

# Characterization of Pythagorean Fuzzy Bi-Interior Ideal and Bi-Quasi-Ideal in $\Gamma$ -Semirings

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## Abstract:

In this paper, we introduce the Pythagorean fuzzy bi-interior-ideals and Pythagorean fuzzy bi-quasi-ideals in  $\Gamma$  - semiring. More over we prove the every Pythagorean fuzzy left and right ideals are Pythagorean fuzzy bi-interior -ideal in  $\Gamma$  - semiring. Also we study the notion of Pythagorean fuzzy bi-quasi-ideal in  $\Gamma$  - semiring and characterize Pythagorean fuzzy bi-quasi-ideal in  $\Gamma$  - semiring.

**Keywords:** Fuzzy set, Bi-ideal, Semiring

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## 1. Introduction

As a generalization of ring, the notion of a  $\Gamma$  - ring was introduced by Nobusawa [21] in 1964 and Iseki [7, 8, 9] studied the ideal theory in semiring. In 1995, Murali Krishna Rao [23, 24] introduced the notion of a  $\Gamma$  - semiring as a generalization of  $\Gamma$ -ring, ring, ternary semiring and semiring. Ahsan et.al [1] introduced the concept of fuzzy semirings. The concept of soft set was established by Molodtsov [20], which deals with parametrized values of the alternative. Maji et al. [17, 18, 19] investigated the soft set views on decision-making issues and defined some important concepts for soft set with their properties. Maji et al.[18] offered the notion of the fuzzy soft set by merging two existing notions fuzzy sets and soft sets. Peng et al. [22] protracted the idea of intuitionistic fuzzy soft set to Pythagorean fuzzy soft set by upgrading the conditions. The fuzzy set was studied by Zadeh's[36] in his seminal paper. Pythagorean fuzzy sets [34][35] characterized by the condition that the sum of the squares of membership and non-membership degrees is less than or equal to one, have been extensively investigated. Numerous authors have explored the algebraic properties of Pythagorean fuzzy ideals.

This paper is structured into three sections. The initial two sections provide an introduction and lay down the preliminary concepts. The third section deals with Pythagorean fuzzy bi-interior ideal and its properties in  $\Gamma$  - semiring. Also characterize Pythagorean fuzzy soft bi-interior ideal. Fourth section deals with the Pythagorean fuzzy bi-quasi-ideal and prove some important properties.

## 2. Preliminaries

This section deals with the basic definitions.

A semiring is a set  $S$  with two binary operations  $+$  and  $.$  on  $S$  called addition and multiplications such that, (i)  $(S, +)$  is a semigroup, (ii) $(S, .)$  is a semigroup and (iii)  $a(b + c) = ab + ac$  and  $(a + b)c = ac + bc$  for all  $a, b, c \in S$ .

A nonempty subset  $A$  of a semiring  $S$  is called a left (right) ideal of  $S$  if  $A$  is closed under addition and  $SA \subseteq A$  ( $AS \subseteq A$ ).  $A$  is an ideal of  $S$  if it is both a left and a right ideal of the semiring  $S$ .

**Definition 2.1** [3] If  $(S, +)$  and  $(\Gamma, +)$  be two commutative semigroups then  $S$  is called a  $\Gamma$  - semiring if there exists a structure  $S \times \Gamma \times S$  denoted by  $\alpha\gamma\beta$  for all  $\alpha, \beta \in S$  and  $\gamma \in \Gamma$  satisfying the following properties,

1.  $\alpha\gamma(\beta + \nu) = \alpha\gamma\beta + \alpha\gamma\nu$ ,
2.  $(\beta + \nu)\gamma\alpha = \beta\gamma\alpha + \nu\gamma\alpha$ ,
3.  $\alpha(\gamma + \gamma_1)\nu = \alpha\gamma\nu + \alpha\gamma_1\nu$ ,
4.  $\alpha\gamma(\beta\gamma_1\nu) = (\alpha\gamma\beta)\gamma_1\nu$  for all  $\alpha, \beta, \nu \in S$  and  $\gamma, \gamma_1 \in \Gamma$ .

**Definition 2.2** [3] Define addition in the following way  $A, B \in S, \gamma \in \Gamma$ , let  $A\gamma B$  denote the ideal generated by  $\{\alpha\gamma\beta/\alpha, \beta \in S\}$ . Then  $S$  is a  $\Gamma$  - semiring.

**Definition 2.3** [3] A  $\Gamma$  - semiring  $S$  is said to be commutative if  $\alpha\gamma\beta = \beta\gamma\alpha$ , for all  $\alpha, \beta \in S$  and  $\gamma \in \Gamma$ .

**Definition 2.4** [3] A  $\Gamma$ -semiring  $S$  is said to have a zero element if  $0\beta\alpha = 0 = \alpha\beta 0$  and  $\alpha + 0 = \alpha = 0 + \alpha$ , for all  $\alpha \in S$  and  $\gamma \in \Gamma$ .

**Definition 2.5** [3]  $S$  is said to have a identity element if there exists  $\gamma \in \Gamma$  such that  $l\gamma\alpha = \alpha = \alpha\gamma l$ , for all  $\alpha \in S$ .

**Definition 2.6** [3]  $S$  is said to have a strong identity element if for all  $\alpha \in S$ ,  $l\gamma\alpha = \alpha = \alpha\gamma l$  for all  $\gamma \in \Gamma$ .

**Definition 2.7** [3] A nonempty subset  $R$  of a  $\Gamma$  - semiring  $S$  is said to be a  $\text{sub}\Gamma$  semiring of  $S$  if  $(R, +)$  is a sub semigroup of  $(S, +)$  and  $\alpha\gamma\beta \in R$  for all  $\alpha, \beta \in R$  and  $\gamma \in \Gamma$ .

**Definition 2.8** [3] A nonempty subset  $R$  of a  $\Gamma$  - semiring  $S$  is called an ideal if  $\alpha, \beta \in R$  implies  $\alpha + \beta \in R$  and  $a \in R, \alpha \in S$  and  $\gamma \in \Gamma$  implies  $\alpha\gamma a \in R$  and  $a\alpha\gamma \in R$ .

**Definition 2.9** [11] Let  $U$  be the universe and  $E$  be the set of parameters. Let  $P(U)$  denote the power set of  $U$  and  $A \subset E$ . A pair  $(F, A)$  is called a soft set over  $U$ , where  $F$  is a mapping given by  $F: A \rightarrow P(U)$ .

**Definition 2.10** [23] A nonempty set  $A$  of  $S$  is called a  $\Gamma$  - subsemiring of  $S$  if  $(A, +)$  is a subsemigroup of  $(S, +)$  and  $A\Gamma A \subseteq A$ .

**Definition 2.11** [30]  $A$  is called a quasi-ideal of  $S$  if  $A$  is a  $\Gamma$  - subsemiring of  $S$  and  $A\Gamma S \cap S\Gamma A \subseteq A$ .

**Definition 2.12** [30]  $A$  is called a bi-ideal of  $S$  if  $A$  is a  $\Gamma$  - subsemiring of  $S$  and  $A\Gamma S\Gamma A \subseteq A$ .

**Definition 2.13** [30]  $A$  is called an interior-ideal of  $S$  if  $A$  is a  $\Gamma$  - subsemiring of  $S$  and  $S\Gamma A\Gamma S \subseteq A$ .

**Definition 2.14** [30]  $A$  is called a left(right) ideal of  $S$  if  $A$  is a  $\Gamma$  - subsemiring of  $S$  and  $S\Gamma A \subseteq A$  ( $A\Gamma S \subseteq A$ ).

**Definition 2.15** [30]  $A$  is called an ideal of  $S$  if  $A$  is a  $\Gamma$  - subsemiring of  $S$  and  $S\Gamma A \subseteq A, A\Gamma S \subseteq A$ .

**Definition 2.16** [30]  $A$  is called a left(right) bi-quasi-ideal of  $S$  if  $A$  is a  $\Gamma$ -subsemiring of  $S$  and  $A\Gamma S \cap S\Gamma S\Gamma A \subseteq A(A\Gamma S \cap S\Gamma A\Gamma S \subseteq A)$ .

**Definition 2.17** [9] Let  $U$  be the initial universe.  $E$  be the set of parameters and  $FS(U)$  denote the fuzzy power set of  $U$  and  $A \subset E$ . A pair  $(F, A)$  is called a fuzzy soft set over  $U$ , where  $F$  is a mapping given by  $F: A \rightarrow FS(U)$ . A fuzzy soft set is a parameterized family of fuzzy subsets of  $U$ .

**Definition 2.18** [16] Let  $X$  be a non empty set. A Pythagorean Fuzzy Set  $\mathfrak{A}$  in  $X$  is given by  $\mathfrak{A} = \{\alpha, \mathfrak{A}_x(\alpha), \mathfrak{A}_y(\alpha) / \alpha \in X\}$  where  $\mathfrak{A}_x: X \rightarrow [0, I]$  and  $\mathfrak{A}_y: X \rightarrow [0, 1]$  represent the degree of membership and degree of non membership of  $\mathfrak{A}$  respectively. Also,  $\mathfrak{A}_x$  and  $\mathfrak{A}_y$  satisfies the condition  $(\mathfrak{A}_x)^2 + (\mathfrak{A}_y)^2 \leq 1$  for all  $\alpha \in X$ .

**Definition 2.19** [13] Let  $U$  be the initial universe.  $E$  be the set of parameters and  $PFS(U)$  denote the Pythagorean fuzzy power set of  $U$  and  $A \subset E$ . A pair  $(F, A)$  is called a Pythagorean fuzzy soft set over  $U$ , where  $F$  is a mapping given by  $F: A \rightarrow PFS(U)$ . A Pythagorean fuzzy soft set is a parameterized family of fuzzy subsets of  $U$ .

**Definition 2.20** [13] Let  $(F, A)$  and  $(G, B)$  be two Pythagorean fuzzy soft sets over  $U$ . Then the union of  $(F, A)$  is called a Pythagorean fuzzy soft subset of  $(G, B)$  if

1.  $A \subset B$
2.  $F(\alpha)$  is a Pythagorean fuzzy subset of  $G(\alpha)$ , for all  $\alpha \in A$ .

**Definition 2.21** [13] Let  $(F, A)$  and  $(G, B)$  be two Pythagorean fuzzy soft sets over  $U$ .  $(F, A)AND(G, B)$  denoted by  $(F, A) \wedge (G, B)$ , is defined by

$$(F, A) \wedge (G, B) = (H, A \times B), \text{ where } H(\alpha, \beta) = F(\alpha) \cap G(\beta), \text{ for all } (\alpha, \beta) \in A \times B.$$

**Definition 2.22** [13] Let  $(F, A)$  and  $(G, B)$  be two Pythagorean fuzzy soft sets over  $U$ .  $(F, A)OR(G, B)$  denoted by  $(F, A) \vee (G, B)$ , is defined by

$$(F, A) \vee (G, B) = (H, A \times B), \text{ where } H(\alpha, \beta) = F(\alpha) \cup G(\beta), \text{ for all } (\alpha, \beta) \in A \times B.$$

**Definition 2.23** [13] The intersection of two Pythagorean fuzzy soft sets  $(F, A)$  and  $(G, B)$  over a universe  $U$  is a Pythagorean fuzzy soft set denoted by  $(H, C)$ , where  $C = A \cap B$  and  $H(\alpha) =$

$$\begin{cases} F(\alpha) & \text{if } \alpha \in A - B \\ G(\alpha) & \text{if } \alpha \in B - A \\ \min\{F(\alpha), G(\alpha)\} & \text{if } \alpha \in A \cap B \end{cases}$$

For all  $\alpha \in C$ . It is denoted by  $(H, C) = (F, A) \cap (F, B)$ .

**Definition 2.24** [13] The union of two Pythagorean fuzzy soft sets  $(F, A)$  and  $(G, B)$  over a universe  $U$  is a Pythagorean fuzzy soft set denoted by  $(H, C)$ , where  $C = A \cup B$  and

$$H(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in A - B \\ G(\alpha) & \text{if } \alpha \in B - A \\ \min\{F(\alpha), G(\alpha)\} & \text{if } \alpha \in A \cup B \end{cases}$$

For all  $\alpha \in C$ . It is denoted by  $(H, C) = (F, A) \cup (F, B)$ .

**Definition 2.25** [13] Let  $(F, A)$  and  $(G, B)$  be two Pythagorean fuzzy soft sets over  $U$  such that  $A \cup B \neq \emptyset$ . The bi-union of  $(F, A)$  and  $(G, B)$  is defined to be the Pythagorean fuzzy soft set  $(H, C)$ , where  $C = A \cup B$  and  $H(\alpha) = F(\alpha) \cup G(\alpha)$  for all  $\alpha \in C$ . It is denoted by  $(H, C) = (F, A) \sqcup (G, B)$ .

**Definition 2.26** [13] Let  $(F, A)$  and  $(G, B)$  be two Pythagorean fuzzy soft sets over  $U$  such that  $A \cap B \neq \emptyset$ . The bi-union of  $(F, A)$  and  $(G, B)$  is defined to be the Pythagorean fuzzy soft set  $(H, C)$ , where  $C = A \cap B$  and  $H(\alpha) = F(\alpha) \cap G(\alpha)$  for all  $\alpha \in C$ . It is denoted by  $(H, C) = (F, A) \sqcap (G, B)$ .

**Definition 2.27** [13] Let  $(F, A)$  and  $(G, B)$  two Pythagorean fuzzy soft sets over a universe  $U$ . The product of  $(F, A)$  and  $(G, B)$  is defined to be the Pythagorean fuzzy soft set denoted by  $(F \circ G, C)$ , where  $C = A \cup B$  and

$$A_{x(F \circ G)(\alpha)}(i) = \begin{cases} A_{x(F)(\alpha)}(i) & \text{if } \alpha \in A - B \\ A_{x(G)(\alpha)}(i) & \text{if } \alpha \in B - A \\ \sup_{i=ab} \min\{A_{x(F)(\alpha)}(i), A_{x(G)(\alpha)}(i)\} & \text{if } \alpha \in A \cap B \end{cases}$$

and

$$A_{y(F \circ G)(\alpha)}(i) = \begin{cases} A_{y(F)(\alpha)}(i) & \text{if } \alpha \in A - B \\ A_{y(G)(\alpha)}(i) & \text{if } \alpha \in B - A \\ \inf_{i=ab} \max\{A_{y(F)(\alpha)}(i), A_{y(G)(\alpha)}(i)\} & \text{if } \alpha \in A \cap B \end{cases}$$

For all  $\alpha \in C$  and  $i \in U$ . It is denoted by  $(F \circ G, C) = (F, A) \circ (G, B)$ .

**Definition 2.28** [28] A fuzzy subset  $A$  of  $S$  is called a fuzzy bi-interior-ideal if  $SAS \cap ASA \subseteq A$ .

### 3. Pythagorean fuzzy Bi-interior-ideals in $\Gamma$ -semiring

This section deals with the Pythagorean fuzzy bi-interior-ideals in  $\Gamma$ -semiring  $S$ .

**Definition 3.1** A PFS  $A = (A_\mu, A_\nu)$  of  $S$  is said to be a PFBII of  $S$  if the following conditions are holds:

1.  $A_\mu(x + y) \geq \min\{A_\mu(x), A_\mu(y)\}$   
 $A_\nu(x + y) \leq \max\{A_\nu(x), A_\nu(y)\}$
2.  $\chi_S \circ A_\mu \circ \chi_S \cap A_\mu \circ \chi_S \circ A_\mu \supseteq A_\mu$   
 $\chi_S \circ A_\nu \circ \chi_S \cap A_\nu \circ \chi_S \circ A_\nu \subseteq A_\nu$ .

**Theorem 3.2** Every PF left ideal of  $S$  is a PFBII of  $S$ .

Proof. Let  $A$  be a PF left ideal of  $S$  and  $x \in S, \alpha, \beta \in \Gamma$ .

$$\begin{aligned} \chi_S \circ A_\mu(x) &= \sup_{x=a\alpha b} \left\{ \min\{\chi_S(a), A_\mu(b)\} \right\} \\ &= \sup_{x=a\alpha b} \left\{ \min\{1, A_\mu(b)\} \right\} \\ &= \sup_{x=a\alpha b} \left\{ A_\mu(b) \right\} \end{aligned}$$

$$\begin{aligned}
 &\geq \sup_{x=a\alpha b} \{A_\mu(ab)\} \\
 &= \sup_{x=a\alpha b} \{A_\mu(x)\} \\
 &= A_\mu(x) \\
 \\
 &A_\mu \circ \chi_S \circ A_\mu(x) = \sup_{x=u\alpha v\beta s} \{ \min\{A_\mu(u), \chi_S \circ A_\mu(v\beta s)\} \} \\
 &\geq \sup_{x=u\alpha v\beta s} \{ \min\{A_\mu(u), A_\mu(v\beta s)\} \} \\
 &= A_\mu(x)
 \end{aligned}$$

Now

$$\begin{aligned}
 \chi_S \circ A_\mu \circ \chi_S \cap A_\mu \circ \chi_S \circ A_\mu(x) &= \min\{\chi_S \circ A_\mu \circ \chi_S, A_\mu \circ \chi_S \circ A_\mu\} \\
 &\geq \min\{\chi_S \circ A_\mu \circ \chi_S, A_\mu(x)\} \\
 &= A_\mu(x)
 \end{aligned}$$

Hence  $\chi_S \circ A_\mu \circ \chi_S \cap A_\mu \circ \chi_S \circ A_\mu \supseteq A_\mu$ .

Next we have to prove for non membership function

$$\begin{aligned}
 \chi_S \circ A_\nu(x) &= \inf_{x=a\alpha b} \{ \max\{\chi_S(a), A_\nu(b)\} \} \\
 &= \inf_{x=a\alpha b} \{ \max\{0, A_\nu(b)\} \} \\
 &= \inf_{x=a\alpha b} \{ A_\nu(b) \} \\
 &\leq \inf_{x=a\alpha b} \{ A_\nu(ab) \} \\
 &= \inf_{x=a\alpha b} \{ A_\nu(x) \} \\
 &= A_\nu(x)
 \end{aligned}$$

$$\begin{aligned}
 A_\nu \circ \chi_S \circ A_\nu(x) &= \inf_{x=u\alpha v\beta s} \{ \max\{A_\nu(u), \chi_S \circ A_\nu(v\beta s)\} \} \\
 &\leq \inf_{x=u\alpha v\beta s} \{ \max\{A_\nu(u), A_\nu(v\beta s)\} \} \\
 &= A_\nu(x)
 \end{aligned}$$

Now

$$\begin{aligned}
 \chi_S \circ A_\nu \circ \chi_S \cap A_\nu \circ \chi_S \circ A_\nu(x) &= \max\{\chi_S \circ A_\nu \circ \chi_S, A_\nu \circ \chi_S \circ A_\nu\} \\
 &\leq \max\{\chi_S \circ A_\nu \circ \chi_S, A_\nu(x)\} \\
 &= A_\nu(x)
 \end{aligned}$$

Hence  $\chi_S \circ A_\nu \circ \chi_S \cap A_\nu \circ \chi_S \circ A_\nu \subseteq A_\nu$ .

**Theorem 3.3** Every PF right ideal of  $S$  is a PFBII of  $S$ .

Proof. Let  $A$  be a PF right ideal of  $S$  and  $x \in S, \alpha, \beta \in \Gamma$ .

$$\begin{aligned} A_\mu(x) \circ \chi_S &= \sup_{x=a\alpha b} \{ \min\{A_\mu(a), \chi_S(b)\} \} \\ &= \sup_{x=a\alpha b} \{ \min\{A_\mu(a), 1\} \} \\ &= \sup_{x=a\alpha b} \{ A_\mu(a) \} \\ &\leq \sup_{x=a\alpha b} \{ A_\mu(ab) \} \\ &= \sup_{x=a\alpha b} \{ A_\mu(x) \} \\ &= A_\mu(x) \end{aligned}$$

$$\begin{aligned} A_\mu \circ \chi_S \circ A_\mu(x) &= \sup_{x=u\alpha v\beta s} \{ \min\{A_\mu \circ \chi_S(u\alpha v), A_\mu(s)\} \} \\ &\leq \sup_{x=u\alpha v\beta s} \{ \min\{A_\mu(u\alpha v), A_\mu(s)\} \} \\ &= A_\mu(x) \end{aligned}$$

Now

$$\begin{aligned} \chi_S \circ A_\mu \circ \chi_S \cap A_\mu \circ \chi_S \circ A_\mu(x) &= \min\{\chi_S \circ A_\mu \circ \chi_S, A_\mu \circ \chi_S \circ A_\mu\} \\ &\geq \min\{\chi_S \circ A_\mu \circ \chi_S, A_\mu(x)\} \\ &= A_\mu(x) \end{aligned}$$

Hence  $\chi_S \circ A_\mu \circ \chi_S \cap A_\mu \circ \chi_S \circ A_\mu \supseteq A_\mu$ .

Next we have to prove for non membership function

$$\begin{aligned} A_\nu(x) \circ \chi_S &= \inf_{x=a\alpha b} \{ \max\{A_\nu(a), \chi_S(b)\} \} \\ &= \inf_{x=a\alpha b} \{ \max\{A_\nu(a), 0\} \} \\ &= \inf_{x=a\alpha b} \{ A_\nu(a) \} \\ &\leq \inf_{x=a\alpha b} \{ A_\nu(ab) \} \\ &= \inf_{x=a\alpha b} \{ A_\nu(x) \} \\ &= A_\nu(x) \end{aligned}$$

$$\begin{aligned} A_\nu \circ \chi_S \circ A_\nu(x) &= \inf_{x=u\alpha v\beta s} \{ \max\{A_\nu \circ \chi_S(u\alpha v), A_\nu(s)\} \} \\ &\leq \inf_{x=u\alpha v\beta s} \{ \max\{A_\nu(u\alpha v), A_\nu(s)\} \} \end{aligned}$$

$$= A_v(x)$$

Now

$$\begin{aligned} \chi_S \circ A_v \circ \chi_S \cap A_v \circ \chi_S \circ A_v(x) &= \max\{\chi_S \circ A_v \circ \chi_S, A_v \circ \chi_S \circ A_v\} \\ &\leq \max\{\chi_S \circ A_v \circ \chi_S, A_v(x)\} \\ &= A_v(x) \end{aligned}$$

Hence  $\chi_S \circ A_v \circ \chi_S \cap A_v \circ \chi_S \circ A_v \subseteq A_v$ .

**Corollary 3.4** Every PFI is PFBII of S.

Proof. By Theorem 3.2 and 3.4 proof is obvious.

**Theorem 3.5** Let  $B$  be a nonempty subset of  $S$ . Then  $B$  is a bi-interior-ideal of  $S \Leftrightarrow \chi_B$  is an PFBII of  $S$ .

Proof. Assume that  $B$  is a bi-interior-ideal of  $S$ . Then  $\chi_B$  is an  $PF$  sub- $\Gamma$  semiring of  $S$ . By hypothesis we've  $S\Gamma B\Gamma S \cap B\Gamma S\Gamma B \subseteq B$ .

Then,

$$\chi_B \circ \chi_S \circ \chi_B \cap \chi_B \circ \chi_S \circ \chi_B = \chi_{S\Gamma B\Gamma S} \cap \chi_{B\Gamma S\Gamma B} = \chi_{S\Gamma B\Gamma S \cap B\Gamma S\Gamma B} \subseteq \chi_B$$

Hence  $\chi_B$  is a  $PFBII$  of  $S$ .

Conversely, let us assume that  $\chi_B$  is a  $PFBII$  of  $S$ . Then  $B$  is a sub- $\Gamma$  semiring of  $S$ . We have

$$\begin{aligned} \chi_S \circ \chi_B \circ \chi_S \cap \chi_B \circ \chi_S \circ \chi_B &\subseteq \chi_B \\ \chi_{S\Gamma B\Gamma S} \cap \chi_{B\Gamma S\Gamma B} &\subseteq \chi_B \\ \chi_{SBS \cap BSB} &\subseteq \chi_B \\ SBS \cap BSB &\subseteq B \end{aligned}$$

Hence  $B$  is a bi-interior-ideal of  $S$ .

**Theorem 3.6** Let  $B$  be a nonempty subset of  $S$ . Then  $B$  is a PFBII of  $S \Leftrightarrow$  the nonempty level subset of  $B$  is a bi-interior-ideal of  $S$  for every  $t \in [0, I]$ .

Proof. Assume that  $B$  is a  $PFBII$  of  $S$ .

$$B_{\mu_t} \neq \phi, t \in [0, 1] \text{ and } a, b \in B_{\mu_t}$$

Then,  $B_{\mu_t}(a) \geq t, B_{\mu_t}(b) \geq t$

$$B_{\mu_t}(a + b) \geq \min\{B_{\mu_t}(a), B_{\mu_t}(b)\} \geq t$$

$a + b \in B_{\mu_t}$ .

Let  $x \in S\Gamma B_{\mu_t}\Gamma S \cap B_{\mu_t}\Gamma S\Gamma B_{\mu_t}$ .

Then  $x = b\alpha a\beta u = c\gamma d\delta e, b, u, d \in S$  and  $a, c, e \in B_{\mu_t}, \alpha, \beta, \gamma, \delta \in \Gamma$

$$\chi_S \circ B_{\mu_t} \circ \chi_S \geq t$$

and  $B_{\mu_t} \circ \chi_S \circ B_{\mu_t} \geq t$ .

$$B_{\mu_t}(x) \geq t$$

Hence  $x \in B_{\mu_t}$ . Conversely, suppose that  $B_{\mu_t}$  is a bi-interior-ideal of  $S$  for all  $t \in [0,1]$

Let  $a, b \in S$ ,  $\alpha \in \Gamma$ ,  $B_{\mu_t}(a) = t_1$ ,  $B_{\mu_t}(b) = t_2$  and  $t_1 \geq t_2$ . Then  $a, b \in B_{\mu_t}$ . Then,  $a + b \in B_{\mu_t}$ .

Therefore  $B_{\mu_t} \geq t_2 = \min\{B_{\mu_t}(a), B_{\mu_t}(b)\}$ . Hence we have,  $S\Gamma B_{\mu_t}\Gamma S \cap B_{\mu_t}\Gamma S\Gamma B_{\mu_t} \subseteq B_{\mu_t}$ .

Similarly we prove the non membership function.

**Theorem 3.7** If  $A$  and  $B$  are PFBII of  $S$  then  $A \cap B$  is PFBII of  $S$ .

Proof. Let  $A$  and  $B$  are PFBII of  $S$  and  $x, y \in S$  and  $\alpha, \beta \in \Gamma$ .

$$\begin{aligned} (A_\mu \cap B_\mu)(x + y) &= \min\{A_\mu(x + y), B_\mu(x + y)\} \\ &\geq \min\{\min\{A_\mu(x), A_\mu(y)\}, \min\{B_\mu(x), B_\mu(y)\}\} \\ &= \min\{\min\{A_\mu(x), B_\mu(x)\}, \min\{A_\mu(y), B_\mu(y)\}\} \\ &= \min\{(A_\mu \cap B_\mu)(x), (A_\mu \cap B_\mu)(y)\} \end{aligned}$$

$$\begin{aligned} \chi_S \circ (A_\mu \cap B_\mu)(x) &= \sup_{x=a\alpha b} \{\min\{\chi_S(a), A_\mu \cap B_\mu(b)\}\} \\ &= \sup_{x=a\alpha b} \{\min\{\chi_S(a), \min\{A_\mu(b), B_\mu(b)\}\}\} \\ &= \sup_{x=a\alpha b} \{\min\{\min\{\chi_S(a), A_\mu(b)\}, \min\{\chi_S(a), B_\mu(b)\}\}\} \\ &= \min\left\{ \sup_{x=a\alpha b} \{\min\{\chi_S(a), A_\mu(b)\}\}, \sup_{x=a\alpha b} \{\min\{\chi_S(a), B_\mu(b)\}\} \right\} \\ &= \min\{\chi_S \circ A_\mu(x), \chi_S \circ B_\mu(x)\} \\ &= (\chi_S \circ A_\mu) \cap (\chi_S \circ B_\mu)(x) \end{aligned}$$

$$\begin{aligned} (A_\mu \cap B_\mu) \circ \chi_S \circ (A_\mu \cap B_\mu)(x) &= \sup_{x=a\alpha b\beta c} \{\min\{A_\mu \cap B_\mu(a), \chi_S \circ A_\mu \cap B_\mu(b\beta c)\}\} \\ &= \sup_{x=a\alpha b\beta c} \{\min\{(A_\mu \cap B_\mu)(a), \chi_S \circ A_\mu \cap \chi_S \circ B_\mu(b\beta c)\}\} \\ &= \sup_{x=a\alpha b\beta c} \{\min\{\min\{A_\mu(a), B_\mu(a)\}, \min\{(\chi_S \circ A_\mu)(b\beta c), (\chi_S \circ B_\mu)(b\beta c)\}\}\} \\ &= \min\{A_\mu \circ \chi_S \circ A_\mu(x), B_\mu \circ \chi_S \circ B_\mu(x)\} \end{aligned}$$

Therefore

$$(A_\mu \cap B_\mu) \circ \chi_S \circ (A_\mu \cap B_\mu) = A_\mu \circ \chi_S \circ A_\mu \cap B_\mu \circ \chi_S \circ B_\mu$$

Similarly we prove

$$\chi_S \circ A_\mu \cap B_\mu \circ \chi_S = A_\mu \circ \chi_S \circ A_\mu \cap B_\mu \circ \chi_S \circ B_\mu$$

Hence

$$\begin{aligned} &\chi_S \circ A_\mu \cap B_\mu \circ \chi_S \cap A_\mu \cap B_\mu \circ \chi_S \circ A_\mu \cap B_\mu \\ &= (\chi_S \circ A_\mu \circ \chi_S) \cap (A_\mu \circ \chi_S \circ A_\mu) \cap (\chi_S \circ B_\mu \circ \chi_S) \cap (B_\mu \circ \chi_S \circ B_\mu) \supseteq A_\mu \cap B_\mu \end{aligned}$$

Similarly we can prove for non membership

$$(A_\nu \cap B_\nu)(x + y) = \max\{A_\nu(x + y), B_\nu(x + y)\}$$

$$\leq \max\{\max\{A_\nu(x), A_\nu(y)\}, \max\{B_\nu(x), B_\nu(y)\}\}$$

$$= \max\{\max\{A_\nu(x), B_\nu(x)\}, \max\{A_\nu(y), B_\nu(y)\}\}$$

$$= \max\{(A_\nu \cap B_\nu)(x), (A_\nu \cap B_\nu)(y)\}$$

$$\chi_S \circ (A_\nu \cap B_\nu)(x) = \inf_{x=a\alpha b} \left\{ \max\{\chi_S(a), A_\nu \cap B_\mu(b)\} \right\}$$

$$= \inf_{x=a\alpha b} \left\{ \max\left\{ \chi_S(a), \max\{A_\nu(b), B_\mu(b)\} \right\} \right\}$$

$$= \inf_{x=a\alpha b} \left\{ \max\left\{ \max\{\chi_S(a), A_\nu(b)\}, \max\{\chi_S(a), B_\nu(b)\} \right\} \right\}$$

$$= \max\left\{ \inf_{x=a\alpha b} \left\{ \max\{\chi_S(a), A_\nu(b)\} \right\}, \inf_{x=a\alpha b} \left\{ \max\{\chi_S(a), B_\nu(b)\} \right\} \right\}$$

$$= \max\{\chi_S \circ A_\nu(x), \chi_S \circ B_\nu(x)\}$$

$$= (\chi_S \circ A_\nu) \cap (\chi_S \circ B_\nu)(x)$$

$$(A_\nu \cap B_\nu) \circ \chi_S \circ (A_\nu \cap B_\nu)(x) = \inf_{x=a\alpha b\beta c} \left\{ \max\{A_\nu \cap B_\nu(a), \chi_S \circ A_\nu \cap B_\nu(b\beta c)\} \right\}$$

$$= \inf_{x=a\alpha b\beta c} \left\{ \max\{(A_\nu \cap B_\nu)(a), \chi_S \circ A_\nu \cap \chi_S \circ B_\nu(b\beta c)\} \right\}$$

$$= \inf_{x=a\alpha b\beta c} \left\{ \max\left\{ \max\{A_\nu(a), B_\nu(a)\}, \min\{(\chi_S \circ A_\nu)(b\beta c), (\chi_S \circ B_\nu)(b\beta c)\} \right\} \right\}$$

$$= \max\{A_\nu \circ \chi_S \circ A_\nu(x), B_\nu \circ \chi_S \circ B_\nu(x)\}$$

Therefore

$$(A_\nu \cap B_\nu) \circ \chi_S \circ (A_\nu \cap B_\nu) = A_\nu \circ \chi_S \circ A_\nu \cap B_\nu \circ \chi_S \circ B_\nu$$

Similarly we prove

$$\chi_S \circ A_\nu \cap B_\nu \circ \chi_S = A_\nu \circ \chi_S \circ A_\nu \cap B_\nu \circ \chi_S \circ B_\nu$$

Hence

$$\chi_S \circ A_\nu \cap B_\mu \circ \chi_S \cap A_\nu \cap B_\nu \circ \chi_S \circ A_\nu \cap B_\mu$$

$$= (\chi_S \circ A_\nu \circ \chi_S) \cap (A_\nu \circ \chi_S \circ A_\nu) \cap (\chi_S \circ B_\nu \circ \chi_S) \cap (B_\nu \circ \chi_S \circ B_\nu) \subseteq A_\nu \cap B_\nu$$

#### 4 Pythagorean fuzzy Bi-quasi-ideals in $\Gamma$ -semiring

This section deals with the Pythagorean fuzzy bi-interior-ideals in  $\Gamma$ -semiring  $S$ .

**Definition 4.1** A PFS  $A = (A_\mu, A_\nu)$  of  $S$  is said to be a PFLBQI of  $S$  if the following conditions are holds:

1.  $A_\mu(x + y) \geq \min\{A_\mu(x), A_\mu(y)\}$   
 $A_\nu(x + y) \leq \max\{A_\nu(x), A_\nu(y)\}$
2.  $\chi_S \circ A_\mu \cap A_\mu \circ \chi_S \circ A_\mu \supseteq A_\mu$   
 $\chi_S \circ A_\nu \cap A_\nu \circ \chi_S \circ A_\nu \subseteq A_\nu$

**Definition 4.2** A PFS  $A = (A_\mu, A_\nu)$  of  $S$  is said to be a PFRBQI of  $S$  if the following conditions are holds:

1.  $A_\mu(x + y) \geq \min\{A_\mu(x), A_\mu(y)\}$   
 $A_\nu(x + y) \leq \max\{A_\nu(x), A_\nu(y)\}$
2.  $A_\mu \circ \chi_S \cap A_\mu \circ \chi_S \circ A_\mu \supseteq A_\mu$   
 $A_\nu \circ \chi_S \cap A_\nu \circ \chi_S \circ A_\nu \subseteq A_\nu$ .

**Definition 4.3** A PFS  $A = (A_\mu, A_\nu)$  of  $S$  is said to be a PFBQI of  $S$  if it is both Pythagorean fuzzy left bi-quasi-ideal and right bi-quasi-ideal of  $S$ .

**Theorem 4.4** Every PF left ideal of  $S$  is a PFLBQI of  $S$ .

Proof. Let  $A$  be a PF left ideal of  $S$  and  $x \in S, \alpha, \beta \in \Gamma$ .

$$\begin{aligned} \chi_S \circ A_\mu(x) &= \sup_{x=a\alpha b} \left\{ \min\{\chi_S(a), A_\mu(b)\} \right\} \\ &= \sup_{x=a\alpha b} \{A_\mu(b)\} \\ &\geq \sup_{x=a\alpha b} \{A_\mu(a\alpha b)\} \\ &= \sup_{x=a\alpha b} \{A_\mu(x)\} \\ &= A_\mu(x) \end{aligned}$$

$$\begin{aligned} A_\mu \circ \chi_S \circ A_\mu(x) &= \sup_{x=u\alpha v\beta s} \left\{ \min\{A_\mu(u\alpha v) \circ \chi_S, A_\mu(s)\} \right\} \\ &\geq \sup_{x=u\alpha v\beta s} \left\{ \min\{A_\mu(u\alpha v), A_\mu(s)\} \right\} \\ &= A_\mu(x) \end{aligned}$$

Hence  $\chi_S \circ A_\mu \cap A_\mu \circ \chi_S \circ A_\mu \supseteq A_\mu$ .

Next we have to prove for non membership function

$$\begin{aligned} \chi_S \circ A_\nu(x) &= \inf_{x=aab} \{ \max\{\chi_S(a), A_\nu(b)\} \} \\ &= \inf_{x=aab} \{ A_\nu(b) \} \\ &\leq \inf_{x=aab} \{ A_\nu(aab) \} \\ &= \inf_{x=aab} \{ A_\nu(x) \} \\ &= A_\nu(x) \end{aligned}$$

$$\begin{aligned} A_\nu \circ \chi_S \circ A_\nu(x) &= \inf_{x=uav\beta s} \{ \max\{A_\nu(u), \chi_S \circ A_\nu(v\beta s)\} \} \\ &\leq \inf_{x=uav\beta s} \{ \max\{A_\nu(u), A_\nu(v\beta s)\} \} \\ &= A_\nu(x) \end{aligned}$$

Hence  $\chi_S \circ A_\nu \cap A_\nu \circ \chi_S \circ A_\nu \subseteq A_\nu$ .

**Theorem 4.5** Every PF right ideal of  $S$  is a PFRBQI of  $S$ .

Proof. Let  $A$  be a PF right ideal of  $S$  and  $x \in S, \alpha, \beta \in \Gamma$ .

$$\begin{aligned} A_\mu \circ \chi_S(x) &= \sup_{x=aab} \{ \min\{A_\mu(a), \chi_S(b)\} \} \\ &= \sup_{x=aab} \{ A_\mu(a) \} \\ &\geq \sup_{x=aab} \{ A_\mu(aab) \} \\ &= \sup_{x=aab} \{ A_\mu(x) \} \\ &= A_\mu(x) \end{aligned}$$

$$\begin{aligned} A_\mu \circ \chi_S \circ A_\mu(x) &= \sup_{x=uav\beta s} \{ \min\{A_\mu \circ \chi_S(uav), A_\mu(s)\} \} \\ &\geq \sup_{x=uav\beta s} \{ \min\{A_\mu(uav), A_\mu(s)\} \} \\ &= A_\mu(x) \end{aligned}$$

Hence  $A_\mu \circ \chi_S \cap A_\mu \circ \chi_S \circ A_\mu \supseteq A_\mu$ .

Next we have to prove for non membership function

$$\begin{aligned} A_\nu \circ \chi_S(x) &= \inf_{x=aab} \{ \max\{A_\nu(a), \chi_S(b)\} \} \\ &= \inf_{x=aab} \{ A_\nu(a) \} \\ &\leq \inf_{x=aab} \{ A_\nu(aab) \} \\ &= \inf_{x=aab} \{ A_\nu(x) \} \end{aligned}$$

$$\begin{aligned}
 &= A_\nu(x) \\
 A_\nu \circ \chi_S \circ A_\nu(x) &= \inf_{x=u\alpha\nu\beta s} \{ \max\{A_\nu \circ \chi_S(u\alpha\nu), A_\nu(s)\} \} \\
 &\leq \inf_{x=u\alpha\nu\beta s} \{ \max\{A_\nu(u\alpha\nu), A_\nu(s)\} \} \\
 &= A_\nu(x).
 \end{aligned}$$

Hence  $A_\nu \circ \chi_S \cap A_\nu \circ \chi_S \circ A_\nu \subseteq A_\nu$ .

**Theorem 4.6** Every PF left ideal of  $S$  is a PFRBQI of  $S$ .

Proof. Let  $A$  be a PF left ideal of  $S$  and  $x \in S, \alpha, \beta \in \Gamma$ .

$$\begin{aligned}
 \chi_S \circ A_\mu(x) &= \sup_{x=a\alpha b} \{ \min\{\chi_S(a), A_\mu(b)\} \} \\
 &= \sup_{x=a\alpha b} \{ A_\mu(b) \} \\
 &\geq \sup_{x=a\alpha b} \{ A_\mu(a\alpha b) \} \\
 &= \sup_{x=a\alpha b} \{ A_\mu(x) \} \\
 &= A_\mu(x)
 \end{aligned}$$

$$\begin{aligned}
 A_\mu \circ \chi_S \circ A_\mu(x) &= \sup_{x=u\alpha\nu\beta s} \{ \min\{A_\mu(s), \chi_S \circ A_\mu(\nu\beta s)\} \} \\
 &\geq \sup_{x=u\alpha\nu\beta s} \{ \min\{A_\mu(s), A_\mu(\nu\beta s)\} \} \\
 &= A_\mu(x)
 \end{aligned}$$

Now

$$\begin{aligned}
 A_\mu \circ \chi_S \cap A_\mu \circ \chi_S \circ A_\mu(x) &= \min\{A_\mu \circ \chi_S, A_\mu \circ \chi_S \circ A_\mu(x)\} \\
 &\geq \min\{A_\mu \circ \chi_S(x), A_\mu(x)\} \\
 &= A_\mu(x).
 \end{aligned}$$

Hence  $A_\mu$  is a PFRBQI of  $S$ .

Similarly we can prove  $A_\nu$  is a PFRBQI of  $S$ .

**Theorem 4.7** Every PF right ideal of  $S$  is a PFLBQI of  $S$ .

Proof. Proof is straight forward.

**Corollary 4.8** Every PF left(right) ideal of  $S$  is a PFRBQI of  $S$ .

**Theorem 4.9** Let  $B$  be a nonempty subset of  $S$ . Then  $B$  is a right bi-quasi-ideal of  $S \Leftrightarrow \chi_B$  is an PFRBQI of  $S$ .

Proof. Assume that  $B$  is a right bi-quasi-ideal of  $S$ . Then  $\chi_B$  is an PF subsemiring of  $S$ .

By hypothesis we've  $S\Gamma B \cap B\Gamma S\Gamma B \subseteq B$ . Then,

$$\begin{aligned} & \chi_S \circ \chi_B \cap \chi_B \circ \chi_S \circ \chi_B \\ &= \chi_{S\Gamma B} \cap \chi_{B\Gamma S\Gamma B} \\ &= \chi_{S\Gamma B \cap B\Gamma S\Gamma B} \subseteq \chi_B \end{aligned}$$

Hence  $\chi_B$  is an *PFRBQI* of  $S$ .

Conversely, let us assume that  $\chi_B$  is a *PFRBQI* of  $S$ . Then  $B$  is a subsemiring of  $S$ . We have

$$\begin{aligned} & \chi_S \circ \chi_B \cap \chi_B \circ \chi_S \circ \chi_B \subseteq \chi_B \\ & \chi_{S\Gamma B} \cap \chi_{B\Gamma S\Gamma B} \subseteq \chi_B \end{aligned}$$

Hence  $B$  is a bi-quasi-ideal of  $S$ .

**Theorem 4.10** Let  $B$  be a nonempty subset of  $S$ . Then  $B$  is a left bi-quasi-ideal of  $S \Leftrightarrow \chi_B$  is an *PFLBQI* of  $S$ .

Proof. Assume that  $B$  is a left bi-quasi-ideal of  $S$ . Then  $\chi_B$  is an *PF* sub  $\Gamma$  semi ring of  $S$ .

By hypothesis we've  $B\Gamma S \cap B\Gamma S\Gamma B \subseteq B$ . Then,

$$\begin{aligned} & \chi_B \circ \chi_S \cap \chi_B \circ \chi_S \circ \chi_B \\ &= \chi_{B\Gamma S} \cap \chi_{B\Gamma S\Gamma B} \\ &= \chi_{B\Gamma S \cap B\Gamma S\Gamma B} \subseteq \chi_B \end{aligned}$$

Hence  $\chi_B$  is an *PFRBQI* of  $S$ .

Conversely, let us assume that  $\chi_B$  is a *PFRBQI* of  $S$ . Then  $B$  is a sub  $\Gamma$  semi ring of  $S$ . We have

$$\begin{aligned} & \chi_B \circ \chi_S \cap \chi_B \circ \chi_S \circ \chi_B \subseteq \chi_B \\ & \chi_{B\Gamma S} \cap \chi_{B\Gamma S\Gamma B} \subseteq \chi_B \end{aligned}$$

Hence  $B$  is a bi-quasi-ideal of  $S$ .

**Theorem 4.11** If  $A$  and  $B$  are *PFLBQI* of  $S$  then  $A \cap B$  is *PFLBQI* of  $S$ .

Proof. Let  $A$  and  $B$  are *PFLBQI* of  $S$  and  $x, y \in S$  and  $\alpha, \beta \in \Gamma$ .

$$\begin{aligned} (A_\mu \cap B_\mu)(x + y) &= \min\{A_\mu(x + y), B_\mu(x + y)\} \\ &\geq \min\{\min\{A_\mu(x), A_\mu(y)\}, \min\{B_\mu(x), B_\mu(y)\}\} \\ &= \min\{\min\{A_\mu(x), B_\mu(x)\}, \min\{A_\mu(y), B_\mu(y)\}\} \\ &= \min\{(A_\mu \cap B_\mu)(x), (A_\mu \cap B_\mu)(y)\} \end{aligned}$$

$$\begin{aligned} \chi_S \circ (A_\mu \cap B_\mu)(x) &= \sup_{x=\alpha\alpha b} \{\min\{\chi_S(a), A_\mu \cap B_\mu(b)\}\} \\ &= \sup_{x=\alpha\alpha b} \{\min\{\chi_S(a), \min\{A_\mu(b), B_\mu(b)\}\}\} \end{aligned}$$

$$\begin{aligned}
 &= \sup_{x=a\alpha b} \left\{ \min \left\{ \min \{ \chi_S(a), A_\mu(b) \}, \min \{ \chi_S(a), B_\mu(b) \} \right\} \right\} \\
 &= \min \left\{ \sup_{x=a\alpha b} \left\{ \min \{ \chi_S(a), A_\mu(b) \} \right\}, \sup_{x=a\alpha b} \left\{ \min \{ \chi_S(a), B_\mu(b) \} \right\} \right\} \\
 &= \min \{ \chi_S \circ A_\mu(x), \chi_S \circ B_\mu(x) \} \\
 &= (\chi_S \circ A_\mu) \cap (\chi_S \circ B_\mu)(x) \\
 (A_\mu \cap B_\mu) \circ \chi_S \circ (A_\mu \cap B_\mu)(x) &= \sup_{x=a\alpha b\beta c} \left\{ \min \{ A_\mu \cap B_\mu(a), \chi_S \circ A_\mu \cap B_\mu(b\beta c) \} \right\} \\
 &= \sup_{x=a\alpha b\beta c} \left\{ \min \{ (A_\mu \cap B_\mu)(a), \chi_S \circ A_\mu \cap \chi_S \circ B_\mu(b\beta c) \} \right\} \\
 &= \sup_{x=a\alpha b\beta c} \left\{ \min \left\{ \min \{ A_\mu(a), B_\mu(a) \}, \min \{ (\chi_S \circ A_\mu)(b\beta c), (\chi_S \circ B_\mu)(b\beta c) \} \right\} \right\} \\
 &= \min \{ A_\mu \circ \chi_S \circ A_\mu(x), B_\mu \circ \chi_S \circ B_\mu(x) \}
 \end{aligned}$$

Therefore

$$(A_\mu \cap B_\mu) \circ \chi_S \circ (A_\mu \cap B_\mu) = A_\mu \circ \chi_S \circ A_\mu \cap B_\mu \circ \chi_S \circ B_\mu$$

Hence

$$\begin{aligned}
 \chi_S \circ A_\mu \cap B_\mu \circ \chi_S \cap A_\mu \cap B_\mu \circ \chi_S \circ A_\mu \cap B_\mu \\
 &= (\chi_S \circ A_\mu \circ \chi_S) \cap (A_\mu \circ \chi_S \circ A_\mu) \cap (\chi_S \circ B_\mu \circ \chi_S) \cap (B_\mu \circ \chi_S \circ B_\mu) \supseteq A_\mu \cap B_\mu
 \end{aligned}$$

Similarly we can prove for non membership

$$\begin{aligned}
 (A_\nu \cap B_\nu)(x + y) &= \max \{ A_\nu(x + y), B_\nu(x + y) \} \\
 &\leq \max \{ \max \{ A_\nu(x), A_\nu(y) \}, \max \{ B_\nu(x), B_\nu(y) \} \} \\
 &= \max \{ \max \{ A_\nu(x), B_\nu(x) \}, \max \{ A_\nu(y), B_\nu(y) \} \} \\
 &= \max \{ (A_\nu \cap B_\nu)(x), (A_\nu \cap B_\nu)(y) \} \\
 \chi_S \circ (A_\nu \cap B_\nu)(x) &= \inf_{x=a\alpha b} \left\{ \max \{ \chi_S(a), A_\nu \cap B_\nu(b) \} \right\} \\
 &= \inf_{x=a\alpha b} \left\{ \max \left\{ \chi_S(a), \max \{ A_\nu(b), B_\nu(b) \} \right\} \right\} \\
 &= \inf_{x=a\alpha b} \left\{ \max \{ \max \{ \chi_S(a), A_\nu(b) \}, \max \{ \chi_S(a), B_\nu(b) \} \} \right\} \\
 &= \max \left\{ \inf_{x=a\alpha b} \left\{ \max \{ \chi_S(a), A_\nu(b) \} \right\}, \inf_{x=a\alpha b} \left\{ \max \{ \chi_S(a), B_\nu(b) \} \right\} \right\} \\
 &= \max \{ \chi_S \circ A_\nu(x), \chi_S \circ B_\nu(x) \} \\
 &= (\chi_S \circ A_\nu) \cap (\chi_S \circ B_\nu)(x)
 \end{aligned}$$

$$\begin{aligned}
 (A_\nu \cap B_\nu) \circ \chi_S \circ (A_\nu \cap B_\nu)(x) &= \inf_{x=aab\beta c} \{ \max\{A_\nu \cap B_\nu(a), \chi_S \circ A_\nu \cap B_\nu(b\beta c)\} \} \\
 &= \inf_{x=aab\beta c} \{ \max\{(A_\nu \cap B_\nu)(a), \chi_S \circ A_\nu \cap \chi_S \circ B_\nu(b\beta c)\} \} \\
 &= \inf_{x=aab\beta c} \{ \max\{ \max\{A_\nu(a), B_\nu(a)\}, \max\{(\chi_S \circ A_\nu)(b\beta c), (\chi_S \circ B_\nu)(b\beta c)\} \} \} \\
 &= \max\{A_\nu \circ \chi_S \circ A_\nu(x), B_\nu \circ \chi_S \circ B_\nu(x)\}
 \end{aligned}$$

Therefore

$$(A_\nu \cap B_\nu) \circ \chi_S \circ (A_\nu \cap B_\nu) = A_\nu \circ \chi_S \circ A_\nu \cap B_\nu \circ \chi_S \circ B_\nu$$

Hence

$$\begin{aligned}
 &\chi_S \circ A_\nu \cap B_\mu \circ \chi_S \cap A_\nu \cap B_\nu \circ \chi_S \circ A_\nu \cap B_\mu \\
 &= (\chi_S \circ A_\nu \circ \chi_S) \cap (A_\nu \circ \chi_S \circ A_\nu) \cap (\chi_S \circ B_\nu \circ \chi_S) \cap (B_\nu \circ \chi_S \circ B_\nu) \subseteq A_\nu \cap B_\nu.
 \end{aligned}$$

**Theorem 4.12** If  $A$  and  $B$  are PFRBQI of  $S$  then  $A \cap B$  is PFRBQI of  $S$ .

Proof. Let  $A$  and  $B$  are PFRBQI of  $S$  and  $x, y \in S$  and  $\alpha, \beta \in \Gamma$ .

$$\begin{aligned}
 (A_\mu \cap B_\mu)(x + y) &= \min\{A_\mu(x + y), B_\mu(x + y)\} \\
 &\geq \min\{\min\{A_\mu(x), A_\mu(y)\}, \min\{B_\mu(x), B_\mu(y)\}\} \\
 &= \min\{\min\{A_\mu(x), B_\mu(x)\}, \min\{A_\mu(y), B_\mu(y)\}\} \\
 &= \min\{(A_\mu \cap B_\mu)(x), (A_\mu \cap B_\mu)(y)\} \\
 \chi_S \circ (A_\mu \cap B_\mu)(x) &= \sup_{x=a\alpha b} \{ \min\{\chi_S(a), A_\mu \cap B_\mu(b)\} \} \\
 &= \sup_{x=a\alpha b} \{ \min\{\chi_S(a), \min\{A_\mu(b), B_\mu(b)\}\} \} \\
 &= \sup_{x=a\alpha b} \{ \min\{\min\{\chi_S(a), A_\mu(b)\}, \min\{\chi_S(a), B_\mu(b)\}\} \} \\
 &= \min\left\{ \sup_{x=a\alpha b} \{ \min\{\chi_S(a), A_\mu(b)\} \}, \sup_{x=a\alpha b} \{ \min\{\chi_S(a), B_\mu(b)\} \} \right\} \\
 &= \min\{\chi_S \circ A_\mu(x), \chi_S \circ B_\mu(x)\} \\
 &= (\chi_S \circ A_\mu) \cap (\chi_S \circ B_\mu)(x).
 \end{aligned}$$

$$\begin{aligned}
 (A_\mu \cap B_\mu) \circ \chi_S \circ (A_\mu \cap B_\mu)(x) &= \sup_{x=aab\beta c} \{ \min\{A_\mu \cap B_\mu(a), \chi_S \circ A_\mu \cap B_\mu(b\beta c)\} \} \\
 &= \sup_{x=aab\beta c} \{ \min\{(A_\mu \cap B_\mu)(a), \chi_S \circ A_\mu \cap \chi_S \circ B_\mu(b\beta c)\} \} \\
 &= \sup_{x=aab\beta c} \{ \min\{ \min\{A_\mu(a), B_\mu(a)\}, \min\{(\chi_S \circ A_\mu)(b\beta c), (\chi_S \circ B_\mu)(b\beta c)\} \} \}
 \end{aligned}$$

$$= \min\{A_\mu \circ \chi_S \circ A_\mu(x), B_\mu \circ \chi_S \circ B_\mu(x)\}$$

Therefore

$$(A_\mu \cap B_\mu) \circ \chi_S \circ (A_\mu \cap B_\mu) = A_\mu \circ \chi_S \circ A_\mu \cap B_\mu \circ \chi_S \circ B_\mu$$

Hence

$$\begin{aligned} \chi_S \circ A_\mu \cap B_\mu \circ \chi_S \cap A_\mu \cap B_\mu \circ \chi_S \circ A_\mu \cap B_\mu \\ = (\chi_S \circ A_\mu \circ \chi_S) \cap (A_\mu \circ \chi_S \circ A_\mu) \cap (\chi_S \circ B_\mu \circ \chi_S) \cap (B_\mu \circ \chi_S \circ B_\mu) \supseteq A_\mu \cap B_\mu \end{aligned}$$

Similarly we can prove for non membership

$$\begin{aligned} (A_\nu \cap B_\nu)(x + y) &= \max\{A_\nu(x + y), B_\nu(x + y)\} \\ &\leq \max\{\max\{A_\nu(x), A_\nu(y)\}, \max\{B_\nu(x), B_\nu(y)\}\} \\ &= \max\{\max\{A_\nu(x), B_\nu(x)\}, \max\{A_\nu(y), B_\nu(y)\}\} \\ &= \max\{(A_\nu \cap B_\nu)(x), (A_\nu \cap B_\nu)(y)\} \end{aligned}$$

$$\begin{aligned} \chi_S \circ (A_\nu \cap B_\nu)(x) &= \inf_{x=a\alpha b} \{ \max\{ \chi_S(a), A_\nu \cap B_\mu(b) \} \} \\ &= \inf_{x=a\alpha b} \{ \max\{ \chi_S(a), \max\{ A_\nu(b), B_\mu(b) \} \} \} \\ &= \inf_{x=a\alpha b} \{ \max\{ \max\{ \chi_S(a), A_\nu(b) \}, \max\{ \chi_S(a), B_\nu(b) \} \} \} \\ &= \max\{ \inf_{x=a\alpha b} \{ \max\{ \chi_S(a), A_\nu(b) \} \}, \inf_{x=a\alpha b} \{ \max\{ \chi_S(a), B_\nu(b) \} \} \} \\ &= \max\{ \chi_S \circ A_\nu(x), \chi_S \circ B_\nu(x) \} \\ &= (\chi_S \circ A_\nu) \cap (\chi_S \circ B_\nu)(x). \end{aligned}$$

$$\begin{aligned} (A_\nu \cap B_\nu) \circ \chi_S \circ (A_\nu \cap B_\nu)(x) &= \inf_{x=a\alpha b\beta c} \{ \max\{ A_\nu \cap B_\nu(a), \chi_S \circ A_\nu \cap B_\nu(b\beta c) \} \} \\ &= \inf_{x=a\alpha b\beta c} \{ \max\{ (A_\nu \cap B_\nu)(a), \chi_S \circ A_\nu \cap \chi_S \circ B_\nu(b\beta c) \} \} \\ &= \inf_{x=a\alpha b\beta c} \{ \max\{ \max\{ A_\nu(a), B_\nu(a) \}, \max\{ (\chi_S \circ A_\nu)(b\beta c), (\chi_S \circ B_\nu)(b\beta c) \} \} \} \\ &= \max\{ A_\nu \circ \chi_S \circ A_\nu(x), B_\nu \circ \chi_S \circ B_\nu(x) \} \end{aligned}$$

Therefore

$$(A_\nu \cap B_\nu) \circ \chi_S \circ (A_\nu \cap B_\nu) = A_\nu \circ \chi_S \circ A_\nu \cap B_\nu \circ \chi_S \circ B_\nu$$

Hence

$$\begin{aligned} \chi_S \circ A_\nu \cap B_\mu \circ \chi_S \cap A_\nu \cap B_\nu \circ \chi_S \circ A_\nu \cap B_\mu \\ = (\chi_S \circ A_\nu \circ \chi_S) \cap (A_\nu \circ \chi_S \circ A_\nu) \cap (\chi_S \circ B_\nu \circ \chi_S) \cap (B_\nu \circ \chi_S \circ B_\nu) \subseteq A_\nu \cap B_\nu \end{aligned}$$

## 5 Conclusion

This paper deals with the concept of Pythagorean fuzzy bi-interior-ideal, Pythagorean fuzzy soft bi-interior-ideal and Pythagorean fuzzy bi-quasi-ideals in  $\Gamma$ -semirings. Some properties of these ideals are studied. Additionally, we aim to extend this structure to various other algebraic structures and look into its applications in real-life scenarios in future research endeavors.

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