilde{f}(x)|^r dx\right)^{\frac{1}{r}} = O\left(n^{-\alpha+\frac{1}{r}}\right).$$

This completes the proof of theorem 2.

6 Corollaries

6.1 Corollary

If one $(E,q) = 1$, then $(C,1)(E,q)(E,q)$ means reduces to $(C,1)(E,q)$ means.

Hence, Theorem 2 reduces to theorem 1.

6.2 Corollary

When $q = 1$ then $(C,1)(E,q)(E,q)$ means reduces to $(C,1)(E,1)(E,1)$ means.

6.3 Corollary

If $(C,1) = 1$, then $(C,1)(E,q)(E,q)$ means reduces to $(E,q)(E,q)$ means.

7. Conclusion

The result established here is more general form than some earlier existing results in the sense that, one $(E,q) = 1$ our proposed mean reduces to $(C,1)(E,q)$ Mean.

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