Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 4s (2025)

Time-Fractional Hyperbolic Telegraph Equation: A Semi-Analytic Approach Using Modified Adomian Decomposition Elzaki Transform Method

Parmeshwari Aland [1,2], Prince Singh[3]

[1] Research Scholar, Department of Mathematics, Lovely Professional University, Phagwara, Punjab-144411, India; parmeshwarialand20@gmail.com

[2] Assistant Professor, School of Engineering, Ajeenkya DY Patil University, Lohegaon, Pune-412105, India; facultyit415@adypu.edu.in

[3] Department of Mathematics, School of Chemical Engineering and Physical Science., Lovely Professional University, Phagwara, Punjab-144411, India; princesingh16092@gmail.com

Article History:

Received: *25-09-2024*

Revised: *27-11-2024*

Accepted: *07-12-2024*

Abstract:

In this research paper, an approximate analytical solution approach known as the Modified Adomian Decomposition Method with the coupling of Elzaki Transform (MADETM) is deployed for addressing one-dimensional, two- dimensional, and three-dimensional time-fractional hyperbolic telegraph equations. The Caputo derivative operator yields the approximate analytical solution. The impact ness and its accuracy of the adopted method are demonstrated through comparison of the approximate results with the exact solutions, both presented graphically by plotting its surface graph, line graph through analyzing its error. The MADETM proves to be a reliable and efficient tool for deriving approximate and exact solutions for a large class of partial differential equations (PDEs), fractional PDEs, and ordinary differential equations (ODEs). The considered method yields a solution in series form with low computational complexity and swiftly converges towards precise solutions. The outcomes showcase an effective and uncomplicated approach for examining issues across diverse scientific and technological domains.

Keywords: Modified Adomian Decomposition technique, hyperbolic time fractional telegraph equations, Elzaki transform operator

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 4s (2025)

1. Introduction

Fractional order differential equations (FDEs) indeed have gained significance in applied mathematics, being applied in various systems within applied science. These equations provide a substitute approach to non-linear equations and have proven essential in mathematical modelling in fields such as mechanics, process control, complex systems, and technology. Integral equations work as crucial role in efficiently elaborate mathematical problems associated with FDEs and PDEs.

By selecting appropriate integral transformations, one can convert FDEs and PDEs into algebraic equations, simplifying the problem-solving process. Integral transforms offer a convenient method to address the complexity of various types of differential equations. The development and implementation of integral transforms, like Laplace transform, Elzaki transformation, Elzaki-Laplace transform, Shehu transform, and Sawi transform, Natural transforms have been instrumental in advancing research in this area [[1],[2],[3],[4],[5],[9]]

Moreover, research efforts over the past few decades have extensively explored the application of integral transformations to solve fractional and FDEs. These studies have involved various operators like Caputo, Atanga Baleanu, Erdelyi-Kober, He- polynomials and Grunwald-Letnikov operation, Riemann-Liouville types are leading to applications in diverse fields beyond mathematics [[11],[12],[13]]. Multiple integral equations, ODEs, PDEs, and fractional PDEs are solved using these transformations that are offered in the literature.

A range of analytical and numeration techniques will be utilized to solve hyperbolic time. fractional telegraph equations. These methods encompass the Homotopy Perturbation Transform [[14]] Technique, Sinc-Collocation Technique [[15]], Adomian Decomposition Technique [[16]], q-Homotopy Analysis Transform Technique [[17]], Reduction Differential Transform Technique [[18]], Reproducing Kernel Method [[19]] Variational Iteration Method [[20]], and Haar Wavelet technique [[21]]

The communication process is essential in the modern global community. Due to the extensive usage of radio frequency systems and microwave communication, technologies continue to get substantial industrial attention. Importantly, all transmission media experience the signal deficit, which must be measured for each medium. Telegraph equations are employed to analyses electrical signal propagation, random walks, wave propagation, and transmission line cables and similar phenomena. Heaviside introduced the concept of the transmission line, which will be divided into two types: guided and unguided. In guided media, signals are transmitted through physical systems such as copper wires, which carry voltage waves and higher frequency currents. Conversely, the unguided media use magnetic fields to transmit signals across communication channels, employing microwave communication and radio frequency systems, with antennas facilitating the broadcasting and reception of these electromagnetic waves. To increase the efficiency of telegraph transmission, cable transmission media are researched in controlled transmission environments. Direct information propagation between two or more sites is represented by a physical system using a directed transmission medium. Power and signal losses must be predicted or calculated because they are a necessary part of any system to enhance controlled communication [[22]].

In the last few years, fractional order partial differentiation equations (PDEs) gained significant attention due to their extensive application in a variety of technical and scientific domains, from scientists and researchers. The fractional derivative within these models offers a high degree of flexibility, offering a great tool for characterizing the inherited traits of various prototypes and their varying histories. Extensive research has been conducted to develop analytical, semi-analytical and numerical solutions for solving both nonlinear and linear fractional PDEs [[23]].

The time-fractional telegraph equations are deployed in this article with the use of MADETM. Hyperbolic time fractional telegraph equations are deployed in this study through demonstration of the Modified Adomian Decomposition with coupling Elzaki Transformation Method (MADETM). Certain fractional-order telegraph equation models are used to determine the MADETM solutions. The method demonstrates increased precision and efficiency, as evidenced by graphical comparisons with the exact solution. The EDM solutions for fractional-order telegraph equations exhibit a high rate of convergence. Consequently, this technique is promising for simplifying other fractional forms linearly and nonlinearly partial differentiation equations.

- **One-Dimensional fractional order telegraph equation:**

$$D_t^{2\alpha}\,[\emptyset(x\,,t)] + 2\alpha D_t^{\alpha}\,[\emptyset(x\,,t)] + \beta^2 \,=\emptyset_{xx}(x\,,t) + g(x\,,t) \qquad 0 <\propto\,, \qquad x \leq 1$$

With initial conditions: $\emptyset(x\,,0) = f_1(x), \quad \emptyset_t(x\,,0) = f_2(x)$

$$\emptyset(0,t) = f_1(t), \quad \emptyset_x(x,t) = f_2(t)$$

where $\beta \rightarrow$ arbitrary constants and $\emptyset(x,t)$ is an unknown function.

- **Two-Dimensional fractional order telegraph equation**:

$$D_t^{2\alpha}\,[\emptyset(x\,,y\,,t)] + 2\alpha D_t^{\alpha}\,[\emptyset(x\,,y\,,t)] + \beta^2\,[\emptyset(x\,,y\,,t)] = \emptyset_{xx}(x\,,y\,,t) + \emptyset_{yy}(x\,,y\,,t) + g(x\,,y\,,t)$$

$$0 <\propto\leq 1\,, \qquad x = 1$$

With initial conditions: $\emptyset(x,y,0) = g_1(x,y), \quad \emptyset_t(x,y,0) = g_2(x,y)$

- **Three-dimensional fractional order telegraph equation:**

$$D_t^{2\alpha}\ [\emptyset(x.,y,z,t)] + 2\alpha D_t^{\alpha}\ [\emptyset(x,y,z,t)] + \beta^2\ [\emptyset(x.,y,z,t)] = \emptyset_{xx}(x,y,z,t) + \emptyset_{yy}(x,y,z,t) + \emptyset_{zz}(x,y,z,t) + g(x,y,z,t)$$

$$0 < \propto \leq 1\ , \qquad x = 1$$

With initial conditions: $\emptyset(x.,y,z,t) = h_1(x,y,z),\quad \emptyset_t(x.,y,z,t) = h_2(x,y,z)$

The hyperbolic telegraph equation is broadly applied in the signal processing for transmitting wave theory and electric impulses. It has found various applications in biomedical sciences and aerospace. Researchers are particularly interested in solving problems involving fractional derivatives. Fractional-order partial differential equations (PDEs) are essentially a type of integer-order PDEs. Fractional-order methods yield results that converge to those of integer-order methods.

2. Preliminaries and Notations

Elzaki transformations, also called Elzaki integral transformations, are algebraic equation transformations that are used to solve differential equations in ordinary form (ODEs). Elzaki Ali Elzaki, a mathematician from Sudan, introduced it in the 1960s. Heat conduction, fluid dynamical mechanics, and electrical circuits are only a few of the applied scientific and engineering domains where the Elzaki transformation has been effectively used. It offers an alternate strategy for resolving ODEs, especially in situations when existing methods are not easily able to produce analytical solutions. When analytical solutions are hard to come by with other approaches, applying the Elzaki transform to solve FPDEs can be quite helpful. This section presents a basic explanation of Elzaki transformation and offers a framework for transforming the problem into an algebraic form that can be solved using well-established algebraic techniques.

Definition 1: Fundamental Principle of Elzaki Transformation

The exponential form of function in A series, defined by set A expressed as new transformation which renamed as an Elzaki Transformation [[24]]

$$A = \left\{ f(t) : \exists\ M\ , k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}},\ if\ t \in (-1)^j\ X\ [\ 0, \infty) \right\}$$

Regarding the specified function in the set, M is defined as finite or infinite number, as $k_1\ and\ k_2$.

The Elzaki Transform defined as operator $E(\text{g}(\tau))$ in the integral form as follows.

$$E[\ f(\tau)] = v \int_0^{\infty} f(\tau)^{-\ \frac{\tau}{v}}\ d\tau = T\ (v) \quad , \tau \geq 0\ , \quad k_1 \leq v \leq\ k_2$$

The Elzaki Transformation for some functions is defined below [[24]].

	$f(t)$	$E[f(t)] = T(v)$
1	1	v^2
2	t	v^3
3	t^n	$n!\ v^{n+2}$

The following conclusion was established based on the description and fundamental analyses.

$$E[t^n] = n!\, v^{n+2}$$

$$E[f'(t)] = \frac{F(v)}{v} - vf(0)$$

$$E[f''(t)] = \frac{F(\gamma)}{v^2} - f(0) - vf'(0)$$

$$E\left[f^{(n)}(t)\right] = \frac{F(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0)$$

Definition 2: Caputo Fractional Elzaki Transform Operator

The Caputo Fractional operator's Elzaki Transformation is as follows:

$$E\left[\frac{\partial^\alpha}{\partial\tau^\alpha} f(\tau)\right] = \frac{E[f(\tau)]}{v^\alpha} - \sum_{k=0}^{n-1} v^{k-\alpha+2} f^{(k)}(0), \quad n-1 < \alpha \le n \tag{1}$$

3. The methodology for Modified Adomian Decomposition Elzaki Transform (MADETM):

Consider the partial differentiation equation of fractional order non-linearity.

$$D_t^\alpha \emptyset(x,t) + R[\emptyset(x,t)] + N[\emptyset(x,t)] = g(x,t) \qquad x,t \ge 0 \qquad m-1 < \alpha < m \tag{2}$$

With initial condition $\emptyset(x,0) = f(x)$ **(3)**

the Caputo fractional function $\emptyset(x,t)$ defined as:

$$D_t^\alpha \emptyset(x,t) = \frac{\partial^\alpha \emptyset(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-x)^{n-\alpha-1} \frac{\partial^n \emptyset(x,t)}{\partial t^n}\, dt & , n-1 < \alpha < n \\ \dfrac{\partial^n \emptyset(x,t)}{\partial t^n} & \alpha = n \in N \end{cases}$$

Where $D_t^\alpha \emptyset(x,t)$ is Caputo fractional order derivative α, N and are R nonlinear and linear terms respectively, and g is source term.

Taking the Elzaki Transform on both sides of Equation (2)

$$\mathrm{E}[D_t^\alpha \emptyset(x.,t)] + E[R[\emptyset(x.,t)] + N[\emptyset(x,t)]] = E[g(x.,t)] \tag{4}$$

$$\frac{1}{v^\alpha} E[\emptyset(x,t)] - v^{2-\alpha} \emptyset(x,0) = E[g(x,t)] - E\left[R[\emptyset(x,t)] + N[\emptyset(x,t)]\right]$$

$$E[\emptyset(x,t)] = v^2 \emptyset(x,0) + v^\alpha E[g(x,t)] - v^\alpha E\left[R[\emptyset(x,t)] + N[\emptyset(x,t)]\right] \tag{5}$$

From equation (3) its Initial Conditions is: $\emptyset(x,0) = f(x)$

$$E[\emptyset(x,t)] = v^2 f(x) + v^\alpha E[g(x,t)] - v^\alpha E\left[R[\emptyset(x,t)] + N[\emptyset(x,t)]\right]$$

Applying inverse Elzaki Transform on Equation (5)

$$E^{-1}\left[E\ [\emptyset\ (x\ ,t)]\right] = E^{-1}\left[v^{2}\ f(x) + v^{\alpha}\ E[g(x\ ,t)] - \ v^{\alpha}E\left[R\ [\emptyset(x\ ,t)] + N[\emptyset(x\ ,t)]\right]\right]$$

$$\emptyset\ (x\ ,t) = E^{-1}\ [v^{2}\ f(x)] + E^{-1}\left[v^{\alpha}\ E\ [g(x\ ,t)]\right] - E^{-1}\left[v^{\alpha}E\left[R\ [\emptyset(x,t)] + N[\emptyset(x,t)]\right]\right] \quad \textbf{(6)}$$

By applying MADM on right hand side of equation occurs the solution in infinite series given below.

$$\emptyset(x,t) = \sum_{n=0}^{\infty} \emptyset_n(x,t) \quad \textbf{(7)}$$

The non-linear terms N in an Adomian polynomial represented as follows.

$$N\ [\emptyset(x,t)] = \sum_{n=0}^{\infty} A_n$$

Where $A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\ \sum_{i=0}^{\infty}\lambda^i\emptyset_i\right]\right]_{\lambda=0}$; $i = 0,1,2,3, \ldots ....$ **(8)**

The nonlinear terms denoted by N are explained with adapted modified Adomian decomposition technique for

handling nonlinear polynomial system solution. following the utilization of the Elzaki transformation as specified below:

$$\{A_n\} = \{\ N_1(\ s_n) - \ N_1(\ s_{n-1})\} \quad \textbf{(9)}$$

Equation (6) is obtained by substituting Equations (7) and (8)

$$\sum_{n=0}^{\infty}\emptyset_n(x,t) = E^{-1}[v^{2}\ f(x)] + E^{-1}\left[v^{\alpha}\ E\ [g(x,t)]\right] - E^{-1}\left[v^{\alpha}E\left[R[\sum_{n=0}^{\infty}\emptyset_n(x,t)] + [\sum_{n=0}^{\infty}A_n]\right]\right]$$

Since, $E^{-1}(v^{2}\ ) = 1$

$$\sum_{n=0}^{\infty}\emptyset_n(x,t) = f(x) + E^{-1}\left[v^{\alpha}\ E[g(x,t)]\right] - E^{-1}\left[v^{\alpha}E\ \left[R\left[\sum_{n=0}^{\infty}\emptyset_n(x,t)\right] + \left[\sum_{n=0}^{\infty}A_n\right]\right]\right]$$

Analysing both sides of the Equation (9)

$$\emptyset_0(x,t) = f(x) + E^{-1}\left[v^{\alpha}\ E[g(x\ ,t)]\right]$$

$$\emptyset_1(x,t) = -E^{-1}\left[v^{\alpha}E\ [R[\emptyset_0(x,t)] + A_0]\right]$$

$$\emptyset_2(x,t) = -E^{-1}\left[v^{\alpha}E\ [R[\emptyset_1(x,t)] + A_1]\right]$$

.

:

:

$$\emptyset_{n+1}(x,t) = -E^{-1}\left[v^{\alpha}E\ [R[\emptyset_{\mathrm{n}}(x,t)] + A_{\mathrm{n}}]\right] \quad \textbf{(10)}$$

The analytic solution $\emptyset(x,t)$ is finally approximated using truncated series:

$$\emptyset(x,t) = \sum_{n=0}^{\infty} \emptyset_n(x\,,t) = \emptyset_0(x\,,t) + \emptyset_1(x,t) + \emptyset_2(x,t) + \emptyset_3(x\,,t) + \emptyset_4(x,t) + \dots \quad (11)$$

The following different forms of nonlinear one-dimensional, two-dimensional and three-dimensional fraction form telegraph equations are demonstrated with adopted technique for validation of results in the following applications.

4. Application

One-dimensional non-linear telegraph equation:

Example 1. consider one-dimensional nonlinear telegraph equation [330]:

$$D_t^{\alpha}\emptyset(x,t) = \emptyset_{xx}(x\,,t) + \emptyset_t(x\,,t) - \emptyset^2 + x\emptyset\emptyset_x(x\,,t) \text{where} \quad 0 < \propto \leq 2 \quad (12)$$

Initial conditions: $\emptyset(x\,,0) = x, \quad \emptyset_t(x,0) = x$ **(13)**

$$\emptyset_0(x\,,t) = \emptyset(x\,,0) + t\,\emptyset_t(x,0) = x + xt = x(1+t)$$

Apply the Elzaki transformation on Equation (12),

$$E\,[\emptyset\,(x\,,t)] = v^2\,\emptyset(x,0) + v^{\alpha}\left[E[\emptyset_{xx} + \emptyset_t - \emptyset^2 + x\,\emptyset\emptyset_x]\right]$$

Applying inverse Elzaki Transform on above equation

$$\emptyset\,(x,t) = E^{-1}[v^2\,\emptyset(x,0)] + E^{-1}\left[v^{\alpha}\,E\,[\emptyset_{xx} + \emptyset_t - \emptyset^2 + x\,\emptyset\emptyset_x]\right]$$

$$\emptyset\,(x,t) = \emptyset(x,0) + E^{-1}\left[v^{\alpha}\,E\,[R[\emptyset] + N[\emptyset]]\right] \quad \textbf{(14)}$$

Here, $E^{-1}(v^2\,) = 1$; $R[\emptyset] = (\emptyset_{xx} + \emptyset_t)$ and $N[\emptyset] = (x\,\emptyset\emptyset_x - \emptyset^2)$

Appling the MADETM process on equation (14)

$$\emptyset_0(x,t) = \emptyset(x,0) = x(1+t) \quad \textbf{(15)}$$

Appling the recursive series as shown in equation (10),

$$\emptyset_{n+1}(x,t) = E^{-1}\left[v^{\alpha}E\,[R(\,\emptyset_n) + N(\,\emptyset_n)]\right]$$

For $n = 0$

$$\emptyset_1(x,t) = E^{-1}\left[v^{\alpha}E\,[R(\,\emptyset_0) + N[\,\emptyset_0]]\right]$$

Here, $R\,(\,\emptyset_0) = \emptyset_{0xx} + \emptyset_{0t} = x$

$N\,(\,\emptyset_0) = x\,\emptyset_0\emptyset_{0x} - {\emptyset_0}^2 = 0$

Therefore, above equation implies, $\emptyset_1(x,t) = E^{-1}\left[v^{\alpha}E\,[R(\,\emptyset_0) + N[\,\emptyset_0]]\right]$

$$\emptyset_1(x,t) = E^{-1}\left[v^{\alpha}E\,[x+0]\right] = 4e^xE^{-1}\left[v^{\alpha}E\,[x]\right] = xE^{-1}[v^{\alpha}E(1)]$$

$$\emptyset_1(x,t) = x\,E^{-1}(v^{\alpha+2}) = x\,\frac{t^{\alpha}}{\lceil(\alpha+1)}$$

$\emptyset_1(x,t) = x\ \frac{t^{\alpha}}{\lceil(\alpha+1)}$

For $n = 2$

$$\emptyset_2(x,t) = E^{-1}\left[v^{\alpha}E\ [R(\emptyset_1) + N[\emptyset_1]]\right]$$

Here, $R\ (\emptyset_1) = \emptyset_{1xx} + \emptyset_{1t} = x\ \frac{\alpha}{\lceil(\alpha+1)}\ \ t^{\alpha-1}$

$$N\ (\emptyset_1) = x\ \emptyset_0\emptyset_{0x} - {\emptyset_0}^2 = 0$$

$$\emptyset_2(x,t) = E^{-1}\left[v^{\alpha}E\ [R(\emptyset_1) + N[\emptyset_1]]\right] = E^{-1}\left[v^{\alpha}E\ \left[x\ \frac{\alpha}{\lceil(\alpha+1)}\ \ t^{\alpha-1}\right]\right]$$

$$\emptyset_2(x,t) = x\ \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{t^{2\alpha-1}}{\lceil(2\alpha)}$$

Considering $n = 3, 4 \ldots..$

$$\emptyset_3(x,t) = E^{-1}\left[v^{\alpha}E\ [R(\emptyset_2) + N[\emptyset_2]]\right] = x\ \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{(2\alpha-1)\ \lceil(2\alpha-1)}{\lceil(2\alpha)}\frac{t^{3\alpha-1}}{\lceil(3\alpha-1)}$$

$$\emptyset_4(x,t) = E^{-1}\left[v^{\alpha}E\ [R(\emptyset_3) + N[\emptyset_3]]\right] = x\ \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{(2\alpha-1)\ \lceil(2\alpha-1)}{\lceil(2\alpha)}\frac{(3\alpha-1)\ \lceil(3\alpha-1)}{\lceil(3\alpha)}\frac{t^{4\alpha-1}}{\lceil(4\alpha-1)} \quad \textbf{(16)}$$

Therefore, Series representation of the solution $\emptyset\ (x,t)$ is as follows:

$$\emptyset\ (x,t) = \emptyset_0(x,t) + \emptyset_1(x,t) + \emptyset_2(x,t) + \emptyset_3(x,t) + \emptyset_4(x,t) + \cdots \ldots \ldots$$

$$\emptyset\ (x\ ,t) = x(1+t) + x\ \frac{t^{\alpha}}{\lceil(\alpha+1)} + x\ \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{t^{2\alpha-1}}{\lceil(2\alpha)} + x\ \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{(2\alpha-1)\ \lceil(2\alpha-1)}{\lceil(2\alpha)}\frac{t^{3\alpha-1}}{\lceil(3\alpha-1)} + x\ \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{(2\alpha-1)\ \lceil(2\alpha-1)}{\lceil(2\alpha)}\frac{(3\alpha-1)\ \lceil(3\alpha-1)}{\lceil(3\alpha)}\frac{t^{4\alpha-1}}{\lceil(4\alpha-1)} + \cdots \ldots ....$$

$$\emptyset\ (x,t) = x\left[(1+t) + \frac{t^{\alpha}}{\lceil(\alpha+1)} + \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{t^{2\alpha-1}}{\lceil(2\alpha)} + \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{(2\alpha-1)\ \lceil(2\alpha-1)}{\lceil(2\alpha)}\frac{t^{3\alpha-1}}{\lceil(3\alpha-1)} + \frac{\alpha\ \lceil\alpha}{\lceil(\alpha+1)}\frac{(2\alpha-1)\ \lceil(2\alpha-1)}{\lceil(2\alpha)}\frac{(3\alpha-1)\ \lceil(3\alpha-1)}{\lceil(3\alpha)}\frac{t^{4\alpha-1}}{\lceil(4\alpha-1)} \ldots \ldots \ldots.\right]$$

In particular when $\alpha = 2$, the solution is of the form:

$$\emptyset\ (x,t) = x\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \ldots..\right] \quad \textbf{(17)}$$

The exact solution for equation (12) is: $\emptyset\ (x,t) = xe^t$ **(18)**

Example 2. Consider the following one-dimensional nonlinear telegraph equation [25]

$$D_t^{\alpha}\emptyset(x,t) = \emptyset_x(\emptyset^2(x,t).\emptyset_x(x\ ,t)) \quad (19)$$

Initial condition $\emptyset(x\ ,0) = \frac{x+b}{2c}$; where $c > 0, and\ \ b$ is arbitrary constant. (20)

Apply the Elzaki transformation on Equation (19)

$$E[\ D_t^{\alpha}\emptyset(x,t)] = E[2\emptyset(x,t)\ \emptyset_x^2(x,t) + \emptyset^2(x,t)\emptyset_{xx}(x\ ,t)]$$

$$E\ [\emptyset\ (x\ ,t)] = v^2\ \emptyset(x,0) + v^\alpha \left[E[2\emptyset.\ \emptyset_x^2 + \emptyset^2.\emptyset_{xx}]\right] \quad \textbf{(21)}$$

Applying inverse Elzaki Transform on equation (21)

$$\emptyset\ (x,t) = E^{-1}[v^2\ \emptyset(x,0)] + E^{-1}\left[v^\alpha\ E\ [2\emptyset.\emptyset_x^2 + \emptyset^2.\emptyset_{xx}]\right]$$

$$\emptyset\ (x,t) = \emptyset(x,0) + E^{-1}\left[v^\alpha\ E\ [N_1(\emptyset) + N_2(\emptyset)]\right] \quad \textbf{(22)}$$

Here, $E^{-1}(v^2\ ) = 1$; $N_1(\emptyset) = (2\emptyset.\emptyset_x^2)$ and $N_2(\emptyset) = (\emptyset^2.\emptyset_{xx})$

Applying the MADETM process on equation (22)

$$\emptyset_0(x,t) = \emptyset(x,0) = \frac{x+b}{2c} \quad (23)$$

Applying the recursive series as shown in equation (10),

$$\emptyset_{n+1}(x,t) = E^{-1}\left[v^\alpha E\ [N_1(\ \emptyset_n) + N_2\ (\emptyset_n)]\right]$$

For $n = 0$

$$\emptyset_1(x,t) = E^{-1}\left[v^\alpha E\ [N_1(\ \emptyset_0) + N_2(\ \emptyset_0)]\right]$$

$$\emptyset_1(x,t) = E^{-1}\left[v^\alpha E\ \left[2\emptyset_0.\emptyset_{0x}^2 + \emptyset_0^2.\emptyset_{0xx}\right]\right]$$

$$\emptyset_1(x,t) = \frac{x+b}{4c^3}\ E^{-1}(v^{\alpha+2}) = \frac{x+b}{4c^3}\ \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$\emptyset_1(x,t) = \frac{x+b}{4c^3}\ \frac{t^\alpha}{\Gamma(\alpha+1)}$$

For $n = 2$

$$\emptyset_2(x,t) = E^{-1}\left[v^\alpha E\ [N_1(\ \emptyset_1) + N_2(\ \emptyset_1)]\right]$$

$$\emptyset_2(x,t) = E^{-1}\left[v^\alpha E\ \left[2\emptyset_1.\emptyset_{1x}^2 + \emptyset_1^2.\emptyset_{1xx}\right]\right] = E^{-1}\left[v^\alpha E\ \left[\frac{(x+b)}{4c^5}\ \frac{3\ t^\alpha}{2\Gamma(\alpha+1)}\right]\right]$$

$$\emptyset_2(x,t) = \frac{3(x+b)}{8c^5}\ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Considering $n = 3, 4 \ldots..$

$$\emptyset_3(x,t) = E^{-1}\left[v^\alpha E\ [N_1(\ \emptyset_2) + N_2[\ \emptyset_2]]\right] = \frac{4\ (x+b)}{16\ c^5}\ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \quad \textbf{(24)}$$

.

:

Therefore, Series representation of the solution $\emptyset\ (x,t)$ is as follows:

$$\emptyset\ (x,t) = \emptyset_0(\ x,t) + \emptyset_1(\ x,t) + \emptyset_2(\ x,t) + \emptyset_3(\ x,t) + \emptyset_4(\ x,t) + \cdots \ldots \ldots$$

$$\emptyset\,(x\,,t) = \frac{x+b}{2c} + \frac{(x+b)}{4c^3}\,\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{3(x+b)}{8c^5}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4\,(x+b)}{16\,c^5}\,\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \ldots \ldots$$

In particular when $\alpha = 1$, the solution is of the form:

$$\emptyset\,(x,t) = \left[\frac{x+b}{2c} + \frac{(x+b)\,t}{4c^3} + \frac{3(x+b)\,t^2}{16\,c^5} + \frac{4\,(x+b)\;t^3}{64\,c^7} + \cdots \ldots\right] \quad \textbf{(25)}$$

The exact solution for equation (19) is: $\emptyset\,(x,t) = \frac{x+b}{2\sqrt{c^2-t}}, t < c^2$ **(26)**

Example 3. Illustrate the following fractional-order one dimensional telegraph equation [23]

$$D_t^{2\alpha}\emptyset(x,t) + 2\,D_t^{\alpha}\emptyset(x,t) + \emptyset(x,t) = \emptyset_{xx}(x,t) \quad 0 < \alpha \le 1\,,\; x = 1 \quad \textbf{(27)}$$

With initial conditions: $\emptyset(x,0) = e^x, \quad \emptyset_t(x,0) = -2e^x$ **(28)**

Using the Elzaki transformation of Equation (27),

$$E\left[\,D_t^{2\alpha}\emptyset + 2\,D_t^{\alpha}\emptyset + \emptyset\right] = E[\emptyset_{xx}]$$

Applying Elzaki Transform on above equation we get,

$$\frac{1}{v^\alpha}\,E[\emptyset(x,t)] - v^{2-\alpha}\,\emptyset(x,0) - v^{3-\alpha}\,\emptyset_t(x,0) = -E[(\emptyset - \emptyset_{xx}) - 2\,D_t^{\alpha}\emptyset]$$

$$E[\emptyset(x,t)] = v^2\,\emptyset(x,0) + v^3\,\emptyset_t(x,0) - v^\alpha\,E[(\emptyset - \emptyset_{xx}) - 2\,D_t^{\alpha}\emptyset]$$

Applying inverse Elzaki Transform on above equation

$$E^{-1}\left[E\,[\emptyset(x,t)]\right] = E^{-1}\left[v^2\,\emptyset(x,0) + v^3\,\emptyset_t(x,0) - v^\alpha\,E[(\emptyset - \emptyset_{xx}) - 2\,D_t^{\alpha}\emptyset]\right]$$

$$\emptyset(x,t) = E^{-1}\left[v^2\,\emptyset(x,0) + v^3\,\emptyset_t(x,0) - v^\alpha\,E[L(\emptyset) - 2\,D_t^{\alpha}\emptyset]\right]$$

Appling the MADETM process on above equation

$$\emptyset_0(x,t) = \emptyset(0) = e^x(1-2t) \quad (29)$$

Using the recursive series as shown in equation (10),

$$\emptyset_{n+1}(x,t) = E^{-1}\left[v^\alpha E\,[L(\,\emptyset_n) - 2\,D_t^{\alpha}\,\emptyset_n]\right] \quad \textbf{(30)}$$

For $n = 0$

$$\emptyset_1(x,t) = E^{-1}\left[v^\alpha E\,[L(\,\emptyset_0) - 2\,D_t^{\alpha}\,\emptyset_0]\right]$$

Here, $L\,[\emptyset_0] = \emptyset_{0xx} - \emptyset_0 = 0$

Therefore, above equation implies, $\emptyset_1(x,t) = E^{-1}\left[v^\alpha E\,[-2\,D_t^{\alpha}\emptyset_0]\right]$

Consider, $E\,[-2\,D_t^{\alpha}\emptyset_0] = -2\left[\frac{1}{v^\alpha}\,E[\emptyset_0] - v^{2-\alpha}\,\emptyset_0(0)\right]$

$$E\,[-2\,D_t^{\alpha}\emptyset_0] = -2\left[\left[\frac{1}{v^\alpha}\right]\,E[e^x(1-2t)] - v^2\,e^x\right]$$

$$E\,[-2\,D_t^{\alpha}\emptyset_0] = -2e^x\left[\left[\frac{1}{v^\alpha}\right]\,[E(1) - 2E(t)] - v^2\right] = -2e^x\left[\left[\frac{1}{v^\alpha}\right]\,[v^2 - 2v^3\,] - v^2\right]$$

$$E\,[-2\,D_t^{\alpha}\emptyset_0] = -2e^{x}\left[\left[\frac{1}{v^{\alpha}}\right]\,[-2v^{3}\,]\right] = 4e^{x}[\,v^{3-\alpha}\,]$$

Therefore, $\emptyset_1(x,t) = E^{-1}\left[v^{\alpha}\,[4e^{x}(v^{3-\alpha}\,)]\right] = 4e^{x}E^{-1}[\,(v^{3+\alpha}\,)]$

$$\emptyset_1(x,t) = 4e^{x}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

For $n = 1,2,3\,....$

$$\emptyset_2(x,t) = E^{-1}\left[v^{\alpha}E\,[-2\,D_t^{\alpha}\emptyset_1]\right] = -8e^{x}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$\emptyset_3(x,t) = E^{-1}\left[v^{\alpha}E\,[-2\,D_t^{\alpha}\emptyset_2]\right] = 16e^{x}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\emptyset_4(x,t) = E^{-1}\left[v^{\alpha}E\,[-2\,D_t^{\alpha}\emptyset_3]\right] = -32e^{x}\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \qquad \textbf{(31)}$$

.

:

:

Therefore, Series representation of the solution $\emptyset\,(x,t)$ is as follows:

$$\emptyset\,(x,t) = \emptyset_0(x,t)+\emptyset_1(x,t)+\emptyset_2(x,t) + \emptyset_3(x,t) + \emptyset_4(x,t) + \cdots \ldots \ldots$$

$$\emptyset\,(x,t) = e^{x}(1-2t) + 4e^{x}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 8e^{x}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 16e^{x}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 32e^{x}\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \ldots \ldots ....$$

$$\emptyset\,(x,t) = e^{x}\left[(1-2t) + 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 8\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 16\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 32\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \ldots \ldots \ldots.\right]$$

In particular when $\alpha = 1$, the solution is in the form:

$$\emptyset\,(x,t) = e^{x}\left[1 - \frac{2t}{1!} + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \frac{(2t)^5}{5!} \ldots..\right] \qquad \textbf{(32)}$$

The exact solution for equation (27) is: $\emptyset\,(x,t) = e^{x-2t}$ **(33)**

Two-dimensional fractional telegraph equation:

Example 4. Considering the two-dimensional fractional telegraph equation as Follows [23]:

$$D_t^{2\alpha}\emptyset + 3\,D_t^{\alpha}\emptyset + 2\emptyset = \emptyset_{xx} + \emptyset_{yy} \qquad 0 < \alpha \leq 1 \qquad \textbf{(34)}$$

Initial conditions: $\emptyset(x\,,y,0) = e^{\,x+y}$, $\quad \emptyset_t(x,y,0) = -3e^{x+y}$ (35)

Apply the Elzaki transformation of Equation (34),

$$E\left[\,D_t^{2\alpha}\emptyset + 3\,D_t^{\alpha}\emptyset + 2\emptyset\right] = E\left[\emptyset_{\,xx} + \emptyset_{\,yy}\right]$$

$$E\,\left[\,D_t^{2\alpha}\emptyset\right] = -E\left[\emptyset_{xx} + \emptyset_{yy} - 3\,D_t^{\alpha}\emptyset - 2\emptyset\right]$$

$$\frac{1}{v^{\alpha}}\ E[\emptyset(x,\mathrm{y},t)] - v^{2-\alpha}\ \emptyset(x\ ,y\ ,0) - v^{3-\alpha}\ \emptyset_t(x,y,0) = -E\left[(\emptyset_{xx} + \emptyset_{yy} - 2\emptyset) - 3\ D_t^{\alpha}\emptyset\right]$$

$$E[\emptyset(x,\mathrm{y},t)] = v^2\ \emptyset(x,y,0) + v^3\ \emptyset_t(x,y,0) - v^{\alpha}\ E\left[(\emptyset_{xx} + \emptyset_{yy} - 2\emptyset) - 3\ D_t^{\alpha}\emptyset\right]$$

Applying inverse Elzaki Transform.

$$E^{-1}\left[E\ [\emptyset(x\ ,\mathrm{y},t)]\right] = E^{-1}\left[v^2\ \emptyset(x,y,0) + v^3\ \emptyset_t(x,y,0) - v^{\alpha}\ E\left[(\emptyset_{xx} + \emptyset_{yy} - 2\emptyset) - 3\ D_t^{\alpha}\emptyset\right]\right]$$

$$\emptyset(x,\mathrm{y},t) = E^{-1}\left[v^2\ \emptyset(x,y\ ,0) + v^3\ \emptyset_t(x\ ,y,0) - v^{\alpha}\ E\left[(\emptyset_{xx} + \emptyset_{yy} - 2\emptyset) - 3\ D_t^{\alpha}\emptyset\right]\right]$$

$$\emptyset(x,\mathrm{y},t) = E^{-1}\left[v^2\ \emptyset(x,y,0) + v^3\ \emptyset_t(x,y,0) - v^{\alpha}\ E[L(\emptyset) - 3\ D_t^{\alpha}\emptyset]\right] \quad \textbf{(36)}$$

Appling the MADETM process on equation (36)

$$\emptyset_0(x,\mathrm{y},t) = E^{-1}[v^2\ \emptyset(x\ ,y,0\ ) + v^3\ \emptyset_t(x,y,0\ )]$$

$$\emptyset_0\ (x,\mathrm{y},t) = E^{-1}[v^2\ e^{x+y} + v^3\ (-3e^{x+y})]$$

$$\emptyset_0(x,\mathrm{y},t) = E^{-1}[v^2\ e^{x+y} - v^3\ (3e^{x+y})]$$

$$\emptyset_0(x,\mathrm{y},t) = \emptyset(0) = e^{x+y}(1-3t) \quad \textbf{(37)}$$

$$\emptyset_{\ n+1}(x,y,t) = E^{-1}\left[v^{\alpha}E\ [L(\ \emptyset_n) - 3\ D_t^{\alpha}\ \emptyset_n]\right] \quad \textbf{(38)}$$

For $n = 0$

$$\emptyset_1\ (x,y,t) = E^{-1}\left[v^{\alpha}E\ [L(\ \emptyset_0) - 3\ D_t^{\alpha}\ \emptyset_0]\right]$$

Here, $L\ [\emptyset_0] = (\emptyset_{0xx} + \emptyset_{0yy} - 2\emptyset_0) = 0$

Therefore, above equation implies, $\emptyset_1(x,y,t) = E^{-1}\left[v^{\alpha}\ E\ [-3\ D_t^{\alpha}\ \emptyset_0]\right]$

$$\emptyset_1(x,y,t) = -E^{-1}\left[v^{\alpha}E\ [-3\ D_t^{\alpha}\emptyset_0]\right] = 9e^{x+y}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

For $n = 1,2,3$ ....

$$\emptyset_2(x,y,t) = -E^{-1}\left[v^{\alpha}E\ [-3\ D_t^{\alpha}\emptyset_1]\right] = -27e^{x+y}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$\emptyset_3(x,y,t) = -E^{-1}\left[v^{\alpha}E\ [-3\ D_t^{\alpha}\emptyset_2]\right] = 81\ e^{x+y}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\emptyset_4(x,y,t) = -E^{-1}\left[v^{\alpha}E\ [-3\ D_t^{\alpha}\emptyset_3]\right] = -243\ e^{x+y}\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \quad \textbf{(39)}$$

.

:

:

Therefore, Series form $\emptyset\ (x\ ,y\ ,t)$ will be:

$$\emptyset\ (x,y\ ,t) = \ \emptyset_0(x\ ,y,t)+\emptyset_1(x,y\ ,t)+\emptyset_2(x,y,t) +\ \emptyset_3(x,y,t) +\ \emptyset_4(x,y,t) + \cdots \dots \dots$$

$$\emptyset\,(x,y,t) = e^{x+y}(1-3t) + 9e^{x+y}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 27e^{x+y}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 81\,e^{x+y}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 243\,e^{x+y}\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\dots\dots\dots$$

$$\emptyset\,(x,y,t) = e^{x+y}\left[(1-3t) + 9\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 27\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 81\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 243\,\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\dots\dots\dots.\right]$$

When $\alpha = 1$, the approximate solution will be in the form:

$$\emptyset\;(x,y,t) = e^{x+y}\left[1 - \frac{3t}{1!} + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \frac{(3t)^4}{4!} - \frac{(3t)^5}{5!} + \cdots..\right] \quad \mathbf{(40)}$$

The exact solution for equation (34) implies $\emptyset\,(x,y,t) = e^{x+y-3t}$ (41)

Three-dimensional fractional telegraph equation:

Example 5. The 3D telegraph equation of fractional order is to be considered [23]

$$D_t^{2\alpha}\emptyset + 2\,D_t^{\alpha}\emptyset + 3\emptyset = \emptyset_{xx} + \emptyset_{yy} + \emptyset_{zz} \qquad 0 < \alpha \le 1 \quad \mathbf{(42)}$$

With initial conditions: $\emptyset(x,y,z,0) = \sinh x\;\sinh y \sinh z$

$$\emptyset_t(x,y,z,0) = -\sinh x\;\sinh y \sinh z \quad \mathbf{(43)}$$

Apply the Elzaki transformation of Equation (42),

$$E\left[\,D_t^{2\alpha}\emptyset + 2\,D_t^{\alpha}\emptyset + 3\emptyset\right] = E\left[\emptyset_{\,xx} + \emptyset_{\,yy} + \emptyset_{\,zz}\right]$$

$$E\left[\,D_t^{2\alpha}\emptyset\right] = -E\left[\emptyset_{xx} + \emptyset_{yy} + \emptyset_{zz} - 2\,D_t^{\alpha}\emptyset - 3\emptyset\right]$$

$$\frac{1}{v^{\alpha}}\,E[\emptyset(x,\text{y},\text{z},t)] - v^{2-\alpha}\,\emptyset(x\,,y,z,0) - v^{3-\alpha}\,\emptyset_t(x,y\,,z,0) = -E\left[(\emptyset_{xx} + \emptyset_{yy} + \emptyset_{zz} - 3\emptyset) - 2\,D_t^{\alpha}\emptyset\right]$$

$$E[\emptyset(x,\text{y},\text{z},t)] = v^2\,\emptyset(x,y\,,z\,,0) + v^3\,\emptyset_t(x,y,z,0) - v^{\alpha}\,E\left[(\emptyset_{xx} + \emptyset_{yy} + \emptyset_{zz} - 3\emptyset) - 2\,D_t^{\alpha}\emptyset\right]$$

Applying inverse Elzaki Transform

$$E^{-1}\left[E\,[\emptyset(x,\text{y},\text{z},t)]\right] = E^{-1}\left[v^2\,\emptyset(x,y,z,0) + v^3\,\emptyset_t(x,y,z,0) - v^{\alpha}\,E(\emptyset_{xx} + \emptyset_{yy} + \emptyset_{zz} - 3\emptyset) - 2\,D_t^{\alpha}\emptyset\right]$$

$$\emptyset(x,\text{y},\text{z},t) = E^{-1}\left[v^2\,\emptyset(x,y,z,0) + v^3\,\emptyset_t(x,y,z,0) - v^{\alpha}\,E(\emptyset_{xx} + \emptyset_{yy} + \emptyset_{zz} - 3\emptyset) - 2\,D_t^{\alpha}\emptyset\right]$$

$$\emptyset(x,\text{y},\text{z},t) = E^{-1}\left[v^2\,\emptyset(x,y,z,0) + v^3\,\emptyset_t(x,y,z,0) - v^{\alpha}\,E[L(\emptyset) - 2\;D_t^{\alpha}\emptyset]\right] \quad \mathbf{(44)}$$

Appling the MADETM process on equation (44)

$$\emptyset_0(x,\text{y},\text{z},t) = E^{-1}[v^2\,\emptyset(x,y,z,0) + v^3\,\emptyset_t(x,y,z,0)]$$

$$\emptyset_0(x,y,z,t) = E^{-1}[v^2\;\sinh x\;\sinh y \sinh z + v^3\;(-\sinh x\;\sinh y \sinh z)]$$

$$\emptyset_0(x,y,z,t) = E^{-1}[v^2\;\sinh x\;\sinh y \sinh z - v^3\;(\sinh x\;\sinh y \sinh z)]$$

$$\emptyset_0(x,y,z,t) = \emptyset(0) = \sinh x\;\sinh y \sinh z\,(1-t) \quad \mathbf{(45)}$$

$$\emptyset_{n+1}(x,y,z,t) = E^{-1}\left[v^{\alpha}E\left[L(\emptyset_n) - 2D_t^{\alpha}\emptyset_n\right]\right] \quad \textbf{(46)}$$

For $n = 0$

$$\emptyset_1(x,\text{y}, z,t) = E^{-1}\left[v^{\alpha}E\left[L(\emptyset_0) - 2D_t^{\alpha}\emptyset_0\right]\right]$$

Here, $L[\emptyset_0] = (\emptyset_{0xx} + \emptyset_{0yy} + \emptyset_{0zz} - 3\emptyset_0) = 0$

Therefore, above equation implies, $\emptyset_1(x,y,z,t) = E^{-1}\left[v^{\alpha}E\left[-2D_t^{\alpha}\emptyset_0\right]\right]$

$$\emptyset_1(x,y,z,t) = -E^{-1}\left[v^{\alpha}E\left[-2D_t^{\alpha}\emptyset_0\right]\right] = 2\sinh x\ \sinh y \sinh z \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

For $n = 1,2,3$ ....

$$\emptyset_2(x,y,z,t) = -E^{-1}\left[v^{\alpha}E\left[-2D_t^{\alpha}\emptyset_1\right]\right] = -4\sinh x\ \sinh y \sinh z \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$\emptyset_3(x,y,z,t) = -E^{-1}\left[v^{\alpha}E\left[-2D_t^{\alpha}\emptyset_2\right]\right] = 8\sinh x\ \sinh y \sinh z \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\emptyset_4(x,y,z,t) = -E^{-1}\left[v^{\alpha}E\left[-3D_t^{\alpha}\emptyset_3\right]\right] = -16\sinh x\ \sinh y \sinh z \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \quad \textbf{(47)}$$

.

:

:

Therefore, Series form $\emptyset(x,y,z,t)$ will be:

$$\emptyset(x,y,z,t) = \emptyset_0(x,y,z,t) + \emptyset_1(x,y,z,t) + \emptyset_2(x,y,z,t) + \emptyset_3(x,y,z,t) + \cdots \ldots \ldots$$

$$\emptyset(x,y,z,t) = \sinh x\ \sinh y \sinh z\,(1-t) + 2\sinh x\ \sinh y \sinh z \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 4\sinh x\ \sinh y \sinh z \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 8\sinh x\ \sinh y \sinh z \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 16\sinh x\ \sinh y \sinh z \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \ldots \ldots \ldots.$$

$$\emptyset(x,y,z,t) = \sinh x\ \sinh y \sinh z\left[(1-t) + 2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 4\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 8\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - -16\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} \ldots \ldots \ldots.\right]$$

When $\alpha = 1$, the following approximate solution will represent as:

$$\emptyset(x,y,z,t) = \sinh x\ \sinh y \sinh z\left[1 - t + \frac{2t^2}{2!} - \frac{4t^3}{3!} + \frac{8t^4}{4!} - \frac{16t^5}{5!} \ldots \ldots\right] \quad (48)$$

$$\emptyset(x,y,z,t) = \frac{\sinh x\ \sinh y \sinh z}{2}\left[2 - \frac{2t}{1!} + \frac{4t^2}{2!} - \frac{8t^3}{3!} + \frac{16t^4}{4!} - \frac{32t^5}{5!} \ldots \ldots\right]$$

$$\emptyset\ (x,y,z,t)=\frac{\sinh x\ \sinh y \sinh z}{2}\left[2-\frac{2t}{1!}+\frac{(2t)^2}{2!}-\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}-\frac{(2t)^5}{5!}\ldots..\right]$$

$$\emptyset\ (x,y,z,t)=\frac{\sinh x\ \sinh y \sinh z}{2}(1+e^{1-2t})$$

The precise answer to the equation (42) is:

$$\emptyset\ (x,y,t)=\frac{\sinh x\ \sinh y \sinh z}{2}(1+e^{1-2t}) \quad \textbf{(49)}$$

5. Graphical Discussion

In This discussion, the graphical simulation is shown to validate the results between the approximate solution calculated by adopted technique and exact solution exist are expressed for said applications. Example 1, the approximation solution and exact solution outcomes are compared at $t = 1, 2$ *and* 3 at $\alpha = 2$ shown in Figure 1 .Figure 2. . shows the surface graph of the approximate and exact solutions for Example 1 at $\alpha = 2$. The error surface graph for Example 1 is shown in Figure 2.. Additionally, Figure 2. c illustrates a line graph for Example 1, displaying the approximate solution, exact solution, and the absolute error considering $t = 1$. Figure 3. displays the surface graph for Example 2, showcasing both the approximate and exact solutions at $\alpha = 1$ Figure 3. represents the corresponding error surface graph for Example 2. Furthermore, Figure 3. features a line graph for Example 2, which highlights the approximate and exact solutions, as well as the absolute error, evaluated at $t = 1$. In Figure 4. , the surface graph for Example 3 is depicted, showing both the approximate and exact solutions at $\alpha = 1$. The error surface graph for Example 3 is displayed in Figure 4. Moreover, Figure 4. Figure 4. a presents a line graph for Example 3, illustrating the approximate solution, exact solution, and absolute error for $t = 1$. Figure 5. shows the surface graph for Example 4, highlighting both the approximate and exact solutions at $\alpha = 1$. The error surface graph for Example 4 is presented in Figure 5. b, on the other hand, Figure 5. c displays a line graph for Example 5, which represents the approximate solution, exact solution, and absolute error at $t = 1$. Figure 6. a displays the surface graph for Example 5, showcasing both the approximate and exact solutions at $\alpha = 1$ Figure 6. b presents the corresponding error surface graph for Example 5. Furthermore, Figure 6. c features a line graph for Example 5, which highlights the approximate and exact solutions, as well as the absolute error, evaluated at $t = 1$. An analysis of the figures confirm that the proposed method demonstrates significant consistency between the exact and approximate profiles over an extensive time range.

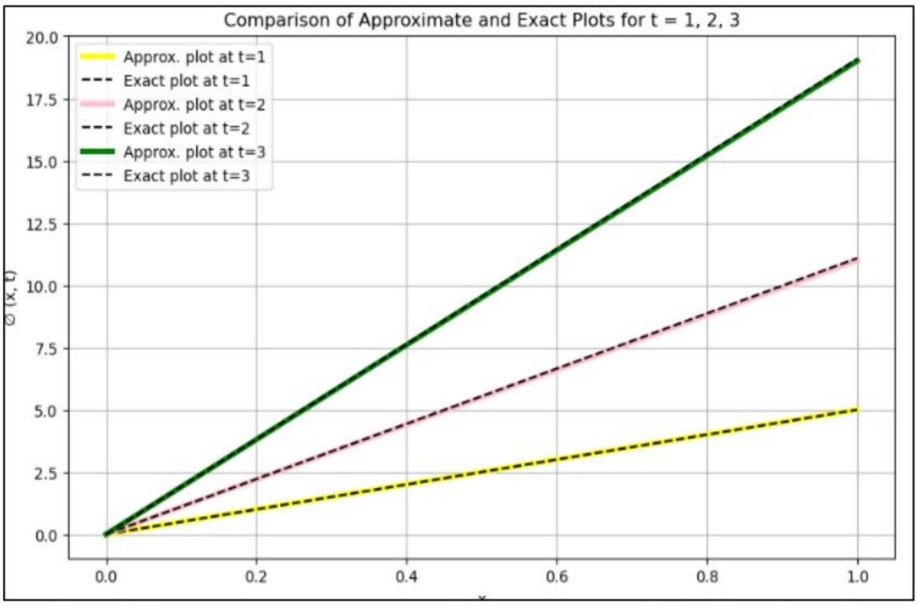

Figure 1. Comparing Approximate solution and Exact solution at t = 1, 2, and 3 for Example 1

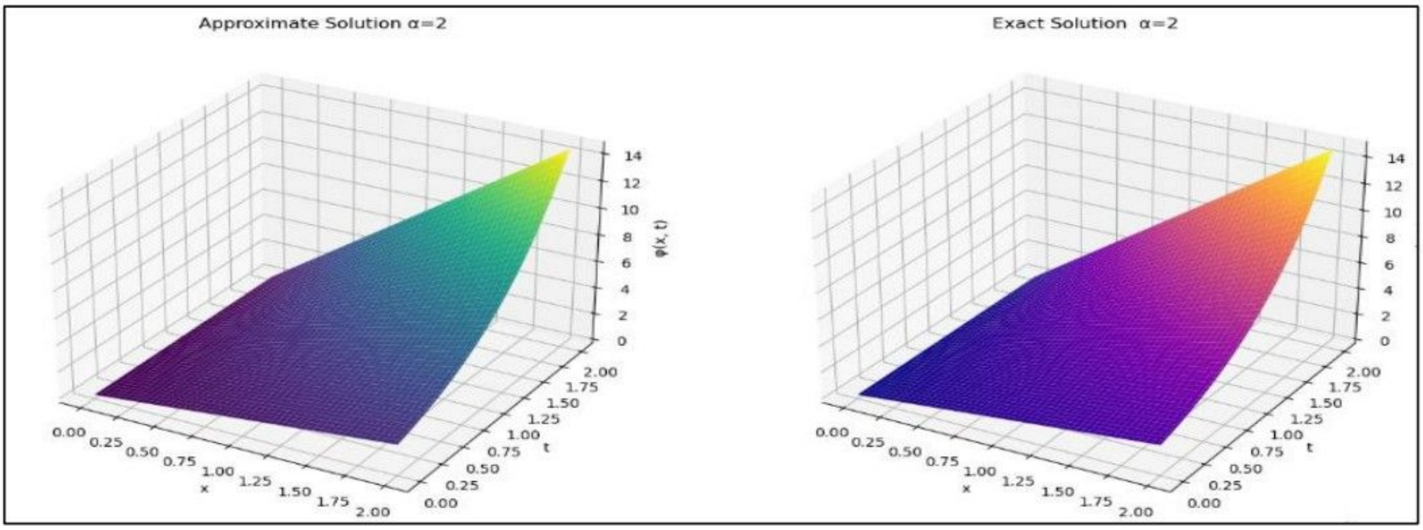

Figure 2. a : Comparison of Approximate solution and Exact solution profiles at $\alpha = 2$, for Example 1

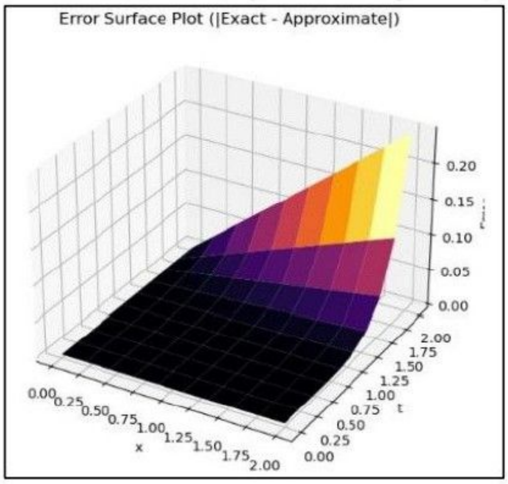

Figure 2.b : Error Plot between Exact- Appro. Solution Example 1

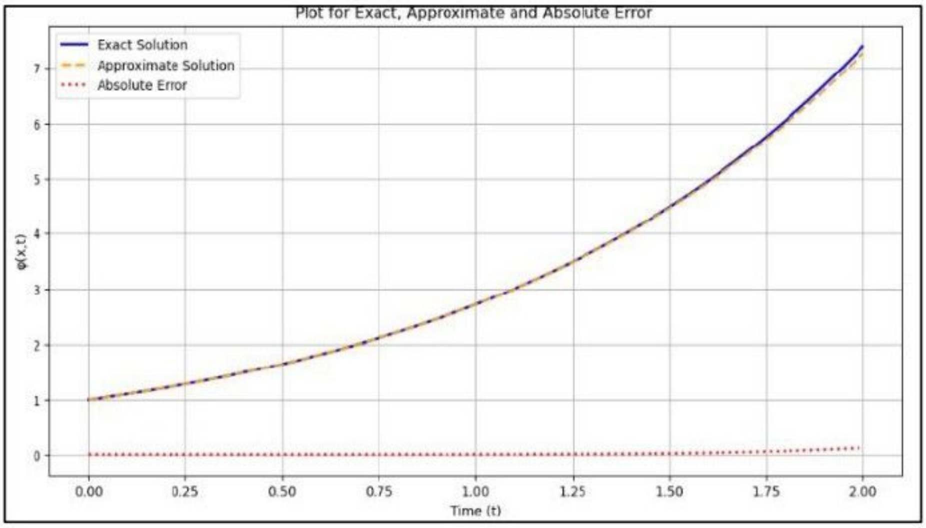

Figure 2. c: Line Plot for Exact, Approximate & Absolute Error Example 1

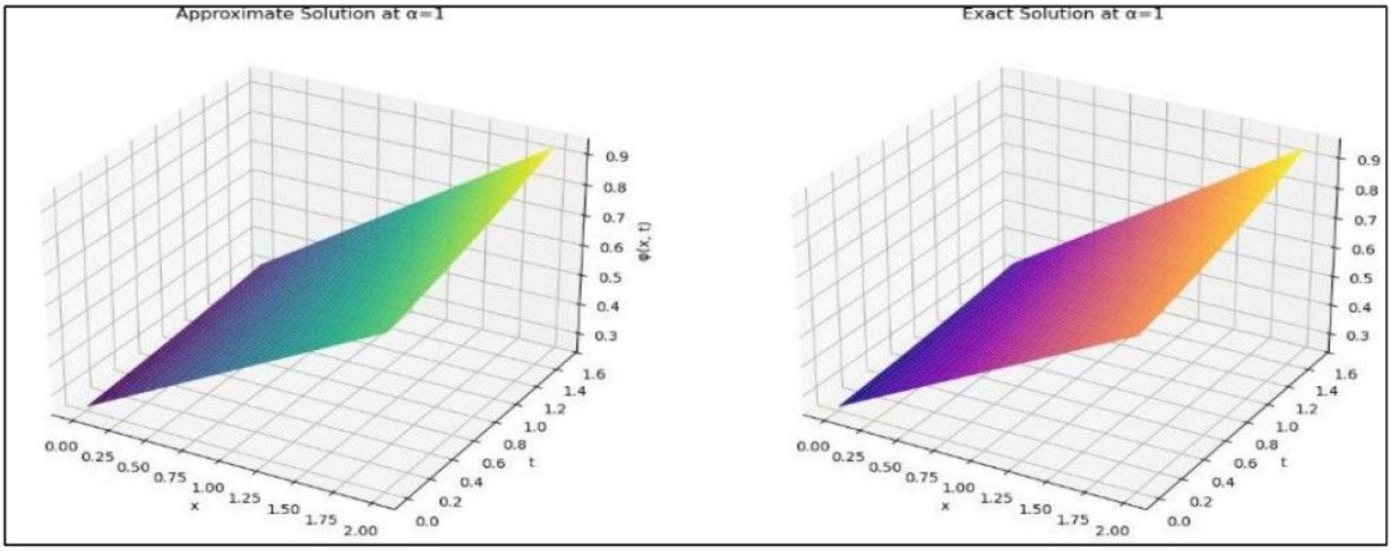

Figure 3. a : Comparison for Approximate solution and Exact solution at $\alpha = 1$ *for Example 2*

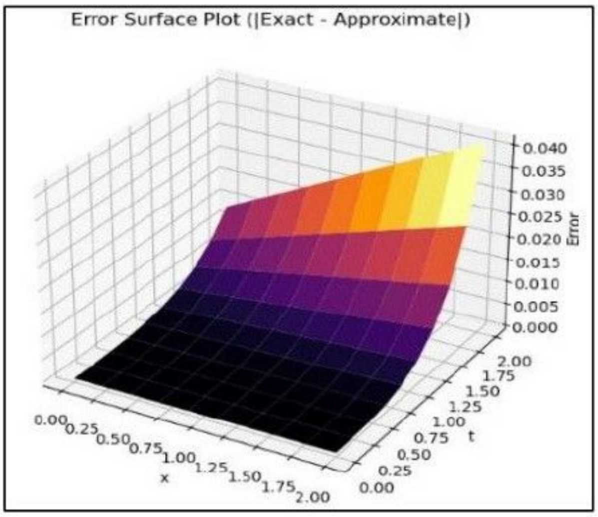

Figure 3. b : Error Plot between Exact- Appro. Solution Example 2

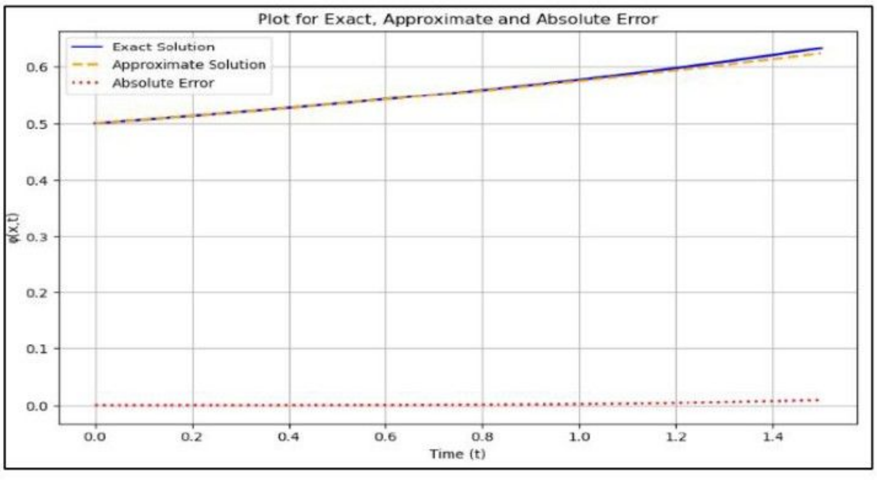

Figure 3. c : Line Plot for Exact, Approximate & Absolute Error Example 2

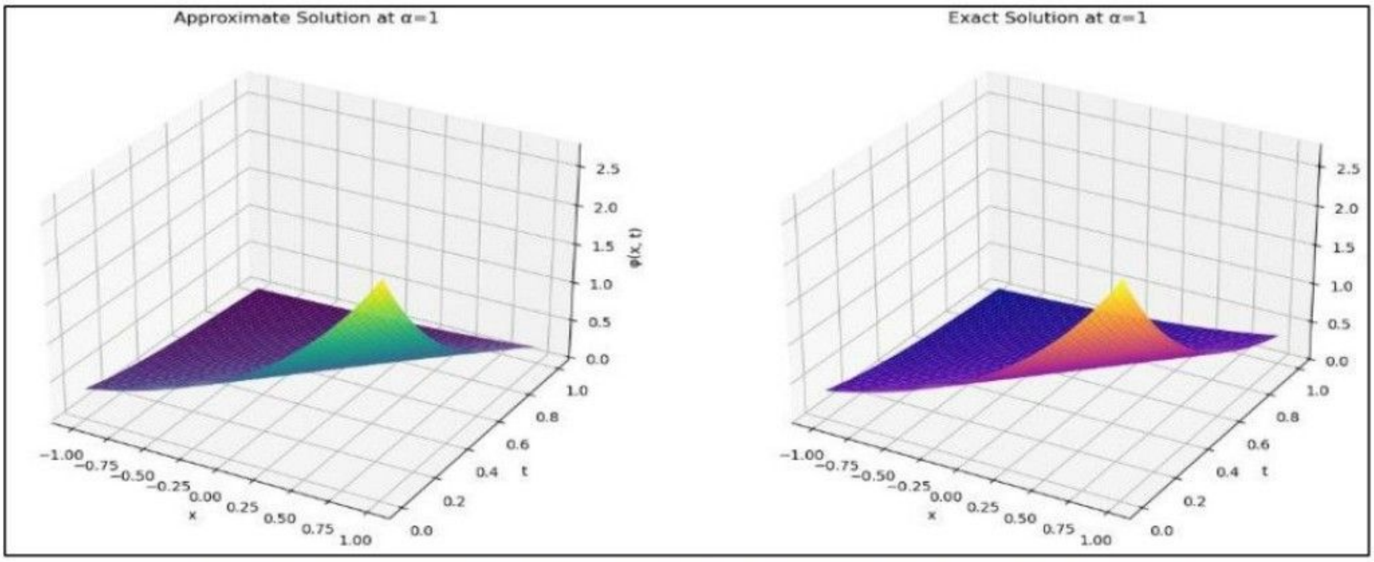

Figure 4. a : Comparison of Approximate solutions and Exact solutions at $\alpha = 1$ for Example 3

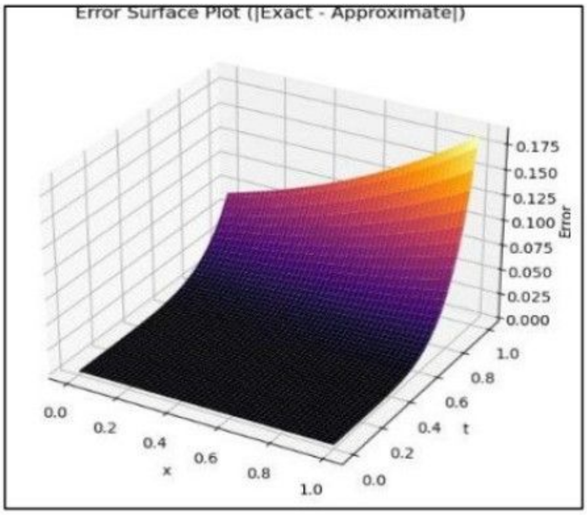

Figure 4. b : Error Plot between Exact- Appro. Solution Example 3

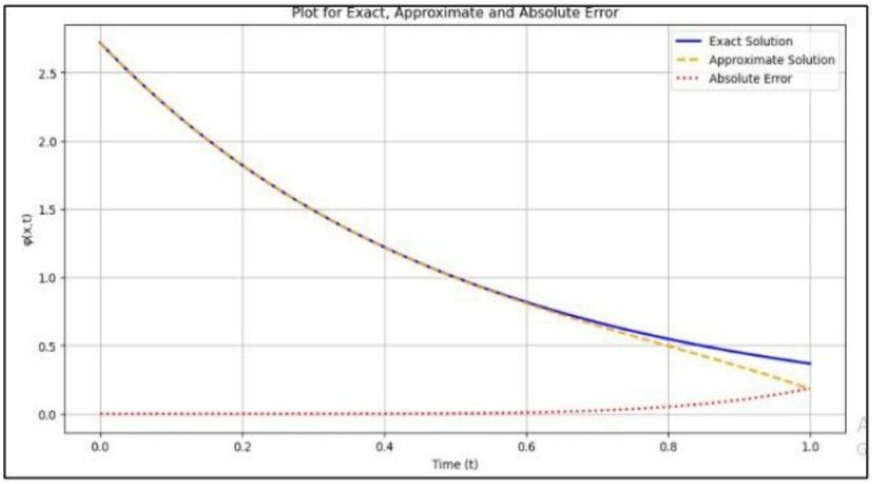

Figure 4. c: Line Plot for Exact, Approximate & Absolute Error Example 3

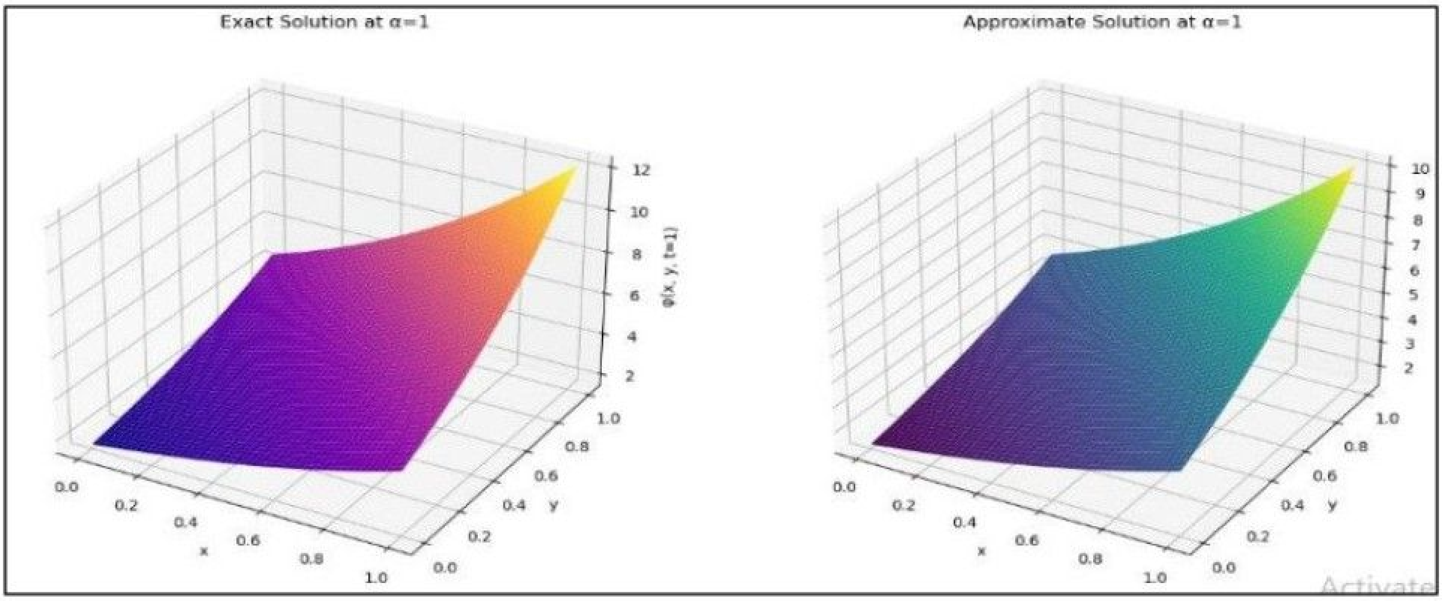

Figure 5. a : Comparison of Approximate solution and Exact solution profiles at $\alpha = 1$ for Example 4

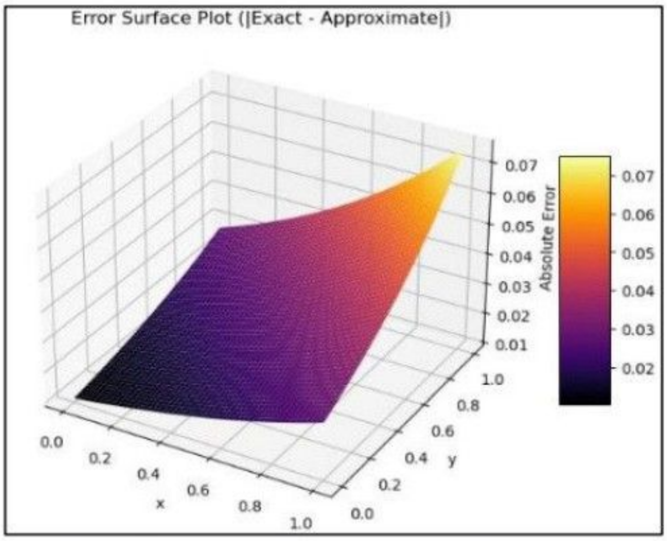

Figure 5. b: Error Plot between Exact- Appro. Solution Example 4

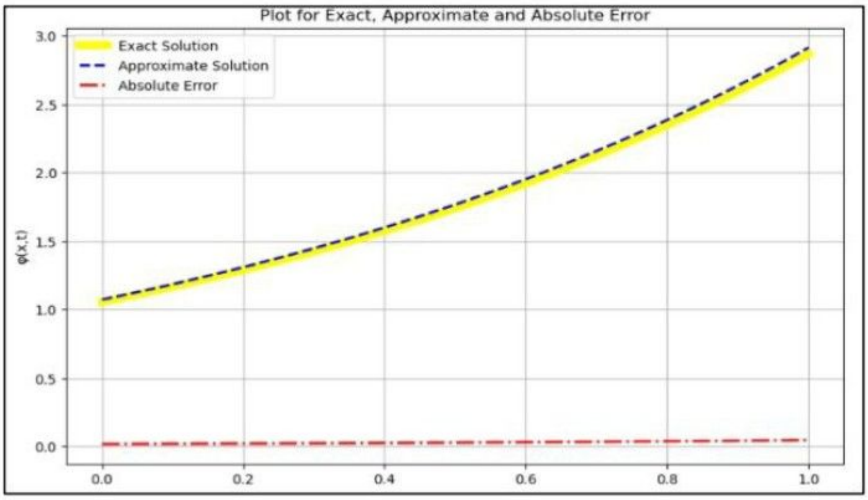

Figure 5. c : Line Plot for Exact, Approximate & Absolute Error Example 4

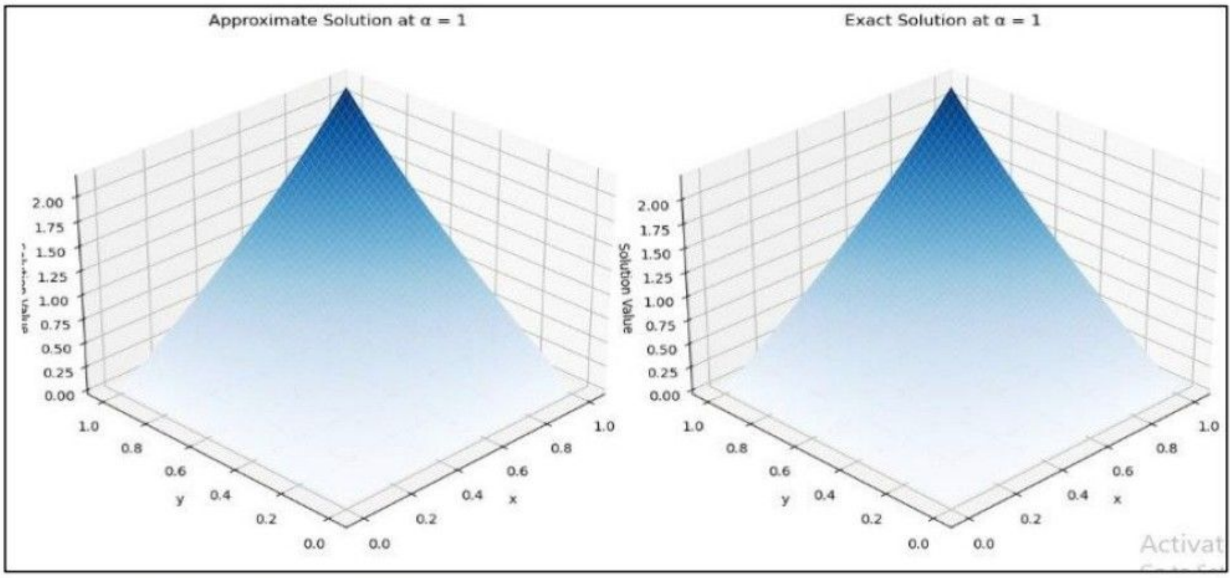

Figure 6. a : Comparing of Approximate solution and Exact solutions at $\alpha = 1$ for Example 5

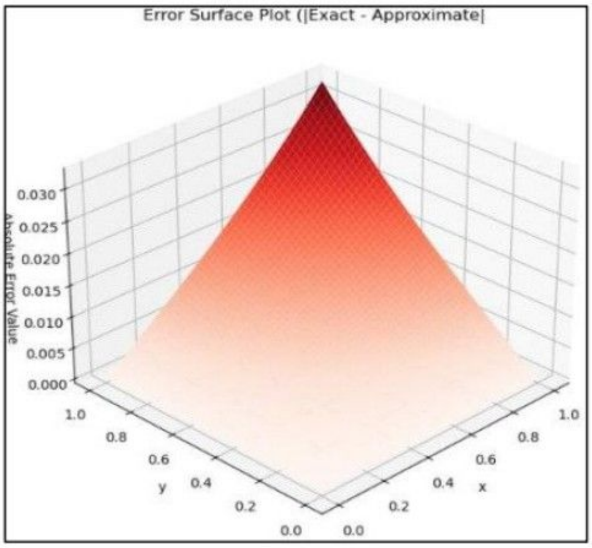

Figure 6. b : Error Plot between Exact- Appro. Solution Example 5

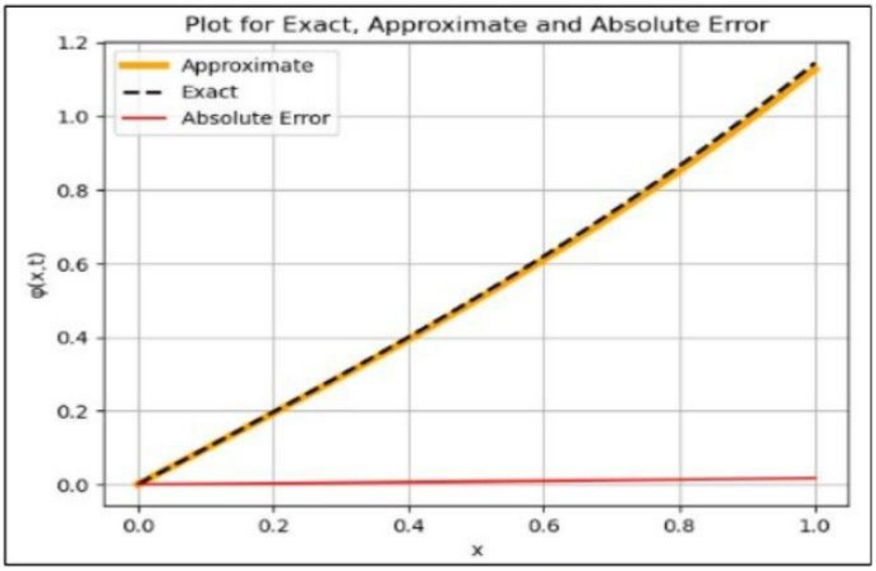

Figure 6. c : Line Plot for Exact, Approximate & Absolute Error Example 5

6. Conclusion

In this research, the Modified Adomian Decomposition Elzaki Transform Technique is employed to analyze time-fractional telegraph equations. This approach provided solutions for one- dimensional, two- dimensional, and three-dimensional time-fractional hyperbolic telegraph equations which deal with linear and non-linear forms of hyperbolic telegraph equations. Numerical calculations are demonstrated to find approximate solution with adopted methodology, and it is compared with exact solution. Graphical representations are performed in the form of a line graph and surface graph to validate the results. Approximate solutions, exact solutions and absolute error line plot is derived expressing it graphically. It's observed that the given fractional-order derivative solutions indicated their convergences to their exact solutions very closely. Additionally, the proposed method proved to be straightforward, efficient, and computationally cost-effective, demonstrating its robustness in handling different nature of fractional-order partial differential equations.

Funding: This research received no external funding for research work.

Data Availability Statement: Not Applicable

Acknowledgments: The authors extend their appreciation to Lovely Professional University, Ajeenkya DY Patil University for their support and guidance.

Conflicts of Interest: The authors declare no conflicts of interest.

References

[1] Ahmadi, Seyed Ahmad Pourreza, Hassan Hosseinzadeh, and AllahBakhsh Yazdani Cherati. "A new integral transform for solving higher order linear ordinary Laguerre and Hermite differential equations." International Journal of Applied and Computational Mathematics 5 (2019): 1-7.

[2] Khan, Zafar H., and Waqar A. Khan. "N-transform-properties and applications." NUST journal of engineering sciences 1.1 (2008): 127-133.

[3] Mahgoub, Mohand M. Abdelrahim, and M. Mohand. "The new integral transform "Sawi Transform"." Advances in Theoretical and Applied Mathematics 14.1 (2019): 81-87

[4] ELtayeb, Hassan, Adem Kılıçman, and Brian Fisher. "A new integral transform and associated distributions." Integral Transforms and Special Functions 21.5 (2010): 367-379.

[5] Elzaki, T. M. "The New Integral Transform"ELzaki Transform". Glob J Pure Appl Math. ISSN 0973-1768, 2011; 7 (1): 57-64."

[6] Kim, Hwajoon. "On the form and properties of an integral transform with strength in integral transforms." Far East Journal of Mathematical Sciences 102.11 (2017): 2831-2844.

[7] Kim, Hwajoon. "The Intrinsic Structure and Properties of Laplace-Typed Integral Transforms." Mathematical Problems in Engineering 2017.1 (2017): 1762729.

[8] Shah, Kamal, Muhammad Junaid, and Nigar Ali. "Extraction of Laplace, Sumudu, Fourier and Mellin transform from the natural transform." J. Appl. Environ. Biol. Sci 5.9 (2015): 108-115.

[9] Watugala, George K. "Sumudu transform: a new integral transform to solve differential equations and control engineering problems." Integrated Education 24.1 (1993): 35-43.

[10] Giannantoni, Corrado. "The problem of the initial conditions and their physical meaning in linear differential equations of fractional order." Applied Mathematics and Computation 141.1 (2003): 87-102.

[11] Javidi, Mohammad, and Nemat Nyamoradi. "Numerical solution of telegraph equation by using LT inversion technique." International journal of advanced mathematical sciences 1.2 (2013): 64-77.

[12] Zhang, Yiqi, et al. "Propagation dynamics of a light beam in a fractional Schrödinger equation." Physical review letters 115.18 (2015): 180403.

[13] Zhang, Yiqi, et al. "PT symmetry in a fractional Schrödinger equation." Laser & Photonics Reviews 10.3 (2016): 526-531.

[14] Javidi, Mohammad, and Nemat Nyamoradi. "Numerical solution of telegraph equation by using LT inversion technique." International journal of advanced mathematical sciences 1.2 (2013): 64-77.

[15] Latifizadeh, Habibolla. "The sinc-collocation method for solving the telegraph equation." J. Comput. Inform 1 (2013): 13-17.

[16] Al-badrani, Hind, et al. "Numerical solution for nonlinear telegraph equation by modified Adomian decomposition method." Nonlinear Analysis and Differential Equations 4.5 (2016): 243-257.

[17] Veeresha, P., and D. G. Prakasha. "Numerical solution for fractional model of telegraph equation by using q-HATM." arXiv preprint arXiv:1805.03968 (2018).

[18] Srivastava, Vineet K., Mukesh K. Awasthi, and Mohammad Tamsir. "RDTM solution of Caputo time fractional-order hyperbolic telegraph equation." AIP advances 3.3 (2013).

[19] Inc, Mustafa, Ali Akgül, and Adem Kiliçman. "Explicit solution of telegraph equation based on reproducing kernel method." Journal of Function Spaces 2012.1 (2012): 984682.

[20] Biazar, J., H. Ebrahimi, and Z. Ayati. "An approximation to the solution of telegraph equation by variational iteration method." Numerical Methods for Partial Differential Equations 25.4 (2009): 797-801.

[21] Erfanian, M., and M. Gachpazan. "A new method for solving of telegraph equation with Haar wavelet." Int. J. Math. Comput. Sci 3.1 (2016): 6-10.

[22] Shah, Nehad Ali, Ioannis Dassios, and Jae Dong Chung. "A decomposition method for a fractional-order multi-dimensional telegraph equation via the Elzaki transform." Symmetry 13.1 (2020): 8.

[23] Kapoor, Mamta, et al. "An analytical approach for fractional hyperbolic telegraph equation using Shehu transform in one, two and three dimensions." Mathematics 10.12 (2022): 1961.

[24] Aland, Parmeshwari, and Prince Singh. "Solution of Time Fractionalnewell–Whitehead–Segal Equation Using Modified Adomian Decomposition Method Elzaki Transformation Method."

[25] Albalawi, Kholoud Saad, Rachana Shokhanda, and Pranay Goswami. "On the solution of generalized time-fractional telegraphic equation." Applied Mathematics in Science and Engineering 31.1 (2023): 2169685.