Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 4s (2025)

Multi Fuzzy Iterated Function System Consisting of Generalized Fuzzy Contraction Mapping in Fuzzy Metric Spaces

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Article History:
Received: *30-09-2024*
Revised: *28-11-2024*
Accepted: *09-12-2024*

Abstract:
The research work in this paper combines two concepts Iterated Function System and Multi Fuzzy Fractal Theory in the formulation of a new Iterated Function System, Multi Fuzzy Fractal Iterated Function System. More precisely, some uniqueness and existence of the attractor were obtained in the new multi fuzzy fractal spaces. Some illustrative examples are given that will be useful for this research purpose. From then on, our results are generalizations of some recent results provided in the literature. We define a methodology for the construction of multi fuzzy IFS in multi fuzzy fractal space consisting of a generalized contraction mapping.

Keywords: $\mathcal{H}$–contraction; attractor; Iterated Function System; Multi Iterated Function System; multi fuzzy fractal space.

1. Introduction

A fuzzy set was invented with a new approach in 1965 by Zadeh [1] as a development to the standard concept of sets and to understand the degree of uncertainty in daily life in purely mathematical style. Kramosil and Machalek [2] estimated the uncertainty in measuring the distance between two sets. George and Veeramani [3, 4] improved their hypothesis by defining the best circumstances in the same setting of the space as one way to get a countable and Hausdorff topology. Grabiec [5] generalized and celebrated the theory of Banach contraction for fuzzy metric space. Lopes and Romaguera [6] established different axioms on compact sets for Hausdorff fuzzy metric space. Qui and Zang [7] established that for a given positive continuous norm, there exists a setting of space for which the given norm is the strongest. Qui et al. [8, 9] obtained interesting results on fixed points for the same setting of space, and Qui et al. [10] generalized the standard Hausdorff metric with a norm and studied its standard axioms. Consequently, various mathematicians have presented and analyzed certain concepts of the same framework of space in distinct forms and also established fixed point results that satisfy a few general contractive conditions, including fascinating consequent theories in the framework of the new notion see, for example, [11, 12, 13, 14, 15, 16, 17].

Fractals and multivalued fractals have broad applications such as image compression, computer graphics, soil mechanics, the digital photography, quantum physics, biomedicine, fluid mechanics etc. The notion of fractal theory was introduced in 1975 by Mandelbrot [18]. The terminologies fractals are defined as a set of a few self-similar pieces that acquire similar features in distinct measures. More precisely, image compression has included some usage of the scheme of IFS on complete metric space. Hutchinson in 1981 gives the definition of IFS [19] which is further investigated by Barnsley, Demko and others [20, 21, 22, 23, 24]. Hutchinson-Barnsley establish remarkable concept of Hutchinson-Barnsley theory and presented the HB operator [19, 20]. By using Banach contraction principle an attractor was established through IFS. In 2009, Singh et al. [25] presented fractals on IFS and discussed the novel concept of multi-functions. Sahu et.al. establish KIFS and collage theorem for Kanan mapping in complete metric space [26]. S C Shrivastava et.al. established IFS for two mappings and commuting mapping and by introducing D-metric space [27, 28]. Iterated Function system established for Riech mapping by many authors [29, 30, 31, 32]. By changing different space and mappings different attractors and IFS was established by many researches. The notion of fuzzy theory is also extended to fractals. Recently, Easwara Moorty and Uthaya Kumar [33, 34] studied the HB operator in fuzzy metric space and introduced an investigation on fractals in such spaces.

A union of a set of many images with distinct scales is known as a multi-fractal. Initially, multifractals were analyzed by physicists in the 1980s. Many authors were attracted to this research area, and they studied and published their research work in this field. In 2001, Al-Shameri [35] presented a generalization of IFS to the new setup of the space. In 2009, Al-Saidi et al. [36] presented a novel generalized space known as the Multi-fuzzy fractal space. In 2011, Uthayakumar and Easwaramoorthy [37] proposed an improvement of HB theory for an IMS of fuzzy contractions in such a system. Recently, Prasad and Katiyar [38] studied HB operators for MIFS in a general setting, and they obtained existing results in fuzzy fractal spaces and multifuzzy fractal spaces.

The intention of this research is to establish a novel MFIFS consisting of generalized fuzzy contraction mapping in a general setting. We extend the result obtained by Uthaya kumar and Gowri shankar [39] in 2015 to the condition of a new type Multi fuzzy IFS consisting of -contractive mapping defined on multifractal space. A generalized contraction was initiated and developed by Wardowski [40] in 2013. This paper is divided into five sections. In the second section includes basic concepts, result and hypothesis for fractals and multi fractals. Third section contains properties of generalized contraction mapping and their attractor for fuzzy metric space. The fourth section, contains a methodology to construct the Multi fuzzy Iterated Function System (MFIFS) consisting of generalized contraction mapping in the sense of Wardowski [40], Al-Saidi et al. [36], and Prasad and Katiyar [38] and prove their some consequent results. In the fifth section, we conclude our outcome.

2. Preliminaries

A. Attractor for fuzzy Metric Space

In this section we discuss required concepts and properties of fuzzy space and fuzzy fractal space used for our findings.

Definition 2.1 [3, 4]: Fuzzy metric space is denoted by the triplet $(X, M, *)$ if X is an arbitrary set, M is a fuzzy set and $*$ is a continuous t-norm on $X \times X \times (0, \infty)$ satisfies the following properties:

1. $M(x, y, t) > 0$.
2. $M(x, M, t) = 1$ iff $x = y$.
3. $M(x, y, t) = M(y, x, t)$.
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.
5. $M(x, y, t.): (0, \infty) \rightarrow [0,1]$ is continuous $x, y, z \in X$ and $s, t > 0$.

Definition 2.2 [3, 41] In fuzzy metric space $(X, M, *)$,

(i) A sequence $\{x_n\}$ is a convergent sequence at a point $x \in X$ with respect to τ_M iff $\lim_{n\to\infty} M(x, x_n, t) = 1$, for every $t > 0$. where τ_M is the topology produced by the fuzzy metric space $(X, M, *)$)

(ii) A sequence $\{x_n\}$ is known as Cauchy sequence if for any $\varepsilon \in (0,1), t > 0, \exists\, n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m > n_0$.

(iii) A fuzzy contraction serves as a map w on X, such that

$$\frac{1}{M(w(x), w(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

for every $x, y \in X, t > 0$, where $0 < k < 1$. The number k is then said to be a fuzzy contractivity ratio of w.

Fuzzy Banach fixed point theorem is presented in formal way in 2002 by Gregori and Sapena [41].

Theorem 2.1[41] Let $(X, M, *)$ be a complete fuzzy metric space for fuzzy contractive mapping $w_n: X \to X$ with contractivity factor k then w_n possesses a unique fixed point.

Definition 2.3[6] Let $\mathcal{K}(X)$ be a set of all non-empty compact subsets of X and $(X, M, *)$ be a fuzzy metric space then Hausdorff fuzzy metric $H_M: \mathcal{K}(X) \times \mathcal{K}(X) \times (0, \infty) \to [0,1]$ defined by

$$H_M(A, B, t) = \min\{M(A, B, t), M(B, A, t)\}$$

where $M(X, B, t) = \sup_{y\in B} M(x, y, t)$ and $M(A, B, t) = \inf_{x\in A} M(x, B, t)$ for every $x \in X$ and $A, B \in \mathcal{K}(X)$.

Then $(\mathcal{K}(X), H_M, *)$ is known as Hausdorff fuzzy metric space where H_M is a fuzzy metric on $\mathcal{K}(X)$.

In 2011, Uthaya kumar and Easwarmoorthy [33] discussed basic properties of HB operator and defined the FIFS and FHB operator for fuzzy metric space.

Lemma 2.1[33] Let $(X_j, M_j, *)$ and $(\mathcal{K}(X_j), H_{M_j}, *)$ for $j = 1, - - - -, N$ be fuzzy metric space and corresponding Hausdorff fuzzy metric space. If $A_j, B_j, C_j, D_j \subset X_j$, for $j = 1, - - - -, N$ then for $t > 0$, $H_{M_j}(A_j \cup B_j, C_j \cup D_j, t) \geq \min\left\{H_{M_j}(A_j, C_j, t), H_{M_j}(B_j, D_j, t)\right\}$, $j = 1, - - - -, N$.

Definition 2.4[33] The system $\{X; w_n, n = 1, - - - -, N\}$ is known as Fuzzy IFS for fuzzy metric space $(X, M, *)$ fuzzy contractive mapping $w_n: X \to X$, with the corresponding contractivity factors $0 < k_n < 1$, where $n = 1, - - - -, N$.

Definition 2.5 [33] Suppose that $(X, M, *)$ be fuzzy metric spaces. Let $\{X; w_n, n = 1, - - - -, N\}$ be a FIFS. Then the Fuzzy HB operator is a function $W: \mathcal{K}(X) \to \mathcal{K}(X)$ such that

$$W(B) = \cup_{n=1}^{N} w_n(B) \qquad \text{for all } B \in \mathcal{K}(X).$$

Definition 2.6 [33] Let $\{X, w_n, n = 1, - - - -, N\}$ be a Fuzzy Iterated Function System (FIFS) for fuzzy metric space $(X, M, *)$ and W be the Fuzzy HB operator of the FIFS. Then there exists unique attractor (fractal) of FIFS, $\mathcal{A}_\infty \in \mathcal{K}(X)$ of the FHB operator W.

B. Attractor in Multi Fuzzy fractal space

This segment defines concepts, properties and consequences of multi-fractal spaces and multi-fuzzy fractal spaces which is required for our outcome.

Definition 2.7[36] Let $\left(X_j, d_j\right)$ be a metric space for each $j \in J$ where J is an indexed set. The product space $\mathcal{X} = \prod_{j\in J} X_j$ is the space that consists of all J-tuples $\left\{X_j\right\}_{j\in J}$ with $x_j \in X_j$. Define a metric $\bar{d}: \mathcal{X} \to R$ as $\bar{d}(x,y) = \max_{j=1,----,N} d_j\left(x_j, y_j\right)$ for all $x, y \in \mathcal{X}$ where $x = (x_1, x_2, ----, x_n), y = \left(y_1, y_2, ----, y_n\right)$ and $x_j, y_j \in X_j$ for $j = 1, ----, N$. Then $\left(\mathcal{X}, \bar{d}\right)$ is a metric space with product metric $\bar{d}$.

Al-Saidi et al [36] presented a basic definition of a multi fuzzy fractal space in the following way.

Definition 2.8[36]Suppose $\mathcal{K}(\mathcal{X}) = \prod_{j=1}^{N} \mathcal{K}\left(X_j\right) = \mathcal{K}(X_1) \times \mathcal{K}(X_2) \times ---- \times \mathcal{K}(X_N)$. Then $(\mathcal{K}(\mathcal{X}), H_{\bar{d}})$ is said to be a multi fuzzy fractal space with metric $H_{\bar{d}}$ represented as

$$H_{\bar{d}}(\mathcal{A}, \mathfrak{B}) = \max_{j=1,2,---,N} \left\{H_{d_j}\left(A_j, B_j\right)\right\}$$

for $\mathcal{A}, \mathfrak{B} \in \mathcal{K}(\mathcal{X})$, where $\mathcal{A} = (A_1, ----, A_N)$ and $\mathfrak{B} = (B_1, ----, B_N)$, H_{d_j} is Hausdorff distance between two sets A_j and B_j.

Theorem2.2[42]: $(\mathcal{K}(\mathcal{X}), H_{\bar{d}})$ be a complete metric space iff $\left(\mathcal{K}\left(X_j\right), H_{d_j}\right)$, where $j = 1, ---, N$ is a complete metric space. Then

Theory of multi IFS and the HB operator of MIFS may be expressed in the following sense.

Definition 2.9[36]: The system

$\left\{X_j, j = 1, ----, N;\ w_{ij}^l: X_j \to X_i, l = 1, ---, m_{ij} \text{and} i, j = 1, ---, N\right\}$ is multi iterated function system with contractivity factor ratio $s = \max\left\{s_{ij}^l, l = 1, ----, m_{ij} \text{ and } i, j = 1, ----, N\right\}$ for complete metric space $\left(X_j, d_j\right), j = 1,2, ----, N$ and contraction mapping $w_{ij}^l: X_j \to X_i$, $l = 1, ---, m_{ij}$ and $i, j = 1, ---, N$.

Definition 2.10[36]: Let in a complete metric space $\left(X_j, d_j\right), j = 1,2, ----, N$, $X_j, j = 1,2, ----, N;\ w_{ij}^l: X_j \to X_i, l = 1, ----, m_{ij}$ and $i, j = 1, ----, N\}$ be an Multi Iterated Function system. Then, the HB operator of MIFS is defined by

$$W(\mathfrak{B}) = \prod_{i=1}^{N} \bigcup_{j=1}^{N} \bigcup_{l=1}^{N} w_{ij}^l\left(B_j\right), \qquad \text{for all } B \in \mathcal{K}(\mathcal{X}).$$

Where function $W: \mathcal{K}(\mathcal{X}) \to \mathcal{K}(\mathcal{X})$Al-Saidi et al [36] consider the above notations and they presented the following result as follows, Suppose $(X_j, d_j), j = 1,2, ----, N$ be given complete, metric spaces. Let $\{X_j, j = 1,2, ----, N;\ w_{ij}^l: X_j \to X_i, l = 1, ----, m_{ij}$ and $i, j = 1, ----, N\}$ be an MIFS.

Theorem 2.3[36]: Consider N complete metric spaces $(X_j, d_j), j = 1, ----, N$. Let $\{X_j, j = 1, ---, N; w_{ij}^l: X_j \to X_i, l = 1, ----, m_{ij}$ and $i, j = 1, ----, N\}$ be an MIFS. Then, the HB operator W is a contraction mapping on $(\mathcal{K}(\mathcal{X}), H_{\bar{d}})$.

In 2017, B. Prasad and K. Katiyar [38] studied the results of Al-Saidi et al. [36] and established the following HB operators in multi-fractal spaces.

Theorem 2.4[38] Suppose (X_j, d_j), $j = 1, ----, N$ are complete metric spaces and $\{X_j, j = 1, -----, m_{ij}$ and $i, j = 1, ----, N;\ w_{ij}^l: X_j \to X_i, l = 1, ----, m_{ij}$ and $i, j = 1, ----, N\}$ be an MFIFS. Then, there exists $\mathcal{A}_\infty \in \mathcal{K}(\mathcal{X})$ known as the attractor of MFIFS, of HB operator W.

George and Veeramani [4] are presented a multi fuzzy fractal space in the following manner.

Definition 2.11[4] Let a complete metric space $(X_j, M_j, *), j = 1, ----, N$ and $\left(\mathcal{K}(X_j), H_{M_j}, *\right), j = 1, ----, N$ is associated Hausdorff fuzzy metric spaces with same t-norm then the space $(\mathcal{K}(\mathcal{X}), H_{\mathcal{M}}, *)$ with metric $(H_M, *)$ is called a multi fuzzy fractal space iff $\mathcal{K}(\mathcal{X}) = \mathcal{K}(X_1) \times \mathcal{K}(X_2) \times ---- \times \mathcal{K}(X_N)$ represented by

$$H_{\mathcal{M}}(\mathcal{A}, \mathfrak{B}, t) = \min\left\{H_{M_j}(A_j, B_j, t), j = 1, ----, N\right\}$$

for all $A_j, B_j \in \mathcal{K}(X_j), \mathcal{A}, \mathfrak{B} \in \mathcal{K}(\mathcal{X})$, where $\mathcal{A} = (A_1, ----, A_N), \mathfrak{B} = (B_1, B_2, ----, B_N)$.

Theorem 2.5[38] The metric space $(\mathcal{K}(\mathcal{X}), H_{\mathcal{M}}, *)$ is a complete fuzzy metric spaces iff $\left(\mathcal{K}(X_j), H_{M_j}, *\right)$ is complete for each $j = 1,2,3, ----, N$.

Prasad and Katiyar [38] extended the results of Al-Saidi et al [36] and they define the MFIFS and HB operator of MFIFS in the following manner.

Definition 2.12[38]: Suppose that $(X_j, M_j, *), j = 1, ----, N$ be complete fuzzy metric spaces. Let $w_{ij}^l: X_j \to X_i, l = 1, ----, m_{ij}$ and $i, j = 1, ----, N$ be fuzzy B-contractions with the corresponding contraction factors $0 < s_{ij}^l < 1$, where $l = 1, ----, m_{ij}$ and $i, j = 1, ----, N$. Then the system $\{X_j, j = 1,2, ----, N;\ w_{ij}^l: X_j \to X_i, l = 1, ----, m_{ij}$ and $i, j = 1,2, ----, N\}$ is said to be a MFIFS along contraction ratio $s = \max\{s_{ij}^l,\ l = 1, ----, m_{ij},\ i, j = 1, ----, N\}$.

Definition 2.13[38] In a complete fuzzy metric space $(X_j, M_j, *), j = 1, ----, N$ assume that $\{X_j, j = 1,2, ---, N; w_{ij}^l: X_j \to X_i, l = 1, ----, m_{ij}$ and $i, j = 1, ----, N\}$ be an MFIFS. Then

$$W(\mathfrak{B}) = \prod_{i=1}^{N} \bigcup_{j=1}^{N} \bigcup_{l=1}^{m_{ij}} w_{ij}^l(B_j) \qquad \text{for all } \mathfrak{B} \in \mathcal{K}(\mathcal{K}).$$

Is called multi fuzzy Hutchinson-Barnsley (MFHB) operator for a function $W: \mathcal{K}(\mathcal{X}) \to \mathcal{K}(\mathcal{X})$.

B. Prasad and K. Katyar [38] presented the fixed point theorem in these way.

Theorem 2.6[38]: Let $(X_j, M_j, *), j = 1, ----, N$ be complete fuzzy metric spaces. Let $\{X_j, j = 1,2, ---, N; w_{ij}^l : X_j \to X_i, l = 1, ----, m_{ij}$ and $i, j = 1, ----, N\}$ be an MFIFS with contractivity ratios $= \max\{s_{ij}^l, l = 1, ----, m_{ij}$ and $i, j = 1, ----, N\}$ and W be the MFHB operator of the MFIFS. Then, there exists only one compact invariant set, also called the attractor (fractal) of MFIFS, $\mathcal{A}_\infty \in \mathcal{K}(X)$ of the MFHB operator W.

1. Generalized fuzzy contraction

We mainly discuss in this section the notion of generalized fuzzy contraction presented by Wardowski [40]. We also discussed the $\mathcal{H}$-IFS theory introduced by R. Uthayakumar and A. Gorishankar [39]. They obtained the attractor as a compact invariant subset generated by the Banach fixed point theorem [43] in M-complete metric space.

Properties of H-contraction

Let $\mathcal{H}$ represents a set of mappings $\eta: (0,1] \to [0, \infty)$ and $s > t \Rightarrow \eta(s) < \eta(t)$ for every $s, t \in (0,1]$.

Definition 3.1[40] In a fuzzy metric space $(X, M, *)$ contraction mapping $f: X \to X$ is known as $\mathcal{H}-$ contractive such that

$$\eta\big(M(f(x), f(y), t)\big) \leq k\eta\big(M(x, y, t)\big) \qquad (1)$$

Where $\eta \in \mathcal{H}$ and $k \in (0,1)$ for every $x, y \in X$ and $t > 0$.

Example 3.1[40] Let $\eta \in \mathcal{H}$ be a mapping and defined by $\eta(t) = \frac{1}{t} - 1, t \in (0,1]$. When the equation (1) changes to

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

for all $x, y \in X$ and $t > 0$.

Proposition 3.1[40] Suppose that $(X, M, *)$ be a fuzzy metric space and $\eta \in \mathcal{H}$. A sequence $\{x_n\}_{n \in N} \subset X$ is M-Cauchy iff for given $\varepsilon > 0, t > 0$ there exists $n_0 \in N$ such that

$$\eta\big(M(X_m, X_n, t)\big) < \varepsilon,$$

for all $m, n \geq n_0$.

Proposition 3.1[40] Suppose that $(X, M, *)$ be a fuzzy metric space and $\eta \in \mathcal{H}$. A sequence $\{x_n\}_{n \in N} \subset X$ is convergent to $x \in X$ iff $\lim_{n \to \infty} \eta\big(M(x_n, x, t)\big) = 0$ for all $t > 0$.

Theorem 3.1[39] Let $(\mathcal{K}(X), H_M, *)$ is a Hausdorff fuzzy metric space in fuzzy metric space $(X, M, *)$ then mapping $w: X \to X$ is a fuzzy $\mathcal{H}$-contraction mapping with respect to $\eta \in \mathcal{H}$ on $(X, M, *)$.

A. Iterated Function System consisting of H-contraction

Definition 3.2[39] Suppose that$(X, M,*)$ be a fuzzy metric space and $w_n: X \to X, n = 1,---,N$ be $N - \mathscr{H}$ contractive mappings. Then the system $\{X; w_n, n = 1,---,N\}$ is called a $\mathscr{H}$-IFS of $\mathscr{H}$–contractions in $(X, M,*)$. The HB operator of the $\mathscr{H}$-IFS is a function $W: \mathcal{K}(X) \to \mathcal{K}(X)$ defined by

$$W(B) = \bigcup_{n=1}^{N} w_n(B), \qquad \text{for all } B \in \mathcal{K}(X)$$

Definition 3.3 [39] Suppose that$(X, M,*)$ be a fuzzy metric space. Let $\{X, w_n, n = 1,2,3,-----,N\}$ be a $\mathscr{H}$-IFS and W be the HB operator of the $\mathscr{H}$-IFS. If W has a unique fixed point A_∞ in $(X, M,*)$, then the set $\mathcal{A}_\infty \in K_0(X)$ is known as the Attractor (or Fractal) produced by the $\mathscr{H}$-IFS of $\mathscr{H}$-contraction.

3. Main Results

Now, we define and establish multifuzzy fractals (attractors) consisting of generalized fuzzy contractions defined on multifuzzy fractal space. Our research work starts with the corresponding result.

Theorem4.1 Assume that $(X_j, M_j,*)$ be a complete fuzzy metric spaces for $j = 1,----,N$ with same t-norm. Let $\left(\mathcal{K}(X_j), H_{M_j},*\right)$ be the corresponding Hausdorff fuzzy fractal spaces. If $w_{ij}: X_j \to X_i, i, j = 1,----,N$ is a fuzzy $\mathscr{H}$-contraction with$\eta \in \mathscr{H}$ and $k_{ij} \in (0,1)$ for $i, j = 1,2,---,N$ on $\left(\mathcal{K}(X_j), H_{M_j},*\right)$. Then w_{ij} is a fuzzy contraction with $\eta \in \mathscr{H}$ and $k_{ij} \in (0,1)$ on $\left(\mathcal{K}(X_j), H_{M_j},*\right)$. That is;

$$\eta\left(H_{M_i}\left(w_{ij}(B_j), w_{ij}(A_j), t\right)\right) \le k_{ij}\eta\left(H_{M_i}(B_i, A_i, t)\right), \qquad \forall A_j, B_j \in \mathcal{K}(X_j)$$

Proof: Let $t > 0$ be a fixed.

Let $A_i, B_i \in \mathcal{K}(X_i)$ and $A_j, B_j \in \mathcal{K}(X_j)$.

We know that, w_{ij} is a fuzzy $\mathscr{H}$-contraction with $\eta \in \mathscr{H}$ on $(X, M,*)$.

Thus, $\exists k_{ij} \in (0,1)$ so that

$$\eta\left(M_i\left(w_{ij}(x_j), w_{ij}(y_j), t\right)\right) \le k_{ij}\eta\left(M_i(x_i, y_i, t)\right), \qquad \forall x_i, y_i \in X_i, \quad x_j, y_j \in X_j$$

$$\eta\left(M_i\left(w_{ij}(x_j), w_{ij}(y_j), t\right)\right) \le k_{ij}\eta\left(M_i(x_i, y_i, t)\right), \qquad \forall x_i \in A_i, y_i \in B_i \text{ and } x_i \in A_j, y_j \in B_j$$

$$\eta\left(\sup_{y_j \in B_j} M_i\left(w_{ij}(x_j), w_{ij}(y_j), t\right)\right) \le k_{ij}\eta\left(\sup_{y_i \in B_i} M_i(x_i, y_i, t)\right) \quad \forall x_i \in A_i, y_i \in B_i, \text{and } x_j \in A_j, y_j \in B_j$$

$$\eta\left(M_i\left(w_{ij}(x_j), w_{ij}(B_j), t\right)\right) \le k_{ij}\eta\left(M_i(x_i, B_i, t)\right), \qquad \forall x_i \in A_i, x_j \in A_j$$

$$\eta\left(\inf_{x_j \in A_j} M_i\left(w_{ij}(x_j), w_{ij}(B_j), t\right)\right) \leq k_{ij}\eta\left(\inf_{x_i \in A_i} M_i(x_i, B_i, t)\right)$$

$$\eta\left(M_i\left(w_{ij}(A_j), w_{ij}(B_j), t\right)\right) \leq k_{ij}\eta\left(M_i(A_i, B_i, t)\right) \quad (1)$$

Similarly, we obtain

$$\eta\left(M_i\left(w_{ij}(B_j), w_{ij}(A_j), t\right)\right) \leq k_{ij}\eta\left(M_i(A_i, B_i, t)\right) \quad (2)$$

Therefore, from (1) and (2), we can say

$$\eta\left(H_{M_i}\left(w_{ij}(B_j), w_{ij}(A_j), t\right)\right) \leq k_{ij}\eta\left(H_{M_i}(B_i, A_i, t)\right).$$

Thus, the desired verification completes.

Corollary 4.1 Consider$(X, M,*)$ be a fuzzy metric space for $j = 1,2,---,N$ along same t-norm. Suppose$\left(\mathcal{K}(X), H_{M,} *\right)$ be the corresponding Hausdorff fuzzy metric space. Assume that$w: X \to X$ be a fuzzy $\mathscr{H}$-contraction function with $\eta \in \mathscr{H}$and $k \in (0,1)$on $(X, M,*)$.If w is a fuzzy contraction with$\eta \in \mathscr{H}$and $k \in (0,1)$on $(X, M,*)$. Then w is a fuzzy contraction with $\eta \in \mathscr{H}$ and $k \in (0,1)$ on $\left(\mathcal{K}(X_j), H_{M_j},*\right)$. That is

$$\eta\left(H_M(w(B), w(A), t)\right) \leq k\eta\left(H_M(B, A, t)\right) \qquad \forall\, A, B \in \mathcal{K}(X) \text{ and } t > 0.$$

Now, we define the contractive operator for MFIFS provided by Prasad and Katiyar [38]. We also explain the uniqueness and existence results of the MFIFS in the setup of fuzzy metric spaces. Our obtained results are inspired by the results of Al-Saidi [36] and Prasad and Katiyar [38].

Definition 4.1 Assume that $\left(X_j, M_j,*\right)$ be a complete fuzzy metric space for $j = 1,---,N$ with same t-norm. Let $w_{ij}^l: X_j \to X_i,\ l = 1,----,m_{ij}$ and $i,j = 1,----,N$be fuzzy $\mathscr{H}$-contractions with$\eta \in \mathscr{H}$ and $k_{ij}^l \in (0,1)$ where $l = 1,2,---,m_{ij};\ i,j = 1,----,N.$ Thenthesystem $\{X_j, j = 1,----,N;\ w_{ij}^l: X_j \to X_i, l = 1,----,m_{ij}$ and $i,j = 1,----,N\}$ is said to bea $\mathscr{H}$-MFIFS on$(X, M,*)$with $\eta \in \mathscr{H}$ and $k = \max\{k_{ij}^l, l = 1,2,----,m_{ij},\ i,j = 1,2,----,N\}$.

Definition4.2 Suppose that$\left(X_j, M_j,*\right)$ be a complete fuzzy metric space for $j = 1,----,N$ be complete metric spaces. Let $\{X_j,\ j = 1,----,N;\ w_{ij}^l: X_j \to X_i,\ l = 1,----,m_{ij}$ and $i,j = 1,----,N\}$ be an $\mathscr{H}$–MFIFS. Then the MFHB operator of the $\mathscr{H}$-IFS is a function $W: \mathcal{K}(\mathcal{X}) \to \mathcal{K}(\mathcal{X})$ such that

$$W(\mathfrak{B}) = \prod_{i=1}^{N} \bigcup_{j=1}^{N} \bigcup_{l=1}^{m_{ij}} w_{ij}^l\left(B_j\right) \qquad \forall\, \mathfrak{B} \in \mathcal{K}(\mathcal{X}).$$

Theorem4.2 Suppose that $\left(X_j, M_j,*\right), j = 1,2,-----,N;$ be complete fuzzy metric spaces. Let $\{X_j, j = 1,2,----,N;\ w_{ij}^l: X_j \to X_i, l = 1,----,m_{ij}$ and $i,j = 1,----,N\}$ be an $\mathscr{H}$-MFIFS with $\eta \in \mathscr{H}$ and $k = \max\{k_{ij}^l, l = 1,----,m_{ij}$ and $i,j = 1,----,N\}$.Then, W is a fuzzy $\mathscr{H}$-contraction on $(\mathcal{K}(X), H_{\mathscr{M}},*)$.

Proof: Fix $t > 0$. Assume that $\mathfrak{B}, \mathcal{C} \in \mathcal{K}(X)$. Then we know that

$$\eta\big(H_{\mathscr{M}}(W(\mathfrak{B}), W(\mathcal{C}), t)\big) = \eta\big(H_{\mathscr{M}}(W(B_1, ----, B_N), W(C_1, ----, C_N), t)\big)$$

$$= \eta\left(H_{\mathscr{M}}\left(\prod_{i=1}^{N}\bigcup_{j=1}^{N}\bigcup_{l=1}^{m_{ij}} w_{ij}^{l}(B_j), \prod_{i=1}^{N}\bigcup_{j=1}^{N}\bigcup_{l=1}^{m_{ij}} w_{ij}^{l}(C_j), t\right)\right) \quad \forall\, B_j, C_j \in X_j$$

$$= \min_{i\in\{1-,---,N\}} \eta\left(H_{M_i}\left(\bigcup_{j=1}^{N}\bigcup_{l=1}^{m_{ij}} w_{ij}^{l}(B_j), \bigcup_{j=1}^{N}\bigcup_{l=1}^{m_{ij}} w_{ij}^{l}(C_j), t\right)\right) \quad \forall\, B_j, C_j \in X_j$$

(By Definition of $H_{\mathscr{M}}$)

$$= \min_{i\in\{1,----,N\}}\left(\min_{j\in\{1,----,N\}} \eta\left(H_{M_{ij}}\left(\bigcup_{l=1}^{m_{ij}} w_{ij}^{l}(B_j), \bigcup_{l=1}^{m_{ij}} w_{ij}^{l}(C_j), t\right)\right)\right) \quad \forall\, B_j, C_j \in X_j$$

(from Lemma (2.1))

$$= \min_{i\in\{1,----,N\}}\left(\min_{j\in\{1,----,N\}}\left(\min_{l\in\{1,----,m_{ij}\}} \eta\left(H_{M_{ij}^{l}}\left(w_{ij}^{l}(B_j), w_{ij}^{l}(C_j), t\right)\right)\right)\right) \quad \forall\, B_j, C_j \in X_j$$

(from Lemma (2.1))

$$\leq \min_{i\in\{1,----,N\}}\left(\min_{j\in\{1,----,N\}}\left(\min_{l\in\{1,----,m_{ij}\}} k_{ij}^{l}\, \eta\left(H_{M_{ij}^{l}}(B_i, C_i, t)\right)\right)\right) \forall\, B_i, C_i \in X_i$$

(by theorem (4.1))

$$\leq \min_{i\in\{1,----,N\}}\left(\min_{j\in\{1,----,N\}} k_{ij}\eta\left(H_{M_{ij}}(B_i, C_i, t)\right)\right) \quad \forall\, B_i, C_i \in X_i$$

$$\leq \min_{i\in\{1,----,N\}} k_i\eta\left(H_{M_i}(B_i, C_i, t)\right) \quad \forall\, B_i, C_i \in X_i$$

$$\leq k\eta\big(H_{\mathscr{M}}(\mathfrak{B}, \mathcal{C}, t)\big)$$

Thus, the desired verification completes.

Corollary 4.2 Suppose that $(X, M.*)$ be a complete fuzzy metric space and $(\mathcal{K}(X), H_M, *)$ be the corresponding Hausdorff fuzzy metric space. Let $w_i: X \to X, i = 1, ----, N$ be a fuzzy $\mathscr{H}$-contraction on $(X, M, *)$. Then the fuzzy Hutchinson-Barnsley operator is also a $\mathscr{H}$-contraction on $(\mathcal{K}(\mathcal{X}), H_M, *)$.

Theorem 4.3 Let $(X_j, M_j, *), j = 1,2, ---, N$ be complete fuzzy metric spaces. Let $\{X_j,\ j = 1,2, ---, N;\ w_{ij}^{l}: X_j \to X_i, l = 1, ----, m_{ij}\ and\ \ i, j = 1, ----, N\}$ be a $\mathscr{H}$-MFIFS with $\eta \in \mathscr{H}$ and $k = max\{k_{ij}^{l}, l = 1, ----, m_{ij}\ and\ i, j = 1, ---- N\}$. Let W be the $\mathscr{H}$-MFHB operator of the $\mathscr{H}$-MFIFS. Then there exists only one compact invariant set $\mathcal{A}_\infty \in \mathcal{K}(X)$, also called the attractor (fractal) of the $\mathscr{H}$-MFHB operator W or, equivalently, W has a unique fixed point namely $\mathcal{A}_\infty \in \mathcal{K}(\mathcal{X})$.

Proof: Let$(X_j, M_j,*), j = 1,2,----,N$ are complete fuzzy metric spaces. Then$(\mathcal{K}(\mathcal{X}), H_{\mathscr{M}},*)$ is also a complete Hausdorff multi fuzzy metric space by theorem (2.5). Also the MFHB operator W is a fuzzy $\mathscr{H}$-contraction by theorem (4.2). Hence, we conclude that W has a unique fixed point, namely $\mathcal{A}_\infty \in K_0(X)$ by generalized Fuzzy Banach contraction theorem (3.3).

Example 4.1 Suppose that $X_1 = \left\{3^{3^n}: n \in N\right\} \cup \{3\}, X_2 = \left\{4^{4^n}: n \in N\right\} \cup \{4\}$ and let t-norm $*$ be the frequent product.

Assume that, a fuzzy set $M_1: X_1^2 \times (0,\infty] \to [0,1]$ and $M_2: X_2^2 \times (0,\infty] \to [0,1]$, given by the formula;

$$M_1(x,y,t) = \begin{cases} \frac{x}{y}\ ; \ ifx \le yforallt \ge 0 \\ \frac{y}{x}; \ ify \le xforallx, y \in X_1 \end{cases}$$

and

$$M_2(x,y,t) = \begin{cases} \frac{x}{y}\ ; \ ifx \le yforallt \ge 0 \\ \frac{y}{x}; \ ify \le xforallx, y \in X_2 \end{cases}$$

Then, $(X_j, M_j,*),\ j = 1,2$ are complete fuzzy metric spaces.

Now, define $w_{11}^1: X_1 \to X_1$ as $w_{11}^1\left(3^{3^n}\right) = 3^{3^{n-1}}$ and $w_{22}^1: X_2 \to X_2$ as $w_{22}^1\left(4^{4^n}\right) = 4^{4^{n-1}}$.

Moreover, for every $m, n \in N, m < n \ and \ t > 0$ we have

$$\eta\left(H_M\left(w_{11}^1\left(3^{3^m}\right), w_{11}^1\left(3^{3^n}\right), t\right)\right) = ln\left(3^{3^{n-1}}/3^{3^{m-1}}\right)$$

$$= \frac{1}{3} ln\left(3^{3^n}/3^{3^m}\right)$$

$$= \frac{1}{3}\eta\left(H_M\left(3^{3^m}, 3^{3^n}, t\right)\right)$$

and,

$$\eta\left(H_M\left(w_{11}^1(3), w_{11}^1\left(3^{3^n}\right), t\right)\right) = ln\left(3^{3^{n-1}}/3\right)$$

$$< \frac{1}{3}\eta\left(H_M\left(3, 3^{3^n}, t\right)\right)$$

Similarly;

$$\eta\left(H_M\left(w_{22}^1\left(4^{4^m}\right), w_{22}^1\left(4^{4^n}\right), t\right)\right) = ln\left(4^{4^{n-1}}/4^{4^{m-1}}\right)$$

$$= \frac{1}{4} ln\left(4^{4^n}/4^{4^m}\right)$$

$$= \frac{1}{4}\eta\left(H_M\left(4^{4^m}, 4^{4^n}, t\right)\right)$$

and,

$$\eta\left(H_M\left(w_{22}^{1}(4), w_{22}^{1}\left(4^{4^n}\right), t\right)\right) = ln\left(4^{4^{n-1}}/4\right)$$
$$< \frac{1}{4}\eta\left(H_M\left(4, 4^{4^n}, t\right)\right)$$

Since; $w_{jj}^{l}, l = 1, j = 1,2$ are $\mathscr{H}$-fuzzy contractive with $k = \frac{1}{3}$ and $\frac{1}{4}$. Therefore, the $\mathscr{H}$ – FHB operator W for $\mathscr{H}$-MFIFS $\left\{X_j, j = 1,2; w_{jj}^{l}: X_j \to X_j, l = 1, j = 1,2\right\}$ has a unique attractor $(3,4) \in X_1 \times X_2$.

Remark 4.1 In theorem (4.1),if we substituting $\mathrm{i}, \mathrm{j} = 1$.Then we get the consequent result of Uthayakumar and Gowrishankar (Theorem (3.1), [39]).

Remark 4.2 We establishes the attractor for multi fuzzy IFS in theorem (4.3) which is an improvement of the leading results concerning Uthayakumar and Gowrishankar [39].

Remark 4.3 Every MIFS consisting of $\mathscr{H}-$ contractions on a fuzzy metric space has a unique attractor.

Remark 4.4 Theorem (4.3) explains that multi fuzzy HB operator for multi fuzzy IFS and its unique ‘attractor’ is a multi-fuzzy fractal. That is known as multi fuzzy fractal for generalized fuzzy contractions.

Remark 4.5 Our analysis explains why $\mathcal{A}_\infty$ is called the attractor of W, namely because it “attracts” all the elements of $\mathcal{K}(\mathcal{X})$.

Remark 4.6 Theorem (4.3) is a valuable addition in our main result. As an application, which will be useful in area of image representation, image processing and image compression.

Remark 4.7 Our leading outcomes are the generalization of the leading consequence of Prasad and Katiyar [38] in which we have replaced B -contraction by generalized contraction mapping.

Remark 4.8 It should be noticed that, example (4.1) will be helpful for finding a unique attractor for generalized multi fuzzy Iterated Function System.

4. Conclusions

In this research, we have studied multi-fuzzy fractal spaces and the properties of generalized contraction mapping. Multi fuzzy fractal spaces are the set of fuzzy bounded subsets, compact, non-empty closed, respectively, and supplied along the Hausdorff multi fuzzy fractal spaces. Moreover, with the completeness of multi-fuzzy fractal spaces, we have numerous existing results of attractors reported in the literature. This will help us build our direction to construct the multi-fuzzy IFS consisting of generalized contraction mapping in the multi-fuzzy fractal spaces. Consequently, we have defined multifuzzy fractals in multifuzzy fractal spaces consisting of generalized contraction mappings for an MFIFS. Roughly, we say that an MFIFS consisting of generalized contraction mapping in a complete multifuzzy fractal space possesses a unique attractor, and also that the sequence of iterative mappings produced by any closed bounded set or compact can converge to this attractor. A methodology for constructing multifuzzy IFS in multifuzzy fractal spaces consisting of a generalized contraction mapping is presented.

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