Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 4s (2025)

Extorial in RL Circuits and Heat Flows

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Article History:

***Received:** 21-09-2024*

***Revised:** 24-11-2024*

***Accepted:** 05-12-2024*

Abstract:

The newly defined ℓ-extorial function was used in this paper to generate the solution of the RL circuit. When an inductor and a resistor are linked across a voltage source, the resulting circuit is known as an RL circuit. Depending on how the resistor and inductor are connected, this circuit may be in series or parallel. One of the solutions to RL's mathematical difference problem is the extorial function. To solve the RL circuit, we thus develop the theory of the extorial function and employ it.

Keywords: ℓ-extorial function, ℓ-Delta operator, RL circuit, Heat equation.

1. Introduction

A difference equation is an equation that contains sequence differences. There are various types of difference equations namely ordinary, delay, advanced, neutral, quasilinear, half linear, etc. These equations occur in numerous settings and forms, both in mathematics itself and its applications to Biology, Computer Science, Digital Signal Processing, Economics, Statistics and other fields.

The fractional sum of a function f (or ν^{th} order delta integration) is defined by

$$(\Delta_a^{-\nu}u)(\kappa)=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{\kappa-\nu}\frac{\Gamma(\kappa-s)}{\Gamma(\kappa-s-(\nu-1))}u(s), \qquad (1)$$

where $\nu>0$, f is defined for $s=a \ mod(1)$ and $\Delta^{-\nu}f$ is defined for $\kappa=a+\nu \ mod(1)$. The basic theory of difference equations is based on the difference operator Δ defined as $\Delta u(\kappa)=u(\kappa+1)-u(\kappa)$, where $\{u(\kappa)\}$ is a sequence or a function of κ of numbers. Many authors ([7],[9]) have suggested the definition of generalized difference operator Δ_ℓ on real valued function u defined on $\mathbb{R}=(-\infty,\infty)$ as

$$\Delta_\ell u(\kappa)=u(\kappa+\ell)-u(\kappa),\ \kappa\in\mathbb{R},\ \ell>0. \qquad (2)$$

E. Thandapani, M.Maria Susai Manuel, G.B.A Xavier [8] considered the definition of Δ_ℓ as given in (2) and developed the theory of difference equations in a different direction. If there exists a function v such that $\Delta_\ell v(\kappa)=u(\kappa)$, then we call this function v as $\Delta_\ell^{-1}v$. Hence, for $\kappa\in\mathbb{R}=\bigcup_{0\le j<\ell}\mathbb{N}_\ell(j)$,

$$\text{if } \Delta_\ell v(\kappa)=u(\kappa), \text{then } v(\kappa)=\Delta_\ell^{-1}u(\kappa)+c_j, \qquad (3)$$

where c_j is constant for all κ in each $\mathbb{N}_\ell(j)=\{j,j+\ell,j+2\ell,\dots\}$, $j=\kappa-[\frac{\kappa}{\ell}]\ell$.

In 1989, Miller and Rose introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. In 1984, Jerzy Popenda [4] introduced a particular type of difference operator on u as $\Delta_\alpha u(\kappa) = u(\kappa+1) - \alpha u(\kappa)$, In 2011, M.Maria Susai Manuel, et.al, [5] extended the operator Δ_α to generalized α −difference operator as $\underset{\alpha(\ell)}{\Delta} v(\kappa) = v(\kappa+\ell) - \alpha v(\kappa)$ for real valued function v. In 2014, the authors in [1] have applied the q-difference operator defined by $\Delta_q v(\kappa) = v(q\kappa) - v(\kappa)$ and delta operator $\underset{\kappa(\ell)}{\Delta}$ with variable coefficients defined by $\underset{\kappa(\ell)}{\Delta} v(\kappa) = v(\kappa+\ell) - \kappa v(\kappa)$, $\ell \neq 0 \in \mathbb{R}$. Also the generalized difference operator with n-shift values $l = (\ell_1, \ell_2, \ell_3, \ldots, \ell_n) \neq 0$ on a real valued function $v\colon \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\underset{(\ell)}{\Delta} v(\kappa) = v(\kappa_1+\ell_1, \kappa_2+\ell_2, \ldots, \kappa_n+\ell_n) - v(\kappa_1, \kappa_2, \ldots, \kappa_n). \qquad (4)$$

This operator $\underset{(\ell)}{\Delta}$ becomes generalized partial difference operator if some $\ell_i = 0$. The equations involving $\underset{(\ell)}{\Delta}$ with atleast one $\ell_i = 0$ is called generalized partial difference equation. for one shift value, we take $\Delta_{(\ell)}$ as Δ_ℓ. By defining the inverse Δ_ℓ^{-1}, many interesting results on sum of partial sums of higher power of arithmetic and geometric functions and applications in numerical methods (see [8, 6, 2, 3, 10, 11, 12, 13]) are obtained. The difference operator defined in (2) becomes the usual difference operator Δ when $\ell = 1$. We obtain several results on factorial function by applying Δ_ℓ^{-1}.

The fractional sum of a function f (or v^{th} order delta integration) is defined by

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$$\text{if } \Delta_\ell v(\kappa) = u(\kappa), \text{then } v(\kappa) = \Delta_\ell^{-1} u(\kappa) + c_j, \qquad (3)$$

where c_j is constant for all κ in each $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \ldots\}$, $j = \kappa - [\frac{\kappa}{\ell}]\ell$.

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$\underset{\alpha(\ell)}{\Delta} v(\kappa) = v(\kappa+\ell) - \alpha v(\kappa)$ for real valued function v. In 2014, the authors in [1] have applied the q-difference operator defined by $\Delta_q v(\kappa) = v(q\kappa) - v(\kappa)$ and delta operator $\underset{\kappa(\ell)}{\Delta}$ with variable coefficients defined by $\underset{\kappa(\ell)}{\Delta} v(\kappa) = v(\kappa+\ell) - \kappa v(\kappa), \ell \neq 0 \in \mathbb{R}$.

Also the generalized difference operator with n-shift values $l = (\ell_1, \ell_2, \ell_3, \ldots, \ell_n) \neq 0$ on a real valued function $v: \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\underset{(\ell)}{\Delta} v(\kappa) = v(\kappa_1+\ell_1, \kappa_2+\ell_2, \ldots, \kappa_n+\ell_n) - v(\kappa_1, \kappa_2, \ldots, \kappa_n). \tag{4}$$

This operator $\underset{(\ell)}{\Delta}$ becomes generalized partial difference operator if some $\ell_i = 0$. The equations involving $\underset{(\ell)}{\Delta}$ with atleast one $\ell_i = 0$ is called generalized partial difference equation. for one shift value, we take $\Delta_{(\ell)}$ as Δ_ℓ. By defining the inverse Δ_ℓ^{-1}, many interesting results on sum of partial sums of higher power of arithmetic and geometric functions and applications in numerical methods (see [8, 6, 2, 3, 10, 11, 12, 13]) are obtained. The difference operator defined in (2) becomes the usual difference operator Δ when $\ell = 1$. We obtain several results on factorial function by applying Δ_ℓ^{-1}.

2. The ℓ - Extorial function and its Properties

The ℓ-Extorial function is arrived by replacing the polynomial κ^n by polynomial factorial function $\kappa_\ell^{(n)}$ in the exponential function e^κ. The formal definition of extorial function is given below.

Definition 2.1. The ℓ-extorial function denoted as $e(\kappa_\ell^{(n)})$ is defined as

$$e(\kappa_\ell^{(n)}) = 1 + \frac{\kappa_\ell^{(n)}}{1!} + \frac{\kappa_\ell^{(2n)}}{2!} + \frac{\kappa_\ell^{(3n)}}{3!} + \cdots + \infty, \tag{5}$$

where $|\ell| \leq 1$ and $n, \kappa \in \mathbb{R}$.

Definition 2.2. For $\ell \in (-1,1)$ and $\kappa \in \mathbb{R}$, the n^{th} order ℓ-extorial function denoted as $e_n(\kappa_\ell)$ is defined as

$$e_n(\kappa_\ell) = 1 + \frac{\kappa_\ell^{(n)}}{n!} + \frac{\kappa_\ell^{(2n)}}{(2n)!} + \frac{\kappa_\ell^{(3n)}}{(3n)!} + \cdots + \infty. \tag{6}$$

From the definition of extorial function, we obtain following lemma.

Lemma 2.3. For any real κ and $\ell, n \in \mathbb{N}$, we have

(i) $e_n(-\kappa_\ell) = e_n(\kappa_{-\ell})$ if n $is\ even\ \&\ 1 - \frac{\kappa_{(-\ell)}^{(n)}}{n!} + \frac{\kappa_{(-\ell)}^{(2n)}}{(2n)!} - \frac{\kappa_{(-\ell)}^{(3n)}}{(3n)!} + \cdots$ if n $is\ odd$

and

(ii) $e_n\big(-\kappa_{(-\ell)}\big) = e_n(\kappa_\ell)$ if n $is\ even\ \&\ 1 - \frac{(\kappa)_\ell^{(n)}}{n!} + \frac{(\kappa)_\ell^{(2n)}}{2n!} - \frac{(\kappa)_\ell^{(3n)}}{3n!} + \cdots$ if n $is\ odd$

Lemma 2.4. Let $\kappa \in \mathbb{R}$ and $n, \ell \in \mathbb{N}$. Then, we have

$$\Delta_\ell e_n(\kappa_\ell) = \ell \textstyle\sum_{m=1}^{\infty} \frac{\kappa_\ell^{(mn-1)}}{(mn-1)!}, \ nm \neq 1.$$

Proof. We shall prove this by induction method

$e_2(\kappa_\ell) = 1 + \frac{\kappa_\ell^{(2)}}{2!} + \frac{\kappa_\ell^{(4)}}{4!} + \frac{\kappa_\ell^{(6)}}{6!} + \cdots + \infty$

$$\Delta_\ell e_2(\kappa_\ell) = \Delta_\ell \frac{\kappa_\ell^{(2)}}{2!} + \Delta_\ell \frac{\kappa_\ell^{(4)}}{4!} + \Delta_\ell \frac{\kappa_\ell^{(6)}}{6!} + \cdots + \infty = \ell \left[\frac{\kappa_\ell^{(1)}}{1!} + \frac{\kappa_\ell^{(3)}}{3!} + \frac{\kappa_\ell^{(5)}}{5!} + \cdots \right]$$

$e_3(\kappa_\ell) = 1 + \frac{\kappa_\ell^{(3)}}{3!} + \frac{\kappa_\ell^{(6)}}{6!} + \frac{\kappa_\ell^{(9)}}{9!} + \cdots + \infty$

$$\Delta_\ell e_3(\kappa_\ell) = \Delta_\ell \frac{\kappa_\ell^{(3)}}{3!} + \Delta_\ell \frac{\kappa_\ell^{(6)}}{6!} + \Delta_\ell \frac{\kappa_\ell^{(9)}}{9!} + \cdots + \infty = \ell \left[\frac{\kappa_\ell^{(2)}}{2!} + \frac{\kappa_\ell^{(5)}}{5!} + \frac{\kappa_\ell^{(8)}}{8!} + \cdots \right]$$

In general, we find for $n \geq 1$

$\Delta_\ell e_n(\kappa_\ell) = \ell \left[\frac{\kappa_\ell^{(n-1)}}{(n-1)!} + \frac{\kappa_\ell^{(2n-1)}}{(2n-1)!} + \frac{\kappa_\ell^{(3n-1)}}{(3n-1)!} + \cdots \right] = \ell \sum_{m=1}^{\infty} \frac{\kappa_\ell^{(mn-1)}}{(mn-1)!}$

Lemma 2.5. For any positive integer m, we have $\Delta_\ell^m e_m(\kappa_\ell) = \ell^m e_m(\kappa_\ell)$.

Proof. $\Delta_\ell e_1(\kappa_\ell) = 0 + \Delta_\ell \frac{\kappa_\ell^{(1)}}{1!} + \Delta_\ell \frac{\kappa_\ell^{(2)}}{2!} + \Delta_\ell \frac{\kappa_\ell^{(3)}}{3!} + \cdots = \ell e_1(\kappa_\ell)$.

$\Delta_\ell e_2(\kappa_\ell) = 0 + \Delta_\ell \frac{\kappa_\ell^{(2)}}{2!} + \Delta_\ell \frac{\kappa_\ell^{(4)}}{4!} + \Delta_\ell \frac{\kappa_\ell^{(6)}}{6!} + \cdots = \frac{2\ell\kappa_\ell(1)}{2!} + \frac{4\ell\kappa_\ell(3)}{4!} + \frac{6\ell\kappa_\ell(5)}{6!} + \cdots$

$\Delta_\ell^2 e_2(\kappa_\ell) = \frac{2\ell(\ell\kappa_\ell^{(0)})}{2!} + \frac{4\ell(3\ell\kappa_\ell^{(2)})}{4!} + \frac{6\ell(5\ell\kappa_\ell^{(4)})}{6!} + \cdots = \ell^2 e_2(\kappa_\ell)$,

which yields $\Delta_\ell^m e_m(\kappa_\ell) = \ell^m e_m(\kappa_\ell)$.

Lemma 2.6. For positive m and real κ, we have $\Delta_\ell^{(-m)} e_m(\kappa_\ell) = \frac{e_m(\kappa_\ell)}{\ell^m}$, $\ell \in \mathbb{N}$.

Proof. From the lemma 5, we find $\Delta_\ell^m e_m(\kappa_\ell) = \ell^m e_m(\kappa_\ell)$.

Taking Δ_ℓ^{-m} on both sides, we get $\Delta_\ell^{-m}(\Delta_\ell^m e_m(\kappa_\ell)) = \Delta_\ell^{-m}(\ell^m e_m(\kappa_\ell))$,

which gives $\Delta_\ell^{(-m)} e_m(\kappa_\ell) = \frac{e_m(\kappa_\ell)}{\ell^m}$.

Definition 2.7. For $|\ell| < 1$, and $n \in \mathbb{N}$, $e_{(-n)}(k_\ell)$ is defined as

$$e_{(-n)}(\kappa_\ell) = 1 + \frac{1}{n!}\frac{1}{\kappa_\ell^{(n)}} + \frac{1}{(2n)!}\frac{1}{\kappa_\ell^{(2n)}} + \frac{1}{(3n)!}\frac{1}{\kappa_\ell^{(3n)}} + \cdots + \infty. \tag{7}$$

Lemma 2.8. For $\ell \in (-1,1)$ and positive κ, we have

$\Delta_\ell e_{(-n)}(\kappa_\ell) = -\ell[\frac{1}{(n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(n+1)}} + \frac{1}{(2n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(2n+1)}} \quad + \frac{1}{(3n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(3n+1)}} + \cdots]$

Proof. Putting $n = 1$ in (7), we get

$$\begin{aligned} e_{(-1)}(\kappa_\ell) &= 1+\frac{1}{1!}\frac{1}{\kappa_\ell^{(1)}}+\frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}}+\frac{1}{3!}\frac{1}{\kappa_\ell^{(3)}}+\cdots+\infty\Delta_\ell e_{(-1)}(\kappa_\ell) \\ &= 1+\Delta_\ell\frac{1}{1!}\frac{1}{\kappa_\ell^{(1)}}+\Delta_\ell\frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}}+\Delta_\ell\frac{1}{3!}\frac{1}{\kappa_\ell^{(3)}}+\cdots+\infty \\ &= -\ell\left[\frac{1}{(\kappa+\ell)_\ell^{(2)}}+\frac{1}{1!}\frac{1}{(\kappa+\ell)_\ell^{(3)}}+\frac{1}{2!}\frac{1}{(\kappa+\ell)_\ell^{(4)}}+\cdots\right]. \end{aligned}$$

Putting $n=2$ in (7), we get

$$\begin{aligned} e_{(-2)}(\kappa_\ell) &= 1+\frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}}+\frac{1}{4!}\frac{1}{\kappa_\ell^{(4)}}+\frac{1}{6!}\frac{1}{\kappa_\ell^{(6)}}+\cdots+\infty\Delta_\ell e_{(-2)}(\kappa_\ell) \\ &= 1+\Delta_\ell\frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}}+\Delta_\ell\frac{1}{4!}\frac{1}{\kappa_\ell^{(4)}}+\Delta_\ell\frac{1}{6!}\frac{1}{\kappa_\ell^{(6)}}+\cdots+\infty \\ &= -\ell\left[\frac{1}{1!}\frac{1}{(\kappa+\ell)_\ell^{(3)}}+\frac{1}{3!}\frac{1}{(\kappa+\ell)_\ell^{(5)}}+\frac{1}{5!}\frac{1}{(\kappa+\ell)_\ell^{(7)}}+\cdots\right]. \end{aligned}$$

Putting $n=3$ in (7), we get

$$\begin{aligned} e_{(-3)}(\kappa_\ell) &= 1+\frac{1}{3!}\frac{1}{\kappa_\ell^{(3)}}+\frac{1}{6!}\frac{1}{\kappa_\ell^{(6)}}+\frac{1}{9!}\frac{1}{\kappa_\ell^{(9)}}+\cdots+\infty\Delta_\ell e_{(-3)}(\kappa_\ell) \\ &= 1+\Delta_\ell\frac{1}{3!}\frac{1}{\kappa_\ell^{(3)}}+\Delta_\ell\frac{1}{6!}\frac{1}{\kappa_\ell^{(6)}}+\Delta_\ell\frac{1}{9!}\frac{1}{\kappa_\ell^{(9)}}+\cdots+\infty \\ &= -\ell\left[\frac{1}{2!}\frac{1}{(\kappa+\ell)_\ell^{(4)}}+\frac{1}{5!}\frac{1}{(\kappa+\ell)_\ell^{(7)}}+\frac{1}{8!}\frac{1}{(\kappa+\ell)_\ell^{(10)}}+\cdots\right]. \end{aligned}$$

In general,

$$\Delta_\ell e_{(-n)}(\kappa_\ell) = -\ell[\tfrac{1}{(n-1)!}\tfrac{1}{(\kappa+\ell)_\ell^{(n+1)}}+\tfrac{1}{(2n-1)!}\tfrac{1}{(\kappa+\ell)_\ell^{(2n+1)}}+\tfrac{1}{(3n-1)!}\tfrac{1}{(\kappa+\ell)_\ell^{(3n+1)}}+\cdots].$$

3. Current Flows in RL Circuit

Consider a RL circuit by using the Kirchhoff's circuit rule. The differential equation connecting voltage V, resistance R, current I and induction L in series is given by first order linear difference equation

$$V = RI(\kappa)+L\frac{dI(\kappa)}{d\kappa}. \tag{8}$$

The discrete analogue of (8) is assumed by replacing $dI(\kappa)=\Delta I(\kappa)$, where $\Delta I(\kappa)=I(\kappa+1)-I(\kappa)$ and $d\kappa=1$ in (8).

The corresponding difference equation for the current flows in RL series circuit in the discrete case takes the form, at time κ

$$v(\kappa)=RI(\kappa)+L\Delta I(\kappa). \tag{9}$$

Due to the resistance of conductor, heat temperature may be raised in the RL circuit. In that case, we need to modify the difference equation (9). In that case the difference equation (9) becomes fractional difference equation as

$$V = RI(\kappa) + L\Delta^{\nu} I(\kappa), (0 < \nu < 1). \tag{10}$$

Which can be expressed as

$$\frac{V(\kappa) - RI(\kappa)}{L} = \Delta^{\nu} I(\kappa)$$

The corresponding discrete integral equations

$$\Delta^{-\nu}\left(\frac{V(\kappa) - RI(\kappa)}{L}\right) = I(\kappa)$$

By applying γ^{th} order delta sum given by (1),

$$\frac{1}{L\Gamma(\nu)} \sum_{s=0}^{\kappa-\nu} \frac{\Gamma(\kappa - s)}{\Gamma(\kappa - s - (\nu - 1))} (V(s) - RI(s)) = I(\kappa), \tag{11}$$

The corresponding fractional difference equation for de-energizing in RL circuit is obtained by putting $v(\kappa) = 0$. In this case, we get

$$0 = RI(\kappa) + L\Delta^{\nu} I(\kappa). \tag{12}$$

Which is the same as

$$\Delta^{\nu} I(\kappa) = -\frac{RI(\kappa)}{L}.$$

$$I(\kappa) = \Delta^{-\nu}\left(-\frac{RI(\kappa)}{L}\right) = -\frac{R}{L}\Delta^{-\nu} I(\kappa)$$

By applying fractional order delta integration (1), we obtain

$$I(\kappa) = \frac{R}{L\Gamma(\nu)} \sum_{s=0}^{\kappa-\nu} \frac{\Gamma(\kappa - s)}{\Gamma(\kappa - s - (\nu - 1))} I(s), \tag{13}$$

The solution (11) and (13) are summation forms. Through our research, we identifies that these fractional difference equations have exact type solutions, when the initial time a is taken as zero. We obtain exact solution for the equations (9) and (10) using our newly defined extorial functions.

4. Extorial Type Solution of RL Circuit

In this section, we find solution of equation (9) after arriving at some basic results of extorial functions. This extorial function is easily obtained by replacing polynomial κ^n into factorial polynomial in the expansion of exponential function e^{κ}. This function is useful to arrive at solutions for fractional difference equation.

Consider the extorial function $e_1((m\kappa)_{(m)})$ is defined by

$$e_1((m\kappa)_{(m)}) = 1 + \frac{(m\kappa)_m^{(1)}}{1!} + \frac{(m\kappa)_m^{(2)}}{2!} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{(m\kappa)_m^{(r)}}{r!}, \tag{14}$$

where $(m\kappa)_m^{(r)} = (m\kappa)(m\kappa - m) \cdots (m\kappa - (r-1)m)$ for positive integer r, is a falling polynomial factorial. In general, for real index ν, we have

$$e_{(\nu)}((m\kappa)_{(m)}) = 1 + \frac{(m\kappa)_m^{(\nu)}}{1!} + \frac{(m\kappa)_m^{(2\nu)}}{2!} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{(m\kappa)_m^{(r\nu)}}{r!}, \tag{15}$$

where $(m\kappa)_m^{(r\nu)} = (m)^{(r\nu)} \frac{\Gamma(\kappa+1)}{\Gamma(K+1-r\nu)}$ and $\Gamma(.)$ is the gamma function.

Lemma 4.1. If $e_1((m\kappa)_{(m)})$ is an extorial function, then we have

$$\Delta e_1((m\kappa)_{(m)}) = (m) e_1((m\kappa)_{(m)}). \tag{16}$$

Proof. Applying Δ on the extorial function $e_1((m\kappa)_{(m)})$, we arrive at

$$\Delta e_1((m\kappa)_{(m)}) = \Delta(1) + \Delta \frac{(m\kappa)_m^{(1)}}{1!} + \Delta \frac{(m\kappa)_m^{(2)}}{2!} + \cdots + \infty$$

$$= 0 + \frac{(m(\kappa+1))_m^{(1)}}{1!} + \frac{(m\kappa)_m^{(1)}}{1!} + \frac{1}{2!}[(m(\kappa+1))_m^{(2)} - (m\kappa)_m^{(2)}] + \cdots + \infty$$

$$= \frac{(m)}{1!} + \frac{1}{2!}[(m\kappa + m)(m\kappa) - (m\kappa)(m\kappa - m)] + \cdots + \infty$$

$$= m + \frac{2m}{2!}(m\kappa)_m^{(1)} + \frac{3m}{3!}(m\kappa)_m^{(2)} + \cdots + \infty$$

$$= m[1 + \frac{(m\kappa)_m^{(\kappa)}}{1!} + \frac{(m\kappa)_m^{(2)}}{2!} + \cdots + \infty]$$

$\Delta e^{(m\kappa)_m} = (m) e_1((m\kappa)_{(m)})$.

Lemma 4.2. The extorial function $u(\kappa) = e_1((m\kappa)_{(m)})$ is a solution of equation

$$(A\Delta^2 + B\Delta + C)u(\kappa) = 0, \tag{17}$$

if m is a root of the auxiliary equation $Am^2 + Bm + C = 0$.

Proof. If we try $u(\kappa) = e_1((m\kappa)_{(m)})$ as a solution of equation(17), then it should satisfy the equation

$$A\Delta^2 e_1((m\kappa)_{(m)}) + B\Delta e_1((m\kappa)_{(m)}) + C e_1((m\kappa)_{(m)}) = 0. \tag{18}$$

By linear property of Δ and the expansion of $e_1((m\kappa)_{(m)})$, we arrive at

$$\Delta_\ell e_1((m\kappa)_{(m)}) = 0 + (m)\frac{(m\kappa)_{(m)}^{(0)}}{1!} + \frac{2m(m\kappa)_{(m)}^{(1)}}{2!} + \frac{3m(m\kappa)_{(m)}^{(2)}}{3!} +\ldots$$

i.e, $\Delta_\ell e_1((m\kappa)_{(m)}) = m[1 + \frac{(m\kappa)_{(m)}^{(1)}}{1!} + \frac{(m\kappa)_{(m)}^{(2)}}{2!} +\ldots] = m e_1((m\kappa)_{(m)})$,

which yields

$\Delta_\ell^2 e_1((m\kappa)_{(m)}) = (m)\Delta_\ell e_1((m\kappa)_{(m)}) = (m)^2 e_1((m\kappa)_{(m)})$.

Applying the values of $\Delta e_1((m\kappa)_{(m)})$ and $\Delta_\ell^2 e_1((m\kappa)_{(m)})$ in (18), we obtain

$$(Am^2 + Bm + C) e_1((m\kappa)_{(m)}) = 0,$$

Since $e_1((m\kappa)_m) \neq 0$, we get

$$Am^2 + Bm + C = 0. \tag{19}$$

Hence $u(\kappa) = e_1((m\kappa)_{(m)})$ is a solution of (17) when m is a root of (18).

Remark 4.3. The above lemma can be extended to higher order linear difference equation with constant coefficients.

Theorem 4.4. Let I_0 be initial value of $I(\kappa)$ and $\nu = 1$. The de-energizing difference equation (12) for $\nu = 1$ has a solution of the form

$$I(\kappa) = I_0 e_1\left(\left(\frac{-R}{L}\kappa\right)_{(\frac{-R}{L})}\right) \tag{20}$$

where e_1 denotes the extorial function.

Proof. Consider the first order difference equation $L\Delta I(\kappa) + RI(\kappa) = 0$ which is obtained from (12) by taking $\nu = 1$ and its Auxillary equation ML+R=0. The auxiliary equation $mL + R = 0$ has an unique solution $m = \frac{-R}{L}$, $L \neq 0$. Applying Lemma 4 for first order difference equation by taking $A = 0$, $I(t) = I_0 e_1((\frac{-R}{L}\kappa)_{(\frac{-R}{L})})$, which is a solution of the equation (12) for $\nu = 1$.

Theorem 4.5. For $\nu = 1$, the energizing difference equation (10) has a solution

$$I(\kappa) = \frac{V}{L(e^s - 1) + R} + I_0 e_1\left(\left(\frac{-R}{L}\kappa\right)_{(\frac{-R}{L})}\right), \tag{21}$$

where s is a constant.

Proof. Let $I(\kappa) = \frac{V}{c} e^{s\kappa}$ be a solution of equation (10) for $\nu = 1$, where c is to be determined. Since s is a constant, we get $\Delta e^{s\kappa} = e^{s(\kappa+1)} - e^{s\kappa} = (e^s - 1)e^{s\kappa}$ and

$\Delta I(\kappa) = \frac{V}{c}\Delta e^{s\kappa} = \frac{V}{c} e^{s\kappa}(e^s - 1)$.

Substituting $\nu = 1$, $I(\kappa)$ and $\Delta I(\kappa)$ in (10), we arrive

$$I(\kappa)R + L\Delta I(\kappa) = R\frac{V}{c}e^{s\kappa} + L[\frac{V}{c}e^{s\kappa}(e^s - 1)],$$

which yields

$[R + L\Delta]I = \frac{V}{c}[L(e^s - 1 + R)]e^{s\kappa}$.

Hence, taking $c = L(e^s - 1 + R)$, we find a particular solution of equation (10) when $\nu = 1$, as

$$I(\kappa) = \frac{V}{L(e^s - 1 + R)} e^{s\kappa}. \tag{22}$$

Now (21) follows by adding (20) and (22) the proof is complete.

Corollary 4.6. If $I_0 = \frac{-V}{R}$, then the extorial solution of difference equation (9) of the RL circuit is $I(\kappa) = \frac{V}{R} - \frac{V}{R} e_1\left(\left(\frac{-R}{L}\kappa\right)_{(\frac{-R}{L})}\right)$.

Proof. The proof follows by taking $s = 0$ in (21).

Finding the solutions of integer order difference equation is comparatively easier than the fractional order difference equation.

5. Extorial Energizing for RL Circuit

In this section, we derive at the solution of RL circuit model with extorial energizing. Here, we deal with fractional order difference equation also.

Theorem 5.1. The flow of current in the RL circuit creates chaos due to increase of temperature of heat. In this case, the difference equation of RL circuit becomes

$$Ve_1((s\kappa)_s) = RI(\kappa) + L\Delta^{\nu}I(\kappa), (0 < \nu < 1). \tag{23}$$

Equation (23) is ν^{th} order fractional difference equation. When there is no choas in RL circuit, the parameter ν takes integer value.

Theorem 5.2. For $\nu = 1$, the difference equation (23) has a extorial solution

$$I(\kappa) = \frac{Ve_1((s\kappa)_{s\ell})}{L(e^s-1)+R} + I_0e_1(\left(\frac{-R}{L}\kappa\right)_{(\frac{-R}{L})}). \tag{24}$$

Proof. Let $I(\kappa) = \frac{V}{c}e_1((s\kappa)_s)$ be a solution of equation (23) ($\nu = 1$), where c is to be determined. Since s is a constant, from (24), we get $\Delta e_1((s\kappa)_s) = e_1((s(\kappa + 1))_{(s)}) - e_1((s\kappa)_s)$. This gives

$$\Delta I(\kappa) = \frac{V}{c}\Delta e_1((s\kappa)_s) = \frac{V}{c}e_1((s\kappa)_{(s)})(e_1((1)_{(s)}) - 1).$$

Substituting $I(\kappa)$ and $\Delta I(\kappa)$ in the above equation, we arrive

$$I(\kappa)R + L\Delta I(\kappa) = R\frac{V}{c}e_1((s\kappa)_s) + L[\frac{V}{c}e^{(s\kappa)}(e_1(1_{(s)}) - 1)],$$

which yields $[R + L\Delta]I = \frac{V}{c}[L(e_1(\ell_{(s)}) - 1 + R)]e_1((s\kappa)_s)$.

Hence taking $c = L(e_1(1_{(s)}) - 1 + R)$, we find $I(\kappa) = \frac{V}{L(e_1(1_{(s)})-1+R)}e^{s\kappa}$ is a particular solution of equation when $\nu = 1$ and (24) follows.

Theorem 5.3. For For $0 < \nu < 1$, the energizing fractional difference equation

$$Ve_1((s\kappa)_{(s)}) = I(\kappa)R + \Delta^{\nu}I(\kappa), \tag{25}$$

has an extorial solution of the form

$$\frac{Ve_1((s\kappa)_{(s)})}{L(e_1(\ell_{(s)})-1)^{\nu}+R} + I_0e_1((\frac{-R}{L})^{\frac{1}{\nu}}\kappa_{(\frac{-R}{L})^{\frac{1}{\nu}}}). \tag{26}$$

Proof. We try $I(\kappa) = Vce_1((s\kappa)_{(s)})$ as a solution of equation (25), where c is to be determined.

$\Delta I(\kappa) = Vc(e_1(1_{(s)}) - 1)e_1((s\kappa)_{(s)})$, $\Delta^2 I(\kappa) = Vc(e_1(1_{(s)}) - 1)^2 e_1((s\kappa)_{(s)}) \cdots$,

$\Delta^{\nu}I(\kappa) = Vc(e_1(1_{(s)}) - 1)^{\nu}e_1((s\kappa)_{(s)})$ is obtained from $\Delta I(\kappa) = I(\kappa + 1) - I(\kappa)$.

Substituting I and $\Delta^{\nu}I$ in (25), we find

$$I(\kappa)R + \frac{L}{\ell}\Delta^{\nu} I(\kappa) = RVce_1((s\kappa)_{(s)}) + L[Vc(e_1(1_{(s)}) - 1)^{\nu} e_1((s\kappa)_{(s)})]$$

$$= Vc[\tfrac{L}{\ell}(e_1(1_{(s)}) - 1)^{\nu} + R]e_1((s\kappa)_{(s)}).$$

Hence $I(\kappa) = \frac{V}{L(e_1(1_{(s)})-1)^{\nu}+R} e_1((s\kappa)_{(s)})$ is a particular solution of (26).

Thus extorial function is used to obtain the solution to RL-circuit difference equation. Also we have obtained solution of RL circuit of chaos situation represented by fractional order difference equation.

6. Fractional Difference Heat Equation Model

In this section, we apply the alpha and Fibonacci difference operators and obtain new model of heat equations. The solution of these equations are expressed in terms of extorial functions. The materials up to three dimensions i.e., rod, thin plate and medium are taken for study and the transfer of heat is examined. The two operators (alpha and Fibonacci) are used for the study of transfer of heat and are defined accordingly.

Let $\alpha \neq 0$, $l = (1,1,1,\ldots,1)$, $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n) \in \mathbb{R}^n$ and $v(\kappa)$ be a real valued n-variable function defined on $\mathbb{R}^n$. The n-variable α-difference operator, denoted as Δ_{α}, on $v(\kappa)$ is defined by

$$\underset{\alpha}{\Delta} v(\kappa) = v(\kappa_1 + 1, \kappa_2 + 1, \ldots, \kappa_n + 1) - \alpha v(\kappa_1, \kappa_2, \ldots, \kappa_n). \tag{27}$$

This operator becomes partial α-difference operator if we replace by $\kappa_i + 1$ in centain component i. Thus the above definition of the alpha and Fibonacci difference operators and its equations are employed in the forthcoming sections and solutions are derived for heat equations.

Also we present solutions of partial fractional alpha difference equation with polynomial factorial and extorial functions. We also apply these type of solutions to heat flows. In the following lemma, some identities related to alpha difference operator on extorial function are given.

Lemma 6.1. Let $\kappa^{(rn)} \neq 0$, $n \in N$. Then we have the following identities with extorial function:

(i). $\Delta_{\alpha} e_1(\kappa) = e_1(\kappa)[1 + 1 - \alpha]$,

(ii). $\Delta_{\alpha} e_{(-1)}(\kappa) = e_{(-1)}(\kappa)[e_{(-1)} - \alpha]^{(-1)}$,

(iii). $\Delta_{\alpha} e_1((-\kappa)) = e_1((-\kappa))[1 + 1 - \alpha]$, $\kappa > 0$.

Proof. (i). By (27), and applying Δ_{α} on $e_1(\kappa)$, we arrive

$$\Delta_{\alpha} e_1(\kappa) = e_1((\kappa + 1)) - \alpha e_1(\kappa) = e_1(\kappa). e_1 - \alpha e_1(\kappa) = e_1(\kappa)[e_1(1) - \alpha]$$

$$= e_1(\kappa)[1 + \tfrac{1}{1!} + \tfrac{1}{2!} + \cdots - \alpha] = e_1(\kappa)[1 + 1 - \alpha].$$

(ii). By (27), and applying Δ_{α} on $e(\kappa^{(-1)})$, we arrive

$$\Delta_{\alpha} e_{(-1)}(\kappa) = e_{(-1)}((\kappa + 1)) - \alpha e_{(-1)}(\kappa) = e_{(-1)}(\kappa). e_{(-1)} - \alpha e_{(-1)}(\kappa).$$

$$= e_{(-1)}(\kappa)[e_{(-1)} - \alpha]^{(-1)}.$$

(iii). follows from (ii) by replacing κ as $-\kappa$.

Theorem 6.2. If $v(\kappa_1,\kappa_2)=e_1((\kappa_1)).e_1((\kappa_2))$ then we have the identities:

$$(i)\Delta_\alpha v(\kappa_1,\kappa_2)=e_1((\kappa_1)).e_1((\kappa_2))[e_1(1)-\alpha],$$

$$(ii)\Delta_\alpha v(\kappa_1,\kappa_2)=e_1((\kappa_2)).e_1((\kappa_1))[e_1((1))-\alpha].$$

Proof. $(i)\Delta_\alpha v(\kappa_1,\kappa_2)=e_1((\kappa_1))[\Delta_\alpha e_1((\kappa_2))]$

$$=e_1((\kappa_1))[e_1((\kappa_2+1))-\alpha e_1((\kappa_2))]$$

$$=e_1((\kappa_1))e_1((\kappa_2))[e_1(1)-\alpha].$$

In the similar way, the proof of (ii) follows.

Assume that $v(\kappa_1,\kappa_2)$ be the temperature of a rod at position κ_1 at time κ_2, ℓ_1 and ℓ_2 be shift values of κ_1 and κ_2 respectively and γ be the rate of conductivity of rod. When considering impact of external climate change on the rod, the partial α - difference equation of heat flow in the rod becomes fractional α- difference equation

$$\underset{\alpha}{\Delta^\nu} v(\kappa_1,\kappa_2)=\gamma[\underset{\alpha}{\Delta^\nu} v(\kappa_1,\kappa_2)+\underset{\alpha}{\Delta^\nu} v(\kappa_1,\kappa_2)]. \tag{28}$$

Theorem 6.3. If $\gamma=[e_1(1)-\alpha/e_1(\pm(1))-\alpha]$, then the function $v(\kappa_1,\kappa_2)=e_1((\kappa_1)).e_1((\kappa_2))$ is the exact solution of the α- difference equation (28).

Proof. By applying the Theorem 6, we get the proof.

Corollary 6.4. The fractional partial α-difference heat equation (28) has a solution of the form

$v(\kappa_1,\kappa_2)=e_1((\kappa_1)).e_1((\kappa_2))$ if $\gamma=[(e_1(1)-\alpha)^\nu/(e_1(\pm(1))-\alpha)^\nu]$.

Assume that $v(\kappa_1,\kappa_2,\kappa_3)$ be the temperature of a thin plate at position (κ_1,κ_2) at time κ_3. Let (1,1,1) be the shift values of (κ_1,κ_2) and κ_3 and γ be the rate of conductivity of thin plate. The fractional partial α-difference heat equation of thin plate is given by

$$\underset{\alpha}{\Delta^\nu} v(\kappa_1,\kappa_2,\kappa_3)=\gamma\left\{\underset{\alpha}{\Delta^\nu} v(\kappa_1,\kappa_2,\kappa_3)+\underset{\alpha}{\Delta^\nu} v(\kappa_1,\kappa_2,\kappa_3)\right\}. \tag{29}$$

Corollary 6.5. If $\gamma=[1+1-\alpha)^\nu/(e_1(\pm(1)_1)-\alpha)^\nu]$, then the function$v(\kappa)=\prod_{i=1}^{3} e_1(\kappa_{i_{(1_i)}})$ is an exact solution of the fractional partial heat equation (29).

Assume that $v(\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5)$ be the temperature of a medium at position $(\kappa_1,\kappa_2,\kappa_3)$ at time κ_4 and at density κ_5. Let (1,1,1,1,1) be the shift values of $(\kappa_1,\kappa_2,\kappa_3)$, κ_4 and κ_5 and γ be the rate of conductivity of medium. The fractional partial α-difference equation of heat flow in medium is

$$\underset{\alpha}{\Delta^\nu} v(\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5)=\gamma\left\{\underset{\alpha(\pm1)}{\Delta^\nu} v(\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5)\right\}. \tag{30}$$

Corollary 6.6. If $\gamma=[1+1-\alpha)^\nu+e_1(1+1-\alpha)^\nu/(e_1(\pm(1)_1)-\alpha)^\nu]$, then $v(\kappa)=\prod_{i=1}^{5} e_1(\kappa_{i_{(1_i)}})$ is a closed form solution of the fractional partial α-difference equation (30).

7. Conclusion

In conclusion, the resistor and inductor, as fundamental linear and passive circuit elements, form the basis of RL circuits, which can be configured in series or parallel. The mathematical analysis of RL circuits involves differential equations, with solutions often expressed in terms of extorial functions. By developing the theory of extorial functions, we successfully applied it to derive solutions for RL circuits and extended its utility to solving problems in wave motion, demonstrating its versatility and practical significance.

References

[1] Britto Antony Xavier.G, Gerly.T.G and Nasira Begum.H, Finite Series of Polynomials and Polynomial Factorials arising from Generalized q-Difference operator, Far East Journal of Mathematical Sciences,94(1)(2014), 47-63.

[2] Britto Antony Xavier.G, John Borg. S, Meganathan. M, *Discrete heat equation model with shift values*, Applied Mathematics, 2017, 8, 1343-1350.

[3] Weisstein, Eric. *Harmonic Series* MathWorld. Web. 24 August 2014.

[4] Jerzy Popenda and Blazej Szmanda, *On the Oscillation of Solutions of Certain Difference Equations*, Demonstratio Mathematica, XVII(1), (1984), 153 - 164.

[5] Maria Susai Manuel.M, Chandrasekar.V and Britto Antony Xavier.G, *Solutions and Applications of Certain Class of α-Difference Equations*, International Journal of Applied Mathematics, 24(6) (2011), 943-954.

[6] M.Maria Susai Manuel, G.Britto Antony Xavier and V.Chandrasekar, *Theory of Generalized Difference Operator of the n^{th} Kind and Its Application to Number Theory (Part - I),* International Journal of Pure and Applied Mathematics

[7] R.P Agarwal *Difference Equations and Inequalities*, Marceldekker, New York, 2000.

[8] M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thandapani, *Theory of Generalised Difference Operator and Its Applications*, Far East Journal of Mathematical Sciences, Vol.20,No.2(2006), pp 163 - 171.

[9] Ronald E.Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York, 1990.

[10] Ferreira.R. A. C and Torres.D. F. M, *Fractional h-difference equations arising from the calculus of variations, Applicable Analysis and Discrete Mathematics*, 5(1) (2011), 110-121.

[11] G.Britto Antony Xavier, S.U.Vasantha kumar, and B.Mohan, *Sum of two dimensional Fibonacci sequence by solutions of higher order difference equations, Ser.A:Appl.math.inform.and Mech.Vol.8,2 (2016), 131-138*.

[12] G. Britto Antony Xavier, S.John Borg, S.Jaraldpushparaj, *Extorial solutions for fractional and partial difference equations with application,AIP conference proceedings 2095, 030004(2019) https://doi.org/10.1063/1.5097515*.

[13] Britto Antony Xavier.G, M. Meganathan,*Fractional order alpha Laplace and extorial transform by inverse difference operator,AIP conference proceedings 2095, 030016(2019) https://doi.org/10.1063/1.5097527*.

[14] Sathinathan, T., Sherine, V.R. And Xavier, G.B.A., Fractional Order of Alpha-Delta and its Sums on Extorial Functions. *Advances in Mathematics: Scientific Journal* 9 (2020), no.8, 6097–6106.

[15] H. Byeon, M. Abisha, V. R. Sherine, G. B. A. Xavier, S. Prema, V. Govindan, H. Ahmad, D. Piriadarshani, S. El-Morsy, *Journal of Mathematics and Computer Science*, 34(4), 381-393.

[16] Sherine, V.R., Gerly, T. and Xavier, G.B.A., Infinite Series of Fractional Order of Fibonacci Delta Operator and Its Sum. *Advances in Mathematics: Scientific Journal* 9 (2020), no.8, 5891–5900.