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Customer Impatience and Feedback Mechanism in $M^X/G/1$ Retrial Queue with, Unreliable Server, Bernoulli Vacation and Customer Search Strategies

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Abstract:
This article examines a complex retrial queueing system characterized by unreliable batch arrival, three phases of service, customer impatience, feedback mechanism, Bernoulli vacation policy and customer search behaviors. In this system, customer arrivals follow a Poisson process. When the server is idle, one customer from the batch receives service while the others enter a retrial group. The server offers essential service to all the incoming customers. If the server is busy upon arrival, customers may choose to leave (balk) or wait in the retrial group for the server to become available. Customers who retry either rejoin the service if the server is free or abandon the system if their retrial attempts fail (renege). The server may randomly breakdown during any phases of service and the repair starts immediately. After completion of repair, the server continues to serve the interrupted customer in the system. After completing each phase of service, the customer may either opt for optional service, or join the retrial group as a feedback customer, or leave the system. Upon completion of the first essential phase or second optional phase, if the server is idle, it searches for customers if available in the retrial group with a certain probability. After completing the third optional phase, the server goes on a Bernoulli vacation with certain probability. The retrial, service, repair and vacation times are arbitrarily distributed. By using the supplementary variable technique, various system state measures and reliability measures are derived. The effects of various parameters on these system measures are analyzed through numerical examples.

Keywords: Batch arrival, customer impatience, feedback, customer search and Bernoulli vacation.

1. Introduction

In the past few decades, queueing models with retrials have significant contributions in the fields of computer networks and communication systems. The arriving customers are served according to first come, first served basis. In retrial queueing system, the customer who finds the server busy leaves the service area and joins the retrial group to get service after random amount of time.

"Retrial queue" with phase services have been studied by various researchers. Senthil Kumar and Arumuganathan (2010) analyzed $M^X/G/1$ retrial queue with two phase service and two types of repair. Zadeh (2015) investigated a batch arrival multi-phase queueing system with feedback and single vacation policy. Tomar and Shrivatsav (2020) obtained the transient and steady state analysis of unreliable three phase retrial queueing system.

“Impatient customers” generally balk or renege at the system. Arrar et al. (2012) examined the behaviour of batch arrival retrial queue with impatient phenomenon. Sumitha and Udaya Chandrika (2016) modeled two phase retrial queue with impatient customers, Bernoulli vacation and orbital search. Arienzo et al. (2020) analysed group service of impatient customers in retrial queue.

“Breakdowns” are unavoidable in many situations. If the server is experiencing issues, it needs to be repaired. Djellab (2002) presented a retrial queueing system with breakdown. Singh and Kaur (2017) performed the sensitivity analysis of unreliable server retrial queue with optional service, multi-phase repair and Bernoulli vacation. Kuki et al. (2020) suggested retrial queueing system with non-reliable server, collision and impatient customers. Tian et al. (2023) investigated M/M/1 retrial queueing model with breakdown, repair and setup times where the server will be closed down when the system is empty.

After service completion, the unsatisfied customer joins the retrial group as a “feedback” customer. The feedback phenomena occurs in many retrial queueing systems. Shweta Upadhyaya (2014) discussed the retrial queue by considering both the batch arrival process and Bernoulli feedback. Pankaj Sharma (2018) studied retrial queueing system with feedback and modified vacation using generating function approach. Nila and Sumitha (2021) analyzed $M^X/G/1$ retrial queueing system with priority, collision and feedback customers.

“Vacation” generally means the time where the server may not be available due to maintenance. Wang (2012) discussed retrial queue with Bernoulli vacation. Radha et al. (2017) studied unreliable group arrival retrial queue with Bernoulli vacation. Single server constant retrial queue with blocked customers and Bernoulli vacation was analysed by Ke and Wang (2021). Mathavavisakan and Indhira (2023) provided an in-depth analysis, outlining key principles and relevant literature of retrial queueing system with Bernoulli vacations.

“Search of customers” from the retrial group reduces the idle time of the server. Krishnamoorthy et al. (2005) investigated the behaviour of the M/G/1 retrial queueing system with nonpersistent customers and orbital search. Sumitha and Udaya Chandrika (2012) modelled a repairable M/G/1 retrial queue with vacation and orbital search and derived the reliability indices to predict the system behaviour. Murugan et al. (2019) examined the bulk arrival retrial queueing model with orbital search and exponentially distributed multiple working vacations.

2. Model Description

- The unreliable batch arrival retrial queue with three phases of service, balking, reneging, feedback, Bernoulli vacation and search of customers are analysed.
- The customers arrive at the system according to Poisson process with rate Λ. Let J be the size of the arriving batch which is a random variable with $P\{J=k\}=\xi_k$, k=1,2,3… and $\xi_k(Z)$ be the probability generating function with first two moments c_1 and c_2.
- If the server is idle, then one of the customers in the batch gets the service and the rest of the incoming customers from the batch enters the retrial group.
- Otherwise if the server is busy, then the arriving batch may join the retrial group with probability q or leaves the system with complementary probability. During retrials if the primary customer arrives in the system, the retrial customer cancels the attempt for service and return to

retrial group with probability p or renege at the system with complementary probability. The retrial time follows general distribution with distribution function $K(X)$, density function $k(X)$ and Laplace Stieltjes Transform $K^*(s)$ and the hazard rate function $\eta(X)=k(X)/1-K(X)$.

- The server provides three phases of service. The first essential service is provided to all the arriving customers whereas the second and third phases of service are optional.
- After completion of first essential service, the customer leaves the system with probability r_0 or opts for second optional service with probability γ_1 or moves to retrial group with probability $\omega_1 (=1-r_0-\gamma_1)$ for reservice.
- Upon completing the second optional service, the customer leaves the system with probability r_1 or proceeds to third optional service with probability γ_2 or moves to retrial group with probability $\omega_2 (=1-r_1-\gamma_2)$ for reservice.
- After completing the third optional service the customer joins the retrial group with probability ω_3 or leaves the system with complementary probability.
- Service times in all three phases are arbitrarily distributed with distribution function $J_i(X)$, density function $j_i(X)$, Laplace Stieltjes Transform $J_i^*(s)$ with first two moments α_{i1},α_{i2} and the hazard rate function $\alpha_{i1}(X)=j_i(X)/1-J_i(X), i=1,2,3.$
- After the completion of first essential and second optional service, the server searches for customers in the retrial group with probability $\theta_i, i=1,2$ or remain idle with complementary probability.
- At the completion of third optional service, the server may take a single vacation with probability δ or waits for the next customer with complementary probability $\bar{\delta}$. Vacation times are arbitrarily distributed with distribution function $H(X)$, density function $h(X)$, Laplace Stieltjes Transform $H^*(s)$ with first two moments ν_1,ν_2 and the hazard rate function $\nu(X)=v(X)/1-V(X)$.
- In all the three phases of service the server is subjected to breakdown and the repair process starts instantly. The life time of the server in all the phases of service is exponential with rate $\beta_i, i=1,2,3.$ The interrupted customer waits in the system to get service after repair completion. Repair times in all three phases were assumed to be arbitrarily distributed with distribution function $F_i(X)$, density function $f_i(X)$, Laplace Stieltjes Transform $F_i^*(s)$ with first two moments ϕ_{i1},ϕ_{i2} and the hazard rate function $\phi_i(Y)=f_i(Y)/1-F_i(Y), i=1,2,3.$

2.1 System Analysis and Governing Equations

The behaviour of the retrial queueing system at time t can be described by the Markov Process

$$\{N(t), t\geq 0\}=\{S(t), M(t), \varepsilon(t), \varepsilon_0(t), \varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), \varepsilon_4(t), \varepsilon_5(t), \varepsilon_6(t), t\geq 0\}$$

At S(t)=0,1,2,3,4,5,6,7 represents the server states and M(t) represents the number of customers in the retrial group at time t.

E_0 is the probability that the server is idle at time t with no customers in the retrial group.

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$E_n(X,t)dX,\ n \geq 1$ denotes the probability that the server is idle at time t with n customers in the retrial group and the elapsed retrial time is between X and $X+dX$.

$R_{i,n}(X,t)dX,\ n \geq 0,\ i = 1,2,3$ denotes the probability that the server is busy in ith phase of service at time t with n customers in the retrial group and the elapsed service time is between X and $X+dX$.

$F_{i,n}(X,Y,t)dX\ dY,\ n \geq 0,\ i = 1,2,3$ the probability that the server is under repair in ith phase of service at time t with n customers in the retrial group and the elapsed repair time is between Y and $Y+dY$.

$V_n(X,t)dX,\ n \geq 0,$ the probability that the server is on vacation at time t with n customers in the retrial group and the elapsed vacation time is between X and $X+dX$.

The system of equations that governs the behaviour of the system is given below

$$\Lambda E_0 = r_0 \int_0^\infty R_{1,0}(X)\alpha_1(X)dX + r_1 \int_0^\infty R_{2,0}(X)\alpha_2(X)dX + \overline{\omega_3}\,\bar{\delta} \int_0^\infty R_{3,0}(X)\alpha_3(X)dX + \int_0^\infty V_0(X)\nu(X)dX \tag{1}$$

$$\left(\frac{\partial}{\partial X}\right) E_n(X) = -(\Lambda + \eta(X))E_n(X), n \geq 1 \tag{2}$$

$$\left(\frac{\partial}{\partial X}\right) R_{1,0}(X) = -(q\Lambda + \alpha_1(X) + \beta_1)R_{1,0}(X) + \int_0^\infty F_{1,0}(X,Y)\phi_1(Y)dY \tag{3}$$

$$\left(\frac{\partial}{\partial X}\right) R_{1,n}(X) = -(q\Lambda + \alpha_1(X) + \beta_1)R_{1,n}(X) + q\Lambda \sum_{k=1}^{n} \xi_k R_{1,n-k}(X) + \int_0^\infty F_{1,n}(X,Y)\phi_1(Y)dY, n \geq 1 \tag{4}$$

$$\left(\frac{\partial}{\partial X}\right) R_{2,0}(X) = -(q\Lambda + \alpha_2(X) + \beta_2)R_{2,0}(X) + \int_0^\infty F_{2,0}(X,Y)\phi_2(Y)dY \tag{5}$$

$$\left(\frac{\partial}{\partial X}\right) R_{2,n}(X) = -(q\Lambda + \alpha_2(X) + \beta_2)R_{2,n}(X) + q\Lambda \sum_{k=1}^{n} \xi_k R_{2,n-k}(X) + \int_0^\infty F_{2,n}(X,Y)\phi_2(Y)dY, n \geq 1 \tag{6}$$

$$\left(\frac{\partial}{\partial X}\right) R_{3,0}(X) = -(q\Lambda + \alpha_3(X) + \beta_3)R_{3,0}(X) + \int_0^\infty F_{3,0}(X,Y)\phi_3(Y)dY \tag{7}$$

$$\left(\frac{\partial}{\partial X}\right) R_{3,n}(X) = -(q\Lambda + \alpha_3(X) + \beta_3)R_{3,n}(X) + q\Lambda \sum_{k=1}^{n} \xi_k R_{3,n-k}(X) + \int_0^\infty F_{3,n}(X,Y)\phi_3(Y)dY, n \geq 1 \tag{8}$$

$$\left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y}\right) F_{1,0}(X,Y) = -(q\Lambda + \phi_1(Y))F_{1,0}(X,Y) \tag{9}$$

$$\left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y}\right) F_{1,n}(X,Y) = -(q\Lambda + \phi_1(Y))F_{1,n}(X,Y) + q\Lambda \sum_{k=1}^{n} \xi_k F_{1,n-k}(X,Y), n \geq 1 \tag{10}$$

$$\left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y}\right) F_{2,0}(X,Y) = -(q\Lambda + \phi_2(Y))F_{2,0}(X,Y) \tag{11}$$

$$\left(\frac{\partial}{\partial X}+\frac{\partial}{\partial Y}\right)F_{2,n}(X,Y)=-(q\Lambda+\phi_2(Y))F_{2,n}(X,Y)+q\Lambda\sum_{k=1}^{n}\xi_k F_{2,n-k}(X,Y),\ n\geq 1 \tag{12}$$

$$\left(\frac{\partial}{\partial X}+\frac{\partial}{\partial Y}\right)F_{3,0}(X,Y)=-(q\Lambda+\phi_3(Y))F_{3,0}(X,Y) \tag{13}$$

$$\left(\frac{\partial}{\partial X}+\frac{\partial}{\partial Y}\right)F_{3,n}(X,Y)=-(q\Lambda+\phi_3(Y))F_{3,n}(X,Y)+q\Lambda\sum_{k=1}^{n}\xi_k F_{3,n-k}(X,Y),\ n\geq 1 \tag{14}$$

$$\left(\frac{\partial}{\partial X}\right)V_0(X)=-(q\Lambda+\nu(X))V_0(X) \tag{15}$$

$$\left(\frac{\partial}{\partial X}\right)V_n(X)=-(q\Lambda+\nu(X))V_n(X)+q\Lambda\sum_{k=1}^{n}\xi_k V_{n-k}(X),\ n\geq 1 \tag{16}$$

with boundary conditions

$$E_n(0)=r_0\overline{\theta_1}\int_0^\infty R_{1,n}(X)\alpha_1(X)dX+r_1\overline{\theta_2}\int_0^\infty R_{2,n}(X)\alpha_2(X)dX+\overline{\omega_3}\bar{\delta}\int_0^\infty R_{3,n}(X)\alpha_3(X)dx+\bar{\delta}\omega_3\int_0^\infty R_{3,n-1}(X)\alpha_3(X)dX$$
$$+\overline{\theta_1}\omega_1\int_0^\infty R_{1,n-1}(X)\alpha_1(X)dX+\overline{\theta_2}\omega_2\int_0^\infty R_{2,n-1}(X)\alpha_2(X)dX+\int_0^\infty V_n(X)\nu(X)dX,\ n\geq 1 \tag{17}$$

$$R_{1,0}(0)=\Lambda\xi_1E_0+\int_0^\infty E_1(X)\eta(X)dX+\Lambda\bar{p}\int_0^\infty E_1(X)dX+\omega_1\theta_1\int_0^\infty R_{1,0}(X)\alpha_1(X)dX+\omega_2\theta_2\int_0^\infty R_{2,0}(X)\alpha_2(X)dX$$
$$+r_0\theta_1\int_0^\infty R_{1,1}(X)\alpha_1(X)dX+r_1\theta_2\int_0^\infty R_{2,1}(X)\alpha_2(X)dX \tag{18}$$

$$R_{1,n}(0)=\Lambda\xi_{n+1}E_0+\int_0^\infty E_{n+1}(X)\eta(X)dX+\Lambda p\sum_{k=1}^{n}\xi_k\int_0^\infty E_{n-k+1}(X)+\Lambda\bar{p}\sum_{k=1}^{n+1}\xi_k\int_0^\infty E_{n-k+2}(X)dX+\omega_1\theta_1\int_0^\infty R_{1,n}(X)\alpha_1(X)dX$$
$$+\omega_2\theta_2\int_0^\infty R_{2,n}(X)\alpha_2(X)dX+r_0\theta_1\int_0^\infty R_{1,n+1}(X)\alpha_1(X)dX+r_1\theta_2\int_0^\infty R_{2,n+1}(X)\alpha_2(X)dX,\ n\geq 1 \tag{19}$$

$$R_{2,n}(0)=\gamma_1\int_0^\infty W_{1,n}(X)\alpha_1(X)dX,\ n\geq 0 \tag{20}$$

$$R_{3,n}(0)=\gamma_2\int_0^\infty R_{2,n}(X)\alpha_2(X)dX,\ n\geq 0 \tag{21}$$

$$F_{1,n}(X,0)=\beta_1R_{1,n}(X),\ n\geq 0 \tag{22}$$

$$F_{2,n}(X,0)=\beta_2R_{2,n}(X),\ n\geq 0 \tag{23}$$

$$F_{3,n}(X,0)=\beta_3 R_{3,n}(X),\ n\geq 0 \tag{24}$$

$$V_n(0)=\delta\int_0^\infty R_{3,n}(X)\alpha_3(X)dX,\ n\geq 0 \tag{25}$$

The normalizing condition is

$$E_0+\sum_{n=1}^\infty\int_0^\infty E_n(X)dX+\sum_{n=0}^\infty\int_0^\infty R_{1,n}(X)dX+\sum_{n=0}^\infty\int_0^\infty R_{2,n}(X)dX+\sum_{n=0}^\infty\int_0^\infty R_{3,n}(X)dX+\sum_{n=0}^\infty\int_0^\infty\int_0^\infty F_{1,n}(X,Y)dXdY+$$

$$\sum_{n=0}^\infty\int_0^\infty\int_0^\infty F_{2,n}(X,Y)dXdY+\sum_{n=0}^\infty\int_0^\infty\int_0^\infty F_{3,n}(X,Y)dXdY+\sum_{n=0}^\infty\int_0^\infty V_n(X)dX=1$$

Define the probability generating functions

$$E(X,Z)=\sum_{n=1}^\infty E_n(X)Z^n\ ; \qquad R_1(X,Z)=\sum_{n=0}^\infty R_{1,n}(X)Z^n$$

$$R_2(X,Z)=\sum_{n=0}^\infty R_{2,n}(X)Z^n\ ; \qquad R_3(X,Z)=\sum_{n=0}^\infty R_{3,n}(X)Z^n$$

$$F_1(X,Y,Z)=\sum_{n=0}^\infty F_{1,n}(X,Y)Z^n\ ; \qquad F_2(X,Y,Z)=\sum_{n=0}^\infty F_{2,n}(X,Y)Z^n$$

$$F_3(X,Y,Z)=\sum_{n=0}^\infty F_{3,n}(X,Y)Z^n\ ; \qquad V(X,Z)=\sum_{n=0}^\infty V_n(X)Z^n$$

Theorem 2.1

The partial probability generating function of the joint probability distribution for the server being idle, busy with three phases, under repair in three phases and on vacation are respectively given

$$E(Z)=E_0(1-K^*(\Lambda))[ZM_3(Z)+Z\xi(Z)M_2(Z)-Z^2]/U(Z) \tag{26}$$

$$R_1(Z)=\Lambda E_0[Z\xi(Z)-M_1(Z)][1-J_1^*(G_1(q\Lambda-q\Lambda\xi(Z)))]/U(Z)G_1(q\Lambda-q\Lambda\xi(Z)) \tag{27}$$

$$R_2(Z)=\gamma_1\Lambda E_0[Z\xi(Z)-M_1(Z)]J_1^*(G_1(q\Lambda-q\Lambda\xi(Z))[1-J_2^*(G_2(q\Lambda-q\Lambda\xi(Z)))]/U(Z)G_2(q\Lambda-q\Lambda\xi(Z)) \tag{28}$$

$$R_3(Z)=\frac{\gamma_1\gamma_2\Lambda E_0[Z\xi(Z)-M_1(Z)]J_1^*(G_1(q\Lambda-q\Lambda\xi(Z))J_2^*(G_2(q\Lambda-q\Lambda\xi(Z))[1-J_3^*(G_3(q\Lambda-q\Lambda\xi(Z)))]}{U(Z)G_3(q\Lambda-q\Lambda\xi(Z))} \tag{29}$$

$$F_1(Z)=\alpha_1 R_1(Z)[1-F_1^*(q\Lambda-q\Lambda\xi(Z))]/(q\Lambda-q\Lambda\xi(Z)) \tag{30}$$

$$F_2(Z)=\alpha_2 R_2(Z)[1-F_2^*(q\Lambda-q\Lambda\xi(Z))]/(q\Lambda-q\Lambda\xi(Z)) \tag{31}$$

$$F_3(Z)=\alpha_3 R_3(Z)[1-F_3^*(q\Lambda-q\Lambda\xi(Z))]/(q\Lambda-q\Lambda\xi(Z)) \tag{32}$$

$$V(Z)=\delta R_3(Z)[1-V(q\Lambda-q\Lambda\xi(Z))]/(q-q\xi(Z)) \tag{33}$$

$$E_0 = \frac{1+T(1-K^*(\Lambda))(\bar{p}-c_1)-q\Lambda c_1 N-\omega_1-\omega_2\gamma_1-\bar{\delta}\omega_3\gamma_1\gamma_2}{\bar{p}(1-K^*(\Lambda))(T+\Lambda N)+K^*(\Lambda)[\Lambda c_1 N-q\Lambda c_1 N--\omega_1-\omega_2\gamma_1-\bar{\delta}\omega_3\gamma_1\gamma_2-1]} \quad (34)$$

where,

$$U(Z) = Z^2 - [ZK^*(\Lambda)+(1-K^*(\Lambda))\xi(Z)(pZ+\bar{p})]M_2(Z) - ZM_3(Z)$$

$$M_1(Z) = ZK^*(\Lambda)+(1-K^*(\Lambda))\xi(Z)(pZ+\bar{p})$$

$$M_2(Z) = J_1^*(G_1(q\Lambda-q\Lambda\xi(Z)))\{\overline{\theta_1}(r_0+\omega_1 Z)+\gamma_1\overline{\theta_2}(r_1+\omega_2 Z)J_2^*(G_2(q\Lambda-q\Lambda\xi(Z)))+(\overline{\omega_3}\bar{\delta}\gamma_1\gamma_2+\bar{\delta}\omega_3\gamma_1\gamma_2 Z)$$
$$J_2^*(G_2(q\Lambda-q\Lambda\xi(Z)))J_3^*(G_3(q\Lambda-q\Lambda\xi(Z)))+\delta\gamma_1\gamma_2 J_1^*(G_1(q\Lambda-q\Lambda\xi(Z)))J_2^*(G_2(q\Lambda-q\Lambda\xi(Z)))$$
$$J_3^*(G_3(q\Lambda-q\Lambda\xi(Z)))V^*(q\Lambda-q\Lambda\xi(Z))\}$$

$$M_3(Z) = ZJ_1^*(G_1(q\Lambda-q\Lambda\xi(Z)))[\theta_1(r_0+\omega_1 Z)+\gamma_1\theta_2(r_1+\omega_2 Z)J_2^*(G_2(q\Lambda-q\Lambda\xi(Z)))]$$

Proof

To derive the probability generating function, we proceed by multiplying equations (2) to (25) by z^n and summing over suitable powers of n, we get the resulting partial differential equations

$$E(X,Z) = E(0,Z)e^{-\Lambda X}(1-K(X)) \quad (35)$$

$$\left(\frac{d}{dX}+(q\Lambda+\beta_i+\alpha_i(X)-q\Lambda\xi(Z)\right)R_i(X,Z) = \int_0^\infty F_i(X,Y,Z)\phi_i(Y)dY, i=1,2,3 \quad (36)$$

$$F_i(X,Y,Z) = F_i(0,X,Z)e^{-q\Lambda(1-\xi(Z))Y}(1-F_i(Y)), i=1,2,3 \quad (37)$$

$$V(X,Z) = V(0,Z)e^{-q\Lambda(1-\xi(Z))X}(1-V(X)) \quad (38)$$

$$E(0,Z) = r_0\overline{\theta_1}\int_0^\infty R_1(X,Z)\alpha_1(X)dX + r_1\overline{\theta_2}\int_0^\infty R_2(X,Z)\alpha_2(X)dX + \bar{\delta}\overline{\omega_3}\int_0^\infty R_3(X,Z)\alpha_3(X)dX + \bar{\delta}\omega_3 Z\int_0^\infty R_3(X,Z)\alpha_3(X)dX$$
$$+\overline{\theta_1}\omega_1 z\int_0^\infty R_1(X,Z)\alpha_1(X)dX + \overline{\theta_2}\omega_2 z\int_0^\infty R_2(X,Z)\alpha_2(X)dX + \int_0^\infty V(X,Z)\nu(X)dX - \Lambda E_0 \quad (39)$$

$$R_1(0,Z) = \frac{\Lambda\xi(Z)}{Z}E_0 + \frac{1}{Z}\int_0^\infty E(X,Z)\eta(X)dX + \frac{\Lambda p\xi(Z)}{Z}\int_0^\infty E(X,Z)dX + \frac{\Lambda\bar{p}\xi(Z)}{Z^2}\int_0^\infty E(X,Z)dX + \frac{r_0\theta_1}{Z}\int_0^\infty R_1(X,Z)\alpha_1(X)dX$$
$$+\frac{r_1\theta_2}{Z}\int_0^\infty R_2(X,Z)\alpha_2(X)dX + \omega_1\theta_1\int_0^\infty R_1(X,Z)\alpha_1(X)dX + \omega_2\theta_2\int_0^\infty R_2(X,Z)\alpha_2(X)dX \quad (40)$$

$$R_2(0,Z) = \gamma_1\int_0^\infty R_1(X,Z)\alpha_1(X)dX \quad (41)$$

$$R_3(0,Z) = \gamma_2 \int_0^{\infty} R_2(X,Z)\alpha_2(X)dX \tag{42}$$

$$F_i(X,0,Z) = \beta_i R_i(X,Z), i = 1,2,3 \tag{43}$$

$$V(0,Z) = \delta \int_0^{\infty} R_3(X,Z)\alpha_3(X)dX \tag{44}$$

Substituting equation (37) in (36) and solving, we get

$$\left(\frac{d}{dX} + (q\Lambda + \beta_i + \alpha_i(X) - q\Lambda\xi(Z)\right)R_i(X,Z) = \int_0^{\infty} F_i(X,0,Z)F_i^*(q\Lambda - q\Lambda\xi(Z)), i = 1,2,3 \tag{45}$$

Substituting equation (43) in (37) and solving, we obtain

$$F_i(X,Y,Z) = \beta_i R_i(X,Z)e^{-q\Lambda(1-\xi(Z))Y}(1 - F_i(Y)), i = 1,2,3 \tag{46}$$

Substituting equation (43) in (45) and solving, we get

$$R_i(X,Z) = R_i(0,Z)e^{-Gi(q\Lambda - q\Lambda\xi(Z))}(1 - J_i(x)), i = 1,2,3 \tag{47}$$

where,

$$G_i(Z) = q\Lambda - q\Lambda\xi(Z) + \beta_i - \beta_i F_i^*(q\Lambda - q\Lambda\xi(Z)), i = 1,2,3$$

Substituting equation (47) in (43) and solving, we get

$$F_i(X,0,Z) = \beta_i R_i(0,Z)e^{-Gi(q\Lambda - q\Lambda\xi(Z))}(1 - J_i(x)), i = 1,2,3 \tag{48}$$

Substituting equation (47) in (44) and solving, we get

$$V(0,Z) = \delta R_3(0,Z)J_3^*(G_3(q\Lambda - q\Lambda\xi(Z))) \tag{49}$$

Substituting equation (47) in (41) and (42) and solving, we get

$$R_2(0,Z) = \gamma_1 R_1(0,Z)J_1^*(G_1(q\Lambda - q\Lambda\xi(Z))) \tag{50}$$

$$R_3(0,Z) = \gamma_2 R_2(0,Z)J_2^*(G_2(q\Lambda - q\Lambda\xi(Z))) \tag{51}$$

Substituting equation (38) and (47) in (39) and solving, we get

$$\begin{aligned} E(0,Z) = {} & R_1(0,Z)J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))[r_0\overline{\theta_1} + r_1\overline{\theta_2}\gamma_1 J_2^*(G_2(q\Lambda - q\Lambda\xi(Z))) + \gamma_1\gamma_2\overline{\omega_3}\bar{\delta}J_2^*(G_2(q\Lambda - q\Lambda\xi(Z))) \\ & J_3^*(G_3(q\Lambda - q\Lambda\xi(Z))) + \gamma_1\gamma_2\bar{\delta}\omega_3 Z J_2^*(G_2(q\Lambda - q\Lambda\xi(Z)))J_3^*(G_3(q\Lambda - q\Lambda\xi(Z))) + \overline{\theta_1}\omega_1 Z + \overline{\theta_1}\omega_1\gamma_1 Z \\ & J_2^*(G_2(q\Lambda - q\Lambda\xi(Z))) + \delta\gamma_1\gamma_2 J_2^*(G_2(q\Lambda - q\Lambda\xi(Z)))J_3^*(G_3(q\Lambda - q\Lambda\xi(Z)))V^*(q\Lambda - q\Lambda\xi(Z)) - \Lambda E_0 \end{aligned} \tag{52}$$

Using equation (35) and (47) in equation (40) and after some algebraic manipulations we get

$$R_1(0,Z) = \Lambda E_0[Z\xi(Z) - M_1(Z)]/U(Z) \tag{53}$$

Substituting equation (53) in equations (50) and (52) and using the resulting equation in (51), we get

$$E(0,Z)=\Lambda E_0[ZM_3(Z)+Z\xi(Z)M_2(Z)-Z^2]/U(Z) \quad (54)$$

$$R_2(0,Z)=\gamma_1 R_1(0,Z)J_1^*(G_1(q\Lambda-q\Lambda\xi(Z))) \quad (55)$$

$$R_3(0,Z)=\gamma_1\gamma_2 R_1(0,Z)J_1^*(G_1(q\Lambda-q\Lambda\xi(Z)))J_2^*(G_2(q\Lambda-q\Lambda\xi(Z))) \quad (56)$$

Using (53), (55) and (56) in (48) and solving, we get

$$F_1(0,X,Z)=\beta_1 R_1(0,Z)e^{-G_1(q\Lambda-q\Lambda\xi(Z))}(1-J_1(X)) \quad (57)$$

$$F_2(0,X,Z)=\beta_2 R_2(0,Z)e^{-G_2(q\Lambda-q\Lambda\xi(Z))}(1-J_2(X)) \quad (58)$$

$$F_3(0,X,Z)=\beta_3 R_3(0,Z)e^{-G_3(q\Lambda-q\Lambda\xi(Z))}(1-J_3(X)) \quad (59)$$

Substituting equation (56) in equation (49) and solving, we get

$$V(0,Z)=\delta\gamma_1\gamma_2 R_1(0,Z)J_1^*(G_1(q\Lambda-q\Lambda\xi(Z)))J_2^*(G_2(q\Lambda-q\Lambda\xi(Z)))J_3^*(G_3(q\Lambda-q\Lambda\xi(Z))) \quad (60)$$

Substituting expressions of equations $E(0,Z), R_1(0,Z), R_2(0,Z), R_3(0,Z), F_1(0,Z), F_2(0,Z), F_3(0,Z), V(0,Z)$ from (53) to (60) in equations (35), (38), (46) and (47) we get the required results of equations (26) to (33). Using the normalizing equation and applying L'Hospital rule, E_0 is obtained as in equation (34).

2.2 Performance Measures

The probabilities that the server is idle, busy in all the three phases, under repair in all the three phases of service and on vacation are derived and given respectively as

$$E=E_0(1-K^*(\Lambda))[c_1T+q\Lambda c_1N+\omega_1+\omega_2\gamma_1+\bar{\delta}\omega_3\gamma_1\gamma_2-1]/U_1 \quad (61)$$

$$R_1=\Lambda E_0[(1-K^*(\Lambda))\bar{p}+c_1K^*(\Lambda)]\alpha_{11}/U_1 \quad (62)$$

$$R_2=\gamma_1\Lambda E_0[(1-K^*(\Lambda))\bar{p}+c_1K^*(\Lambda)]\alpha_{21}/U_1 \quad (63)$$

$$R_3=\gamma_1\gamma_2\Lambda E_0[(1-K^*(\Lambda))\bar{p}+c_1K^*(\Lambda)]\alpha_{31}/U_1 \quad (64)$$

$$F_1=\beta_1\Lambda E_0[(1-K^*(\Lambda))\bar{p}+c_1K^*(\Lambda)]\alpha_{11}\phi_{11}/U_1 \quad (65)$$

$$F_2=\beta_2\gamma_1\Lambda E_0[(1-K^*(\Lambda))\bar{p}+c_1K^*(\Lambda)]\alpha_{21}\phi_{21}/U_1 \quad (66)$$

$$F_3=\beta_3\gamma_1\gamma_2\Lambda E_0[(1-K^*(\Lambda))\bar{p}+c_1K^*(\Lambda)]\alpha_{31}\phi_{31}/U_1 \quad (67)$$

$$V=\delta\gamma_1\gamma_2\Lambda E_0[(1-K^*(\Lambda))\bar{p}+c_1K^*(\Lambda)]\nu_1/U_1 \quad (68)$$

where,

$$U_1=1+T(1-K^*(\Lambda))(\bar{p}-c_1)-q\Lambda c_1N-\omega_1-\omega_2\gamma_1-\bar{\delta}\omega_3\gamma_1\gamma_2$$

$$T=1-\theta_1+\theta_1\gamma_1-\theta_2\gamma_1+\theta_2\gamma_1\gamma_2$$

$$N=(1+\beta_1\phi_{11})\alpha_{11}+\gamma_1(1+\beta_2\phi_{21})\alpha_{21}+\gamma_1\gamma_2(1+\beta_3\phi_{31})\alpha_{31}+\delta\gamma_1\gamma_2\nu_1$$

Theorem 2.2

The mean number of customers in the retrial group (L_q) and in the system (L_s) is given by

$$L_q = \underset{Z\to 1}{lt} \frac{d}{dZ} Q(Z)$$

$$= \frac{S_2S_3 + S_1S_4}{2S_3^2} \tag{69}$$

$$L_s = L_q + R_1 + R_2 + R_3 + F_1 + F_2 + F_3$$

(70)

Proof

$$S_1 = 2E_0qc_1\{K^*(\Lambda)[q\Lambda c_1N + \omega_1 + \omega_2\gamma_1 + \bar{\delta}\omega_3\gamma_1\gamma_2 + T(c_1 - 1)] - [c_1K^*(\Lambda) + \bar{p}(1 - K^*(\Lambda))(T + \Lambda N)]\}$$

$$\begin{aligned}
S_2 = {} & 3E_0\{K^*(\Lambda)q(c_2 + 2c_1)(qc_1\Lambda N + T(c_1 - 1)) + qc_1K^*(\Lambda)[2q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}(\theta_1 - \theta_1\gamma_1 + \theta_2\gamma_1 - \theta_2\gamma_1\gamma_2 \\
& +\omega_1\theta_1 + \omega_2\theta_2\gamma_1) + 2q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}(\theta_2\gamma_1 - \theta_2\gamma_1\gamma_2 + \omega_2\theta_2\gamma_1) + k_1(\theta_1 - \theta_1\gamma_1 + \theta_2\gamma_1 - \theta_2\gamma_1\gamma_2) \\
& +k_2(\theta_2\gamma_1 - \theta_2\gamma_1\gamma_2) + 2\theta_2\gamma_1(1-\gamma_2)q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} + 2\omega_1\theta_1 + 2\omega_2\theta_2\gamma_1 + c_2T + 2c_1H] \\
& +(2c_1 + c_2)T + 2(1+c_1)H + k_1(1-\theta_1 + \theta_1\gamma_1 - \theta_2\gamma_1 + \theta_2\gamma_1\gamma_2) + \gamma_1k_2(1-\theta_2\gamma_1 + \theta_2\gamma_1\gamma_2) + \gamma_1\gamma_2k_3 \\
& +\delta\gamma_1\gamma_2(q^2\Lambda^2c_1^2\nu_2 + q\Lambda c_2\nu_1) + 2\gamma_1(1-\omega_1)q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} - 2\theta_2\gamma_1(1-\gamma_2) \\
& q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} + 2\gamma_1\gamma_2q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}(q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11} + q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}) \\
& + 2q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}(\bar{\delta}\omega_3\gamma_1\gamma_2 - \omega_1\theta_1 + \omega_2\gamma_1 + \omega_1 - \omega_2\gamma_1\theta_2) + + + q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} + 2\bar{\delta}\omega_3\gamma_1\gamma_2 \\
& (q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} + q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}) + 2\omega_2\gamma_1(1-\theta_2)q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} + 2\omega_2\gamma_1q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11} \\
& (q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} + 2\delta\gamma_1\gamma_2q\Lambda c_1\nu_1(q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11} + q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} + q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}) \\
& - qc_1[(2c_1 + c_2) - (1 - K^*(\Lambda))(2c_1q + c_2)](T + \Lambda N) - [c_1K^*(\Lambda) + \bar{p}(1 - K^*(\Lambda))] + qc_2T + 2qc_1H + k_1 + \gamma_1k_2 \\
& +\gamma_1\gamma_2k_3 + \delta\gamma_1\gamma_2q\Lambda c_1\nu_1 - 2\gamma_1q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}(q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} + \gamma_2q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}) - 2\gamma_1\gamma_2 \\
& (q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}\gamma_2q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31} - 2\delta\gamma_1\gamma_2q\Lambda c_1\nu_1(q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11} + q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21} \\
& +q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31})\}
\end{aligned}$$

$$S_3 = 2U_1(-qc_1)$$

$$S_4 = 3[U_1(-qc_2) + N_4(-qc_1)]$$

Proof

Let $Q(Z) = E_0 + E(Z) + R_1(Z) + R_2(Z) + R_3(Z) + F_1(Z) + F_2(Z) + F_3(Z) + V(Z)$ be the probability generating function of number of customers in the retrial group and $H(Z) = E_0 + E(Z) + Z[R_1(Z) + R_2(Z) + R_3(Z)$ $+F_1(Z) + F_2(Z) + F_3(Z)] + V(Z)$ be the probability generating function of number of customers in the system.

$$Q(Z) = E_0\{(Z\xi(Z) - M_1(Z))[M_2(Z)(q - q\xi(Z)) + (1 - J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))) + \gamma_1 J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))$$
$$(1 - J_2^*(G_2(q\Lambda - q\Lambda\xi(Z)))) + \gamma_1\gamma_2 J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))J_2^*(G_2(q\Lambda - q\Lambda\xi(Z)))(1 - J_3^*(G_3(q\Lambda - q\Lambda\xi(Z)))$$
$$+ \delta\gamma_1\gamma_2 J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))J_2^*(G_2(q\Lambda - q\Lambda\xi(Z)))J_3^*(G_3(q\Lambda - q\Lambda\xi(Z)))(1 - V^*(q\Lambda - q\Lambda\xi(Z))] - ZK^*(\Lambda)$$
$$(q - q\xi(Z))[M_3(Z) + \xi(Z)M_2(Z) - Z]\} / U(Z)(q - q\xi(Z)) \quad (71)$$

$$H(Z) = E_0\{(Z\xi(Z) - M_1(Z))[M_2(Z)(q - q\xi(Z)) + Z((1 - J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))) + \gamma_1 J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))$$
$$(1 - J_2^*(G_2(q\Lambda - q\Lambda\xi(Z)))) + \gamma_1\gamma_2 J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))J_2^*(G_2(q\Lambda - q\Lambda\xi(Z)))(1 - J_3^*(G_3(q\Lambda - q\Lambda\xi(Z))))$$
$$+ \delta\gamma_1\gamma_2 J_1^*(G_1(q\Lambda - q\Lambda\xi(Z)))J_2^*(G_2(q\Lambda - q\Lambda\xi(Z)))J_3^*(G_3(q\Lambda - q\Lambda\xi(Z)))(1 - V^*(q\Lambda - q\Lambda\xi(Z))] - ZK^*(\Lambda)$$
$$(q - q\xi(Z))[M_3(Z) + \xi(Z)M_2(Z) - Z]\} / U(Z)(q - q\xi(Z)) \quad (72)$$

By differentiating the equations (71) and (72) with respect to Z and letting Z=1 we obtain, the mean number of customers in the retrial group (L_q) and mean number of customers in the system (L_S).

Corollary 2.1

- Expected retrial group size when the server is idle in the non-empty system is derived as

$$N_E = \lim_{Z\to 1}\frac{d}{dZ}[E(Z)]$$
$$= \frac{N_2N_3 - N_1N_4}{2[N_3]^2} \quad (73)$$

where,

$$N_1 = E_0(1 - K^*(\Lambda))[c_1T + q\Lambda c_1N + \omega_1 + \omega_2\gamma_1 + \bar{\delta}\omega_3\gamma_1\gamma_2 - 1]$$

$$N_2 = E_0(1 - K^*(\Lambda))\{(2c_1 + c_2T) + 2c_1(q\Lambda c_1N - (\theta_1 - \theta_1\gamma_1 + \theta_2\gamma_1 - \theta_2\gamma_1\gamma_2)q\Lambda c_1(1 + \beta_1\phi_{11})\alpha_{11} - \theta_2\gamma_1(1 - \gamma_2)$$
$$q\Lambda c_1(1 + \beta_2\phi_{21})\alpha_{21} + \omega_1 + \omega_2\gamma_1 + \bar{\delta}\omega_3\gamma_1\gamma_2 - \theta_1\omega_1 - \theta_2\omega_2\gamma_1) + 2(q\Lambda c_1N + \omega_3\gamma_1\gamma_2 + \delta\omega_3\gamma_1\gamma_2 + \omega_1 + \omega_2\gamma_1)$$
$$+ k_1 + \gamma_1k_2 + \gamma_1\gamma_2k_3 + \delta\gamma_1\gamma_2(q^2\Lambda^2c_1^2\nu_2 + q\Lambda c_2\nu_1) + 2\gamma_1\gamma_2q\Lambda c_1((1 + \beta_1\phi_{11})\alpha_{11}(1 + \beta_3\phi_{31})\alpha_{31} + (1 + \beta_2\phi_{21})\alpha_{21}$$
$$(1 + \beta_3\phi_{31})\alpha_{31}) + 2\gamma_1q\Lambda c_1((1 + \beta_1\phi_{11})\alpha_{11}(1 + \beta_2\phi_{21})\alpha_{21}) + 2\bar{\delta}\omega_3\gamma_1\gamma_2q\Lambda c_1((1 + \beta_1\phi_{11})\alpha_{11} + (1 + \beta_2\phi_{21})\alpha_{21}$$
$$+ (1 + \beta_3\phi_{31})\alpha_{31}) + 2\omega_1q\Lambda c_1(1 + \beta_1\phi_{11})\alpha_{11} + 2\omega_2\gamma_1q\Lambda c_1(1 + \beta_1\phi_{11})\alpha_{11} + 2\omega_2\gamma_1q\Lambda c_1(1 + \beta_2\phi_{21})\alpha_{21}$$
$$+ 2\delta\gamma_1\gamma_2q\Lambda c_1\nu_1(q\Lambda c_1[(1 + \beta_1\phi_{11})\alpha_{11} + (1 + \beta_2\phi_{21})\alpha_{21} + (1 + \beta_3\phi_{31})\alpha_{31}]) - 2\}$$

$$N_3 = 1 + T(1 - K^*(\Lambda))(\bar{p} - c_1) - q\Lambda c_1N - \omega_1 - \omega_2\gamma_1 - \bar{\delta}\omega_3\gamma_1\gamma_2$$

$$N_4 = 2 - [(1 - K^*(\Lambda))(c_2 + 2c_1p)]T + 2[K^*(\Lambda) + (1 - K^*(\Lambda))(c_1 + p)]\{q\Lambda c_1N - (\theta_1 - \theta_1\gamma_1 + \theta_2\gamma_1 - \theta_2\gamma_1\gamma_2)$$
$$q\Lambda c_1(1 + \beta_1\phi_{11})\alpha_{11} - \theta_2\gamma_1(1 - \gamma_2)q\Lambda c_1(1 + \beta_2\phi_{21})\alpha_{21} + \omega_1 + \omega_2\gamma_1 + \bar{\delta}\omega_3\gamma_1\gamma_2 - \theta_1\omega_1 - \theta_2\omega_2\gamma_1\}$$
$$- k_1 - \gamma_1k_2 - \gamma_1\gamma_2k_3 - \delta\gamma_1\gamma_2(q^2\Lambda^2c_1^2\nu_2 + q\Lambda c_2\nu_1) + 2\gamma_1q\Lambda c_1((1 + \beta_1\phi_{11})\alpha_{11}(1 + \beta_2\phi_{21})\alpha_{21})$$

$$-2\gamma_1\gamma_2 q\Lambda c_1((1+\beta_1\phi_{11})\alpha_{11}(1+\beta_3\phi_{31})\alpha_{31}+(1+\beta_2\phi_{21})\alpha_{21}(1+\beta_3\phi_{31})\alpha_{31})-2\omega_1 q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}$$
$$-2\bar{\delta}\omega_3\gamma_1\gamma_2 q\Lambda c_1((1+\beta_1\phi_{11})\alpha_{11}+(1+\beta_2\phi_{21})\alpha_{21}+(1+\beta_3\phi_{31})\alpha_{31})-2\omega_1 q\Lambda c_1((1+\beta_1\phi_{11})\alpha_{11}+(1+\beta_2\phi_{21})\alpha_{21})$$
$$-2\delta\gamma_1\gamma_2 q\Lambda c_1\nu_1(q\Lambda c_1[(1+\beta_1\phi_{11})\alpha_{11}+(1+\beta_2\phi_{21})\alpha_{21}+(1+\beta_3\phi_{31})\alpha_{31}])-2\theta_1 q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}$$
$$-2(\theta_1+\theta_1\gamma_1+\theta_2\gamma_1)q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}+2\theta_2\gamma_1\gamma_2 q\Lambda c_1[(1+\beta_1\phi_{11})\alpha_{11}+(1+\beta_2\phi_{21})\alpha_{21}]-2\theta_2\gamma_1$$
$$q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}-2\omega_1\theta_1-2\omega_2\theta_2\gamma_1$$

$$k_1=[\Lambda q c_2+\beta_1(q\Lambda c_2\phi_{11}+q^2\Lambda^2 c_1^2\phi_{12})]\alpha_{11}+[q\Lambda c_1+\beta_1\Lambda c_1\phi_{11}]^2\alpha_{12}$$

$$k_2=[\Lambda q c_2+\beta_2(q\Lambda c_2\phi_{21}+q^2\Lambda^2 c_1^2\phi_{22})]\alpha_{21}+[q\Lambda c_1+\beta_2\Lambda c_1\phi_{21}]^2\alpha_{22}$$

$$k_3=[\Lambda q c_2+\beta_3(q\Lambda c_2\phi_{31}+q^2\Lambda^2 c_1^2\phi_{32})]\alpha_{31}+[q\Lambda c_1+\beta_3\Lambda c_1\phi_{31}]^2\alpha_{32}$$

- Expected retrial group size when the server is busy in first essential service is derived as

$$N_{R_1}=\lim_{Z\to 1}\frac{d}{dZ}[R_1(Z)]$$
$$=\frac{N_6N_7-N_5N_8}{3[N_7]^2} \tag{74}$$

where,

$$N_5=2\Lambda E_0[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)]q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}$$

$$N_6=3\Lambda E_0\{[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)]k_1-[(2c_1+c_2)-(1-K^*(\Lambda))(c_2+2c_1p)]q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}\}$$

$$N_7=2U_1q\Lambda c_1[1+\beta_1\phi_{11}]$$

$$N_8=3[U_1(-q\Lambda c_2-\beta_1(\Lambda c_2\phi_{11}+a^2\Lambda^2c_1^2\phi_{12}))+N_4(-\Lambda c_1(1+\beta_1\phi_{11}))]$$

- Expected retrial group size when the server is busy in second optional service is derived as

$$N_{R_2}=\lim_{Z\to 1}\frac{d}{dZ}[R_2(Z)]$$
$$=\frac{N_{10}N_{11}-N_9N_{12}}{3[N_{11}]^2} \tag{75}$$

where,

$$N_9=2\gamma_1\Lambda E_0[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)]q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}$$

$$N_{10}=3\gamma_1\Lambda E_0\{[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)](k_2+3q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21})$$
$$-[(2c_1+c_2)-(1-K^*(\Lambda))(c_2+2c_1p)]q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}\}$$

$$N_{11}=2U_1q\Lambda c_1[1+\beta_2\phi_{21}]$$

$$N_{12}=3[U_1(-q\Lambda c_2-\beta_2(\Lambda c_2\phi_{21}+a^2\Lambda^2c_1^2\phi_{22}))+N_4(-\Lambda c_1(1+\beta_2\phi_{21}))]$$

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- Expected retrial group size when the server is busy in third optional service is derived as

$$N_{R_3} = \lim_{Z\to 1}\frac{d}{dZ}[R_3(Z)]$$

$$= \frac{N_{14}N_{15} - N_{13}N_{16}}{3[N_{15}]^2} \quad (76)$$

where,

$$N_{13} = 2\gamma_1\gamma_2\Lambda E_0[(1-K^*(\Lambda))(p-1) - c_1K^*(\Lambda)]q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}$$

$$N_{14} = 3\gamma_1\gamma_2\Lambda E_0\{[(1-K^*(\Lambda))(p-1) - c_1K^*(\Lambda)](k_3 + 2q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}$$

$$+ 2q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}) - [(2c_1 + c_2) - (1-K^*(\Lambda))(c_2 + 2c_1p)]q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}\}$$

$$N_{15} = 2U_1q\Lambda c_1[1+\beta_3\phi_{31}]$$

$$N_{16} = 3[U_1(-q\Lambda c_2 - \beta_3(\Lambda c_2\phi_{31} + a^2\Lambda^2c_1^2\phi_{32})) + N_4(-\Lambda c_1(1+\beta_3\phi_{31}))]$$

- Expected retrial group size when the server is under repair in first essential service is derived as

$$N_{F_1} = \lim_{Z\to 1}\frac{d}{dZ}[F_1(Z)]$$

$$= \frac{N_{18}N_{19} + N_{17}N_{20}}{4[N_{19}]^2} \quad (77)$$

where,

$$N_{17} = 6\beta_1\Lambda E_0\{[(1-K^*(\Lambda))(p-1) - c_1K^*(\Lambda)]q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}q\Lambda c_1\phi_{11}\}$$

$$N_{18} = 12\beta_1\Lambda E_0\{[(1-K^*(\Lambda))(p-1) - c_1K^*(\Lambda)](k_1q\Lambda c_1\phi_{11} + q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}(q\Lambda c_2\phi_{11} + q^2\Lambda^2c_1^2\phi_{12}))$$

$$-[(2c_1 + c_2) - (1-K^*(\Lambda))(c_2 + 2c_1p)](q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}q\Lambda c_1\phi_{11})\}$$

$$N_{19} = 6[U_1q\Lambda c_1(1+\beta_1\phi_{11})(-q\Lambda c_1)]$$

$$N_{20} = 12\{U_1[(-q\Lambda c_2 - \beta_1(\Lambda c_2\phi_{11} + a^2\Lambda^2c_1^2\phi_{12}))(-q\Lambda c_1) + (-q\Lambda c_2)q\Lambda c_1(1+\beta_1\phi_{11})] + N_4(-q\Lambda c_1)(-\Lambda c_1(1+\beta_1\phi_{11}))\}$$

- Expected retrial group size when the server is under repair in second optional service is derived as

$$N_{F_2} = \lim_{Z\to 1}\frac{d}{dZ}[F_2(Z)]$$

$$= \frac{N_{22}N_{23} + N_{21}N_{24}}{4[N_{23}]^2} \quad (78)$$

where,

$$N_{21} = 6\beta_2\gamma_1\Lambda E_0\{[(1-K^*(\Lambda))(p-1) - c_1K^*(\Lambda)]q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}q\Lambda c_1\phi_{21}\}$$

$$N_{22} = 12\beta_2\gamma_1\Lambda E_0\{[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)](k_2q\Lambda c_1\phi_{21}+q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}(q\Lambda c_2\phi_{21}+q^2\Lambda^2c_1^2\phi_{22})$$
$$+2(q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}q\Lambda c_1\phi_{21}))-[(2c_1+c_2)-(1-K^*(\Lambda))(c_2+2c_1p)]$$
$$(q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}q\Lambda c_1\phi_{21})\}$$

$$N_{23} = 6[U_1q\Lambda c_1(1+\beta_2\phi_{21})(-q\Lambda c_1)]$$

$$N_{24} = 12\{U_1[(-q\Lambda c_2-\beta_2(\Lambda c_2\phi_{21}+a^2\Lambda^2c_1^2\phi_{22}))(-q\Lambda c_1)+(-q\Lambda c_2)q\Lambda c_1(1+\beta_2\phi_{21})]$$
$$+N_4(-q\Lambda c_1)(-\Lambda c_1(1+\beta_2\phi_{21}))\}$$

- Expected retrial group size when the server is under repair in third optional service is derived as

$$N_{F_3} = \lim_{Z\to 1}\frac{d}{dZ}[F_3(Z)]$$
$$= \frac{N_{26}N_{27}+N_{28}N_{25}}{4[N_{27}]^2} \qquad (79)$$

where,

$$N_{25} = 6\beta_2\gamma_1\gamma_2\Lambda E_0\{[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)]q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}q\Lambda c_1\phi_{31}\}$$

$$N_{26} = 12\beta_2\gamma_1\gamma_2\Lambda E_0\{[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)](k_3q\Lambda c_1\phi_{31}+q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}(q\Lambda c_2\phi_{31}+q^2\Lambda^2c_1^2\phi_{32})$$
$$-[(2c_1+c_2)-(1-K^*(\Lambda))(c_2+2c_1p)](q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}q\Lambda c_1\phi_{31})\}$$

$$N_{27} = 6[U_1q\Lambda c_1(1+\beta_3\phi_{31})(-q\Lambda c_1)]$$

$$N_{28} = 12\{U_1[(-q\Lambda c_2-\beta_3(\Lambda c_2\phi_{31}+a^2\Lambda^2c_1^2\phi_{32}))(-q\Lambda c_1)+(-q\Lambda c_2)q\Lambda c_1(1+\beta_3\phi_{31})]+N_4(-q\Lambda c_1)$$
$$(-\Lambda c_1(1+\beta_3\phi_{31}))\}$$

- Expected retrial group size when the server is on vacation is derived as

$$N_V = \lim_{Z\to 1}\frac{d}{dZ}[V(Z)]$$
$$= \frac{N_{30}N_{31}+N_{32}N_{29}}{3[N_{31}]^2} \qquad (80)$$

where,

$$N_{29} = 2\delta\gamma_1\gamma_2E_0\{[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)]q\Lambda c_1\nu_1\}$$

$$N_{30} = 3\delta\gamma_1\gamma_2E_0\{[(1-K^*(\Lambda))(p-1)-c_1K^*(\Lambda)]((q^2\Lambda^2c_1^2\nu_2+q\Lambda c_2\nu_1)+2q\Lambda c_1\nu_1(q\Lambda c_1(1+\beta_1\phi_{11})\alpha_{11}$$
$$+q\Lambda c_1(1+\beta_2\phi_{21})\alpha_{21}+q\Lambda c_1(1+\beta_3\phi_{31})\alpha_{31}))-[(2c_1+c_2)-(1-K^*(\Lambda))(c_2+2c_1p)]q\Lambda c_1\nu_1\}$$

$$N_{31} = 2(U_1(-qc_1))$$

$$N_{32} = 3[U_1(-qc_1))+N_4(-\Lambda c_1)]$$

2.3 Reliability Measures

Theorem 2.3

Availability of the server at time (t), is the probability that the server is idle or busy with customers is given by

$$A = E_0\{\bar{p}T(1-K^*(\Lambda)) + K^*(\Lambda)(1-q\Lambda c_1 N - \omega_1 - \omega_2\gamma_1 - \bar{\delta}\omega_3\gamma_1\gamma_2) + \Lambda[(1-K^*(\Lambda))\bar{p} + c_1 K^*(\Lambda)]$$

$$[\alpha_{11} + \gamma_1\alpha_{21} + \gamma_1\gamma_2\alpha_{31}]\}/U_1 \quad (81)$$

Proof

The availability of the server can be expressed as

$$A = E_0 + \lim_{Z\to 1}[E(Z) + R_1(Z) + R_2(Z) + R_3(Z)] \quad (82)$$

Substituting the equations (26) to (29) in (82) we get equation (81).

Theorem 2.4

The failure frequency of the server is given by

$$M = \Lambda E_0[(1-K^*(\Lambda))\bar{p} + c_1 K^*(\Lambda)](\beta_1\alpha_{11} + \beta_2\gamma_1\alpha_{21} + \beta_3\gamma_1\gamma_2\alpha_{31}/U_1 \quad (83)$$

Proof

$$M = \lim_{Z\to 1}[\beta_1 R_1(Z) + \beta_2 R_2(Z) + \beta_3 R_3(Z)] \quad (84)$$

Substituting the expressions in (26) to (29) in equation (84), we get (83).

3. Numerical Results

The performance measures are illustrated numerically by using various system measures. For computation we choose the arbitrary parameters as $\Lambda = 2$; $q = 0.7$; $p = 0.6$; $\beta_1 = 0.6$; $\beta_2 = 0.7$; $\beta_3 = 0.8$; $\omega_1 = 0.5$; $\omega_2 = 0.6$; $\omega_3 = 0.7$; $\theta_1 = 0.5$; $\theta_2 = 0.3$; $\eta = 8$; $\delta = 0.9$; $r_0 = 0.2$; $r_1 = 0.1$; $\gamma_1 = 0.3$; $\gamma_2 = 0.3$; $\alpha_1 = 25$; $\alpha_2 = 15$; $\alpha_3 = 10$; $\phi_1 = 10$; $\phi_2 = 8$; $\phi_3 = 6$; $v = 10$; $c_1 = 0.5$; $c_2 = 0.5$. The effects of various parameters on the system measures, E_0 – probability that the server is idle in the empty system, E – probability that the server is idle in the non-empty system, R – probability that the server is busy in service, F – probability that the server is under repair during service, V – probability that the server is on vacation, L_s – the mean system size.

β	ϕ	α	E_0	E	R	F	V	L_s
0.6	3	10	0.7363	0.0229	0.2001	0.0173	0.0235	0.2921
		15	0.7472	0.0215	0.1922	0.0156	0.0236	0.2828
		20	0.7526	0.0208	0.1882	0.0147	0.0236	0.2781
	6	10	0.7390	0.0226	0.2002	0.0148	0.0235	0.2897
		15	0.7490	0.0212	0.1924	0.0138	0.0236	0.2812
		20	0.7540	0.0206	0.1883	0.0134	0.0236	0.2769

	9	10	0.7399	0.0224	0.2003	0.0138	0.0235	0.2889
		15	0.7496	0.0212	0.1924	0.0133	0.0236	0.2807
		20	0.7545	0.0205	0.1883	0.0130	0.0237	0.2765
0.8	3	10	0.7345	0.0231	0.1999	0.0191	0.0235	0.2937
		15	0.7460	0.0216	0.1921	0.0168	0.0236	0.2839
		20	0.7517	0.0209	0.1881	0.0156	0.0236	0.2789
	6	10	0.7381	0.0227	0.2001	0.0157	0.0235	0.2905
		15	0.7484	0.0213	0.1923	0.0144	0.0236	0.2817
		20	0.7536	0.0206	0.1883	0.0139	0.0236	0.2773
	9	10	0.7393	0.0225	0.2003	0.0144	0.0235	0.2895
		15	0.7492	0.0212	0.1924	0.0137	0.0236	0.2810
		20	0.7542	0.0206	0.1883	0.0133	0.0237	0.2768
1	3	10	0.7327	0.0234	0.1998	0.0207	0.0234	0.2953
		15	0.7448	0.0218	0.1921	0.0179	0.0236	0.2850
		20	0.7508	0.0210	0.1880	0.0165	0.0236	0.2797
	6	10	0.7372	0.0228	0.2001	0.0165	0.0235	0.2913
		15	0.7478	0.0214	0.1922	0.0150	0.0236	0.2823
		20	0.7531	0.0207	0.1882	0.0143	0.0236	0.2777
	9	10	0.7387	0.0226	0.2002	0.0150	0.0235	0.2900
		15	0.7488	0.0213	0.1923	0.0140	0.0236	0.2814
		20	0.7539	0.0206	0.1883	0.0136	0.0236	0.2770

Table 1 System Measures by varying β,ϕ and α

Table 1 shows the effect of β,ϕ,α on the performance measures E_0, E, R, F, V and L_s. It is clearly observed that

- E_0 increases with increase in ϕ and α and decreases with increase in β.
- E and L_s increases with increase in β and decreases with increase in ϕ and α.
- R decreases with increase in β and α and increases with increase in ϕ.
- F increases with increase in β and decreases with increase in ϕ and α.
- V increases with increase in α, decreases with increase in β and independent of ϕ.

Influence of parameters Λ and δ on E_0, E and L_s are presented in Figures 1.1 to 1.3. E_0 decreases with increase in Λ and increases with increase in δ. E increases with increase in Λ and decreases with increase in δ. L_s increases for both the increasing values of Λ and δ.

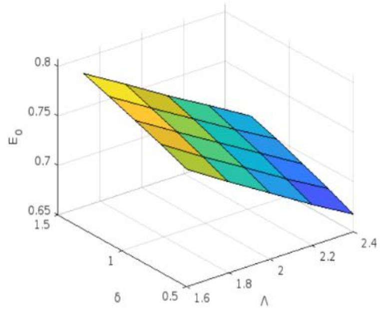

Figure 1.1 Effects of (Λ,δ) on E_0

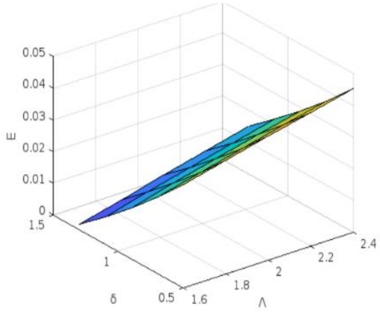

Figure 1.2 Effects of (Λ,δ) on E

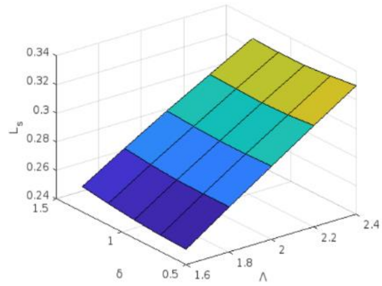

Figure 1.3 Effects of (Λ,δ) on L_s

The effect of parameters β_1 and α_1 on the system measures are given in Figures 1.4 to 1.6. E_0 decreases with increase in β_1 and increases with increase in α_1. E increases with increase in β_1 and decreases with increase in α_1. L_s increases with increase in β_1 and decreases with increase in α_1.

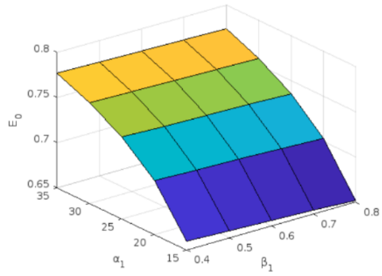

Figure 1.4 Effects of (β_1,α_1) on E_0

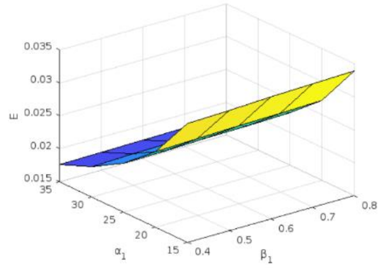

Figure 1.5 Effects of (β_1,α_1) on E

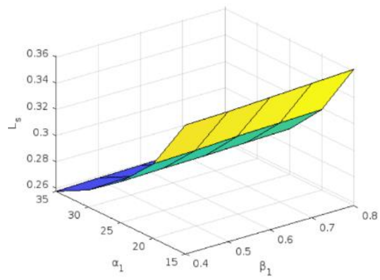

Figure 1.6 Effects of (β_1,α_1) on L_s

4. Conclusion

In this article, batch arrival retrial queueing system with essential and optional phases of service, customer impatience, breakdown, feedback mechanism, Bernoulli vacation and customer search behavior are discussed. Performance measures like the probability that the server is idle, busy, repair in all three phases of service and the server is on vacation are derived. The expected sizes retrial group and the system are derived. Reliability measures such as availability and failure frequency are also obtained. The present investigation can be further extended to cost optimization.

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