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Common Fixed Point Theorem In Complex Valued Extended B-Metric Space

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Abstract:

In this paper, we proved a common fixed point theorem for generalized contractive type maps in complex valued extended b-metric space, which generalized many results in the literature.

Keywords: Fixed point theorem, Contractive type mapping, Complex valued extended b-metric space.

1. Introduction

In 2011, Azam et al. [1] introduced the notion of complex valued metric spaces and proved a common fixed point theorem for a pair of contractive type maps involving rational expressions which is a generalization of the classification Banach fixed point theorem. In 2013, Rao et al. [7] introduced the concept of complex valued b-metric space.

Subsequently, many authors have studied the existence and uniqueness of common fixed point of self-mappings in view of contractive conditions. Some of these observations are described in [1,4-6,8,9].

In 2019, N. Ullah et al. [11] extended the concept of complex valued b-metric space to complex valued extended b-metric space.

The main purpose of this paper is to present a common fixed point result for two self maps satisfying a rational inequality in complex valued extended b-metric space.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex number and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$\begin{aligned} &z_1 \precsim z_2 \\ &\text{iff } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2),\ \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2) \end{aligned} \tag{2.1}$$

Thus $z_1 \precsim z_2$ if one of the following holds:

(i)$\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
(ii)$\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
(iii)$\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
(iv)$\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

We will write if $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (ii), (iii) and (iv) is satisfied: Also we will write $z_1 \prec z_2$ if only (iv) is satisfied.

We can easily check that the following statements are held:

(i)If $a, b \in R$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}$;
(ii)if $0 \precsim z_1 \prec z_2$, then $|z_1| < |z_2|$;
(iii)if $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 2.1 ([1]). Let X be a nonempty set. A function $d: X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i)$0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only for $x = y$;
(ii)$d(x, y) = d(y, x)$;
(iii)$d(x, y) \precsim d(x, z) + d(z, y)$.

The pair (X, d) is called a complex valued metric space.

Example 2.1 ([5]). Let $X = \mathbb{C}$ Define the mapping $d: X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = i|z_1 - z_2| \text{ with } z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \tag{2.2}$$

(X, d) is complex valued metric space.

Example 2.2 ([8]). Let $X = \mathbb{C}$. Define the mapping $d: X \times X \to \mathbb{C}$ by

$$d(x, y) = c^{ik}|x - y|, \text{ where } k \in R,\ \forall x, y \in X \tag{2.3}$$

Then (X, d) is complex valued metric space.

Definition 2.2 ([7]). Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{C}$ is called a complex valued b-metric on X it for all $x, y, z \in X$ the following conditions are satisfied:

(i)$0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii)$d(x, y) = d(y, x)$;
(iii)$d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b-metric space.

Example 2.3 ([7]). Let $X = [0,1]$. Define mapping $d: X \times X \to \mathbb{C}$ by

$$d(x, y) = |x - y|^2 + i|x - y|^2,\ \forall x, y \in X \tag{2.4}$$

Then (X, d) is complex valued b-metric space with $s = 2$.

Definition 2.3 ([7]). Let (X, d) be a complex valued b-metric space. Consider the following

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec s \in C$ such that

$$B(x, r) := \{y \in X: d(x, y) \prec s\} \subseteq A.$$

(ii) A point $x \in X$ is called a limit point of a set A whenever, for every $0 \prec r \in \mathbb{C}$, $B(x,r) \cap A - \{x\} \neq \emptyset$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subbasis for a Hausdorff topology τ on X is a family

$$F = \{B(x,r): x \in X \text{ and } 0 \prec r\}.$$

Definition 2.4 ([11]). Let X be a non-empty set and $\phi: X \times X \to [1,\infty]$. If a mapping $d: X \times X \to \mathrm{C}$ satisfy:

(i)$0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$;
(ii)$d(x,y) = d(y,x)$;
(iii)$d(x,y) \precsim \phi(x,y)[d(x,z) + d(z,y)]$;

for all $x, y, z \in X$ then (X,d) is called a complex valued extended b-metric space.

Example 2.4 ([11]). Let $X = [0,\infty)$ and $\phi: X \times X \to [1,\infty)$ be a function defined by $\phi(x,y) = 1 + x + y$ and $d: X \times X \to \mathbb{C}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ i & \text{if } x \neq y \end{cases} \tag{2.5}$$

Then (X,d) is a complex valued extended b-metric space.

Theorem 2.1 ([1]). Let (X,d) be a complete complex valued metric space and λ, μ be nonnegative real numbers such that $\lambda + \mu < 1$. Suppose that $S, T: X \to X$ are mapping satisfying:

$$d(Sx,Ty) \precsim \lambda d(x,y) + \frac{\mu \cdot d(x,Sx) \cdot d(y,Ty)}{1 + d(x,y)} \tag{2.6}$$

for all $x, y \in X$. Then S, T have a unique Common fired point in X.

Theorem 2.2 ([9]). Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $f, g: X \to X$ be mapping satisfying:

$$d(fx,gy) \precsim \lambda d(x,y) + \frac{\mu \cdot d(x,fx) \cdot d(y,gy)}{1 + d(x,y)} + \frac{\delta \cdot d(y,fx) \cdot d(x,gy)}{1 + d(x,y)} \tag{2.7}$$

where λ, μ, δ nonnegative real numbers with $s\lambda + \mu + \delta < 1$. Then f, g have a unique common fixed point in X.

3. Main Result

Theorem 3.1. Let (X,d) be a complete CVEbMS with $\phi: X \times X \to [1,\infty)$ and $f, g: X \to X$ be two self-maps satisfying

$$d(fx,gy) \precsim A \cdot d(x,y) + B \cdot \frac{d(x,fx) \cdot d(y,gy)}{1 + d(x,y)} + C \cdot \frac{d(y,fx) \cdot d(x,gy)}{1 + d(x,y)}$$

$$+D \cdot \frac{d(x,fx) \cdot d(x,gy)}{1 + d(x,y)} + E \cdot \frac{d(y,fx) \cdot d(y,gy)}{1 + d(x,y)} \tag{3.1}$$

where A, B, C, D, E nonnegative real numbers, with $A + B + C + 2D + 2E < 1$. Then f and g have a unique common fixed point in X.

Proof. For any arbitrary point, $x_0 \in X$. Define a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, \text{ for } n = 0,1,2,3,\ldots \tag{3.2}$$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence.

Let $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1), we have

$$\begin{aligned} d(fx_{2n}, gx_{2n+1}) = \quad & d(x_{2n+1}, x_{2n+2}) \\ \precsim \quad & A \cdot d(x_{2n}, x_{2n+1}) + B \cdot \frac{d(x_{2n}, fx_{2n}) \cdot d(x_{2n+1}, gx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ & +C \cdot \frac{d(x_{2n+1}, fx_{2n}) \cdot d(x_{2n}, gx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ & +D \cdot \frac{d(x_{2n}, fx_{2n}) \cdot d(x_{2n}, gx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ & +E \cdot \frac{d(x_{2n+1}, fx_{2n}) \cdot d(x_{2n+1}, gx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \end{aligned}$$

i.e.,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) \precsim \quad & A \cdot d(x_{2n}, x_{2n+1}) + B \cdot \frac{d(x_{2n}, x_{2n+1}) \cdot d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\ & +C \cdot \frac{d(x_{2n+1}, x_{2n+1}) \cdot d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\ & +D \cdot \frac{d(x_{2n}, x_{2n+1}) \cdot d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\ & +E \cdot \frac{d(x_{2n+1}, x_{2n+1}) \cdot d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\ \Rightarrow \; d(x_{2n+1}, x_{2n+2}) \precsim \quad & A \cdot d(x_{2n}, x_{2n+1}) + B \cdot \frac{d(x_{2n}, x_{2n+1}) \cdot d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\ & +D \cdot \frac{d(x_{2n}, x_{2n+1}) \cdot d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \qquad (3.3) \end{aligned}$$

which implies that

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| \leq \quad & A \cdot |d(x_{2n}, x_{2n+1})| + B \cdot \frac{|d(x_{2n}, x_{2n+1})| \cdot |d(x_{2n+1}, x_{2n+2})|}{|1 + d(x_{2n}, x_{2n+1})|} \\ & +D \cdot \frac{|d(x_{2n}, x_{2n+1})| \cdot |d(x_{2n}, x_{2n+2})|}{|1 + d(x_{2n}, x_{2n+1})|} \qquad (3.4) \end{aligned}$$

Since

$$|1 + d(x_{2n}, x_{2n+1})| > |d(x_{2n}, x_{2n+1})|$$

We get

$$\begin{aligned}
&|d(x_{2n+1},x_{2n+2})| \leq A\cdot|d(x_{2n},x_{2n+1})| + B\cdot|d(x_{2n+1},x_{2n+2})| + D\cdot|d(x_{2n},x_{2n+1})| \\
&+D.|d_{2n+1},x_{2n+2}| \\
&|d(x_{2n},x_{2n+2})| \leq |d(x_{2n},x_{2n+1})|+|\ d(x_{2n+1},x_{2n+2}) \\
&\Rightarrow\ (1-B-D)\cdot|d(x_{2n+1},x_{2n+2})| \leq (A+D)\cdot|d(x_{2n},x_{2n+1})| \\
&\Rightarrow |d(x_{2n+1},x_{2n+2})| \leq \frac{A+D}{1-B-D}|d(x_{2n},x_{2n+1})|.
\end{aligned} \tag{3.5}$$

Similarly, we get

$$|d(x_{2n+2},x_{2n+3})| \leq \tfrac{A+D}{1-B-D}|d(x_{2n+1},x_{2n+2})|. \tag{3.6}$$

$$\begin{aligned}
d(x_{2n+2},x_{2n+3}) = \quad & d(x_{2n+3},x_{2n+2}) \\
= \quad & d(fx_{2n+2},gx_{2n+1}) \\
\precsim \quad & A\cdot d(x_{2n+2},x_{2n+1}) + B\cdot\frac{d(x_{2n+2},fx_{2n+2})\cdot d(x_{2n+1},gx_{2n+1})}{1+d(x_{2n+2},x_{2n+1})} \\
& +C\cdot\frac{d(x_{2n+1},fx_{2n+2})\cdot d(x_{2n+2},gx_{2n+1})}{1+d(x_{2n+2},x_{2n+1})} \\
& +\frac{D\cdot d(x_{2n+2},fx_{2n+2})\cdot d(x_{2n+2},gx_{2n+1})}{1+d(x_{2n+2},x_{2n+1})} \\
& +E\cdot\frac{d(x_{2n+1},fx_{2n+2})\cdot d(x_{2n+1},gx_{2n+1})}{1+d(x_{2n+2},x_{2n+1})} \\
\Rightarrow\ d(x_{2n+2},x_{2n+3}) = \quad & d(fx_{2n+2},gx_{2n+1}) \\
& \precsim A\cdot d(x_{2n+2},x_{2n+1}) + B\cdot\frac{d(x_{2n+2},x_{2n+3})\cdot d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+2},x_{2n+1})} \\
& +C\cdot\frac{d(x_{2n+1},x_{2n+3})\cdot d(x_{2n+2},x_{2n+2})}{1+d(x_{2n+2},x_{2n+1})} \\
& +D\cdot\frac{d(x_{2n+2},x_{2n+3})\cdot d(x_{2n+2},x_{2n+2})}{1+d(x_{2n+2},x_{2n+1})} \\
& +E\cdot\frac{d(x_{2n+1},x_{2n+3})\cdot d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+2},x_{2n+1})}.
\end{aligned} \tag{3.7}$$

which implies that

$$\begin{aligned}
|d\cdot(x_{2n+2},x_{2n+3})| \leq \quad & A\cdot|d(x_{2n+2},x_{2n+1})| + B\cdot\frac{|d(x_{2n+2},x_{2n+3})|\cdot|d(x_{2n+1},x_{2n+2})|}{|1+d(x_{2n+2},x_{2n+1})|} \\
& +E\cdot\frac{|d(x_{2n+1},x_{2n+3})|\cdot|d(x_{2n+1},x_{2n+2})|}{|1+d(x_{2n+2},x_{2n+1})|}
\end{aligned} \tag{3.8}$$

Since

$$|1+d(x_{2n+1},x_{2n+2})| > |d(x_{2n+1},x_{2n+2})|.$$

So, we get

$$|d(x_{2n+2},x_{2n+3})| < A\cdot|d(x_{2n+1},x_{2n+2})| + B\cdot|d(x_{2n+2},x_{2n+3})|$$

$$
\begin{aligned}
&+E \cdot |d(x_{2n+1}, x_{2n+2})| + E \cdot |d(x_{2n+2}, x_{2n+3})| \\
\Rightarrow \quad & (1 - B - E)|d.(x_{2n+2}, x_{2n+3})| \le A \cdot |d(x_{2n+1}, x_{2n+2})| + E \cdot |d(x_{2n+1}, x_{2n+2})| \\
\Rightarrow \quad & |d.(x_{2n+2}, x_{2n+3})| \le \frac{A+E}{1-B-E} \cdot |d(x_{2n+1}, x_{2n+2})|
\end{aligned}
$$

Putting

$$\lambda = \max\left\{\frac{A+D}{1-B-D}, \frac{A+E}{1-B-E}\right\}$$

we obtain that

$$|d(x_{k,}, x_{k+1})| \le \lambda^k |d(x_0, x_1)|, \text{ for some } k \in N \tag{3.9}$$

Now, for $m > n$ and by triangular inequality, we have

$$
\begin{aligned}
d(x_n, x_m) \quad & \precsim \phi(x_n, x_m)[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
& \precsim \phi(x_n, x_m)\lambda^n d(x_0, x_1) + \phi(x_n, x_m) d(x_{n+1}, x_m) \\
& \precsim \phi(x_n, x_m)\lambda^n d(x_0, x_1) + \phi(x_n, x_m)\phi(x_{n+1}, x_m) \\
& \cdot [d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
& \precsim \phi(x_n, x_m)\lambda^n d(x_0, x_1) + \phi(x_n, x_m)\phi(x_{n+1}, x_m) \\
& \cdot \lambda^{n+1} \cdot d[(x_0, x_1) + \phi(x_n, x_m)\phi(x_{n+1}, x_m) \cdot \lambda^{n+1} \cdot d(x_{n+2}, x_m)].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d(x_n, x_m) \precsim \quad & \phi(x_n, x_m)\lambda^n d(x_0, x_1) + \phi(x_n, x_m)\phi(x_{n+1}, x_m)\lambda^{n+1} \cdot d(x_0, x_1) + \cdots \\
& + \phi(x_n, x_m)\phi(x_{n+1}, x_m) \dots \phi(x_{m-1}, x_m)\lambda^{m-1} \cdot d(x_0, x_1)
\end{aligned} \tag{3.10}
$$

which implies that

$$
\begin{aligned}
|d(x_n, x_m)| \le \quad & |d(x_0, x_1)|[\phi(x_n, x_m)\lambda^n + \phi(x_n, x_m)\phi(x_{n+1}, x_m)\lambda^{n+1} + \cdots \\
& + \phi(x_n, x_m)\phi(x_{n+1}, x_m) \dots \phi(x_{m-1}, x_m)\lambda^{m-1}].
\end{aligned}
$$

Since limit $\phi(x_n, x_m)\lambda < 1, n, m \to \infty$, so the series

$$\sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{K} \phi(x_i, x_m) \text{ converges by ratio test for each } m \in N$$

Let

$$x = \sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{K} \phi(x_i, x_m),\ x_n = \sum_{j=1}^{n} \lambda^j \prod_{i=1}^{K} \phi(x_i, x_m), \tag{3.11}$$

Thus for $m > n$, the above inequality can be written as

$$|d(x_n, x_m)| \le |d(x_0, x_1)| \cdot |x_{m-1} - x_n|.$$

Now, by taking the limit as $n, m \to \infty$ we get

$$|d(x_n, x_m)| \to 0 \text{ as } n, m \to \infty$$

Thus $\{x_n\}$ is a Cauchy sequence in X. But X is a complete metric space, so this Cauchy sequence convergent and say converges to x.

i.e., $\lim_{n\to\infty} x_n = x$.

Now, we prove that $fx = x$ i.e., $d(x, fx) = 0$.

On the contrary, suppose that

$$\begin{aligned} &0 \prec v = d(x,fx) = d(fx,x), \\ &0 \prec v \precsim d(fx,x) \\ &\quad \precsim \phi(fx,x)[d(fx,x_{2n+2}) + d(x_{2n+2},x)] \\ &\quad \precsim \phi(fx,x)\cdot[d(fx,gx_{2n+1}) + d(x_{2n+2},x)] \end{aligned}$$

i.e.,

$$\begin{aligned} 0 \prec v \precsim \phi(fx,x) \quad &\Bigg[A\cdot d(x,x_{2n+1}) + B\cdot\frac{d(x,fx)\cdot d(x_{2n+1},gx_{2n+1})}{1+d(x,x_{2n+1})} \\ &+C\cdot\frac{d(x_{2n+1},fx)\cdot d(x,gx_{2n+1})}{1+d(x,x_{2n+1})} + D\cdot\frac{d(x,fx)\cdot d(x_{2n+1},gx_{2n+1})}{1+d(x,x_{2n+1})} \\ &+E\cdot\frac{d(x_{2n+1},fx)\cdot d(x_{2n+1},gx_{2n+1})}{1+d(x,x_{2n+1})} + d(x_{2n+2},x)\Bigg] \end{aligned}$$

i.e.

$$\begin{aligned} 0 \prec v \precsim \phi(fx,x) \quad &\Bigg[A\cdot d(x,x_{2n+1}) + B\cdot\frac{d(x,fx)\cdot d(x_{2n+1},x_{2n+2})}{1+d(x,x_{2n+1})} \\ &+C\cdot\frac{d(x_{2n+1},fx)\cdot d(x,x_{2n+2})}{1+d(x,x_{2n+1})} + D\cdot\frac{d(x,fx)\cdot d(x_{2n+1},x_{2n+2})}{1+d(x,x_{2n+1})} \\ &+E\cdot\frac{d(x_{2n+1},fx)\cdot d(x_{2n+1},x_{2n+2})}{1+d(x,x_{2n+1})} + d(x_{2n+2},x)\Bigg]. \end{aligned}$$

Now, taking the limit as $n \to \infty$ we get $0 \prec v \prec 0$, which implies that $v = 0$. i.e. $d(fx,x) = 0 \Rightarrow fx = x$.

Similarly, we can prove that $gx = x$.

Now, we can show that f and g have unique common fixed point.

On the contrary, suppose that x and y be two common fixed point of f and g.

Now,

$$\begin{aligned} d(x,y) = d(fx,gy) \precsim \quad & A\cdot d(x,y) + B\cdot\frac{d(x,fx)\cdot d(y,gy)}{1+d(x,y)} + C\cdot\frac{d(y,fx)\cdot d(x,gy)}{1+d(x,y)} \\ &+D\cdot\frac{d(x,fx)\cdot d(y,gy)}{1+d(x,y)} + E\cdot\frac{d(y,fx)\cdot d(y,gy)}{1+d(x,y)} \end{aligned}$$

i.e.

$$\begin{aligned} d(x,y) = d(fx,gy) \precsim \quad & A \cdot d(x,y) + B \cdot \frac{d(x,x)\cdot d(y,y)}{1+d(x,y)} + C \cdot \frac{d(y,x)\cdot d(x,y)}{1+d(x,y)} \\ & + D \cdot \frac{d(x,x)\cdot d(y,y)}{1+d(x,y)} + E \cdot \frac{d(y,x)\cdot d(y,y)}{1+d(x,y)} \end{aligned}$$

which implies that

$$\begin{aligned} |d(x,y)| = \quad & |d(fx,gy)| \\ \leq \quad & A \cdot |d(x,y)| + B \cdot \frac{|d(x,x)\cdot d(y,y)|}{|1+d(x,y)|} + C \cdot \frac{|d(y,x)\cdot d(x,y)|}{|1+d(x,y)|} \\ & + D \cdot \frac{|d(x,x)\cdot d(y,y)|}{|1+d(x,y)|} + E \cdot \frac{|d(y,x)\cdot d(y,y)|}{|1+d(x,y)|}. \end{aligned}$$

Since

$$|1+d(x,y)| > |d(x,y)|$$

i.e.

$$\frac{|d(x,y)|}{|1+d(x,y)|} < 1,$$

so we get

$$|d(x,y)| = |d(fx,gy)| < A.|d(x,y)| + B.0 + C.|d(x,y)| + D.0 + E.0$$

i.e.

$$|d(x,y)| = |d(fx,gy)| < A \cdot |d(x,y)| + C.|d(x,y)|$$

i.e.

$$\begin{aligned} & (1-A-C)|d(x,y)| < 0, \text{ a contradiction since } A+B+C+2D+2E < 1 \\ \Rightarrow \quad & A + C < 1 \end{aligned}$$

So, $x = y$, which proves the uniqueness of common fixed point of f and g in X. Corollary 3.1. Let (X,d) be a complete CVEbMS with $\phi: X \times X \to [1,\infty)$ and $f: X \to X$ be self-map satisfying

$$\begin{aligned} d(fx,fy) \precsim \quad & A \cdot d(x,y) + B \cdot \frac{d(x,fx)\cdot d(y,fy)}{1+d(x,y)} + C \cdot \frac{d(y,fx)\cdot d(y,fy)}{1+d(x,y)} \\ & + D \cdot \frac{d(x,fx)\cdot d(x,fy)}{1+d(x,y)} + E \cdot \frac{d(y,fx)\cdot d(y,fy)}{1+d(x,y)} \end{aligned}$$

where A, B, C, D, E nonnegative real numbers, with $A + B + C + 2D + 2E < 1$. Then f has a unique fires point in X.

Proof. Taking $g = f$ in Theorem 3.1.

Corollary 3.2. Let (X, d) be a complete CVEbMS with $\phi: X \times X \to [1, \infty)$ and $f, g: X \to X$ be self-maps satisfying

$$d(fx, gy) \precsim A \cdot d(x, y) + B \cdot \frac{d(x, fx) \cdot d(y, gy)}{1 + d(x, y)} + C \cdot \frac{d(y, fx) \cdot d(x, gy)}{1 + d(x, y)}$$

where A, B, C nonnegative real numbers, with $A + B + C < 1$. Then f and g have a unique common fixed point in X.

Proof. Taking $D = E = 0$ in Theorem 3.1.

Remark 3.1. (i) Theorem 3.1 generalized Theorem 15 of [6] after substituting $C = D = E = 0$ and $\phi(x, y) = s \geq 1$,

(ii) Theorem 3.1 generalized Theorem 10 of [9] after substituting $D = E = 0$ and $\phi(x, y) = s \geq 1$.

(iii) Theorem 3.1 generalized Theorem 1 of [4] after substituting $D = E = 0$ and $\phi(x, y) = 1$.

(iv) Theorem 3.1 generalized Theorem 4 of [1] after substituting $C = D = E = 0$ and $\phi(x, y) = 1$.

(v) Theorem 3.1 generalized Theorem 2 of [10] after substituting $B = C = D = E = 0$ and $\mathbb{C} = R$.

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