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Fixed Points of Generalized - Geraghty Ciric -Rational Type Contraction in B- Metric Spaces

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Abstract:

In this paper we prove the existence and uniqueness of the fixed points generalized ciric type Geraghty rational contractions in b-metric spaces, our results extend some of the known theorems.

Keywords: Fixed point; b-metric space; Geraghty –ciric type contraction.

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Introduction

One of the most important development of nonlinear analysis is fixed point theory. This idea is useful in science and engineering fields. Banach [2] first proposed the principle, one of the fundamental conclusions of conventional functional analysis, in 1922. One well-known and generally accepted outcome of fixed point theory is this idea. In 1973, Geraghty [13] demonstrated the existence of fixed point solutions in the context of full metric spaces [MS] and provided an important expansion of the Banach contraction principle [BCP] by substituting a function with certain qualities for a constant. As you can see from [8, 9, 12, 13] and the references therein, numerous researchers have since expanded and broadened the Geraghty conclusion in different ways. In metric spaces, Ćirić [4,5] demonstrated the Ćirić-type fixed point theorem, which is thought to be one of the most important generalizations of the BCP Definition 1.1 [15]. Let H be a nonempty set and let $t \geq 1$. A mapping $d: H \times H \to R$ is said to be a b-metric space if $\forall\ a,b.c$ in H, the following conditions are satisfied.

(b1) $d\,(a, b)=0$ if and only if $a=b$,
(b2) $d\,(a, b) = d\,(b, a)$

(b3) $d\ (a, c) \leq$ t[$d\ (a, b)$+ $d\ (b, c)$].
In this case, the pair (H, d) is called a b-metric space (with constant s).

Note that every metric space is b-metric for t=1, but the converse is not true.

Let S be the class of functions of non –decreasing functions $\beta: [0, \infty) \to \left[1, \frac{1}{t}\right)$ which satisfy the condition $\lim_{n\to\infty} \beta(t_n) = \frac{1}{t}$ *implies* $\lim_{n\to\infty} t_n = 0$ for some $t \geq 1$.

Geraghty [14] proved the following theorem.

Theorem 1.2. [14] Let (K, d) be a CMS(complete metric space). Let H : $K \to K$ be aself map. If $\exists, \beta \in S$ such that $d(H(u), H(v)) \leq \beta(d(u, v))d(u, v)$ *for all* $u, v \in H$, then f has a unique common fixed point in H .

Definition 1.3. [14] A self map $H : K \to K$ is said to be a *generalizedGeraghty contraction* If $\exists\beta \in S$ such that

$$d(H(r), H(s)) \leq \beta\ (M(r, s))M(r, s) \qquad (1.13.1)$$

$$M(r, s) = \max\{ d(r, s), d(r, Hr), d(s, Hs), (d(r, Hs) + d(s, Hr)/2 \}$$

for all r, s ∈ H.

Definition :1.4 [1] A mapping $H: K \to K$ on a b metric space (K,d, t) with $t \geq 1$ is called a Ciric -type Geraghty contraction mapping If $\exists, \beta \in$ F such that

$$d(Hu, Hv) \leq M(r, s), for\ all\ u, v \in K \text{ Where } M(r, s) = max\ \{\beta\big(d(r, s)\big)d(r, s), \beta\big(d(r, Hr)\big)d(r, Hr), \beta\big(d(s, Hs)\big)d(s, Hs), \beta\big(d(r, Hs)\big)d(r, Hs), \beta\big(d(s, Hr)\big)d(r, Hs)\}$$

Definition1.5 : [1] A mapping $H: K \to K$ on a b metric space (K,d, t) with $t \geq 1$ is called a Ciric -type Geraghty contraction mapping if there exists $\beta \in$ S such that

$$d(Hu, Hv) \leq M(u, v), for\ all\ u, v \in K \text{ Where } M(u, v) = max\ \{\beta\big(d(u, v)\big)d(u, v), \beta\big(d(u, Hu)\big)d(u, Hu), \beta\big(d(v, Hv)\big)d(v, Hv), \beta\big(d(u, Hv)\big)d(u, Hv), \beta\big(d(v, Hu)\big)d(v, Hu)\}$$

Theorem 1.6 :[8,9] Let (M,d) be a CMS and $T: M \to M$ be a Geraghty – ciric –contraction with some $\beta \in$ S , Then T has fixed point and unique .

In 2019, Faraji et.al [8] proved a fixed point theorem with Geraghty –type contractive maps in b-metric spaces .

Theorem 1.7 [8] Let (M,d,v) be a complete- b metric space with $v \geq 1$ and let T $: M \to M$, be a self –mapping and if there exist $\beta \in$ S ,

$$d(Tu, Tv) \leq \beta\big(L(u, v)\big)L(u, v), for\ all\ u, v \in M, \text{ where}$$

$$L(u, v) = \max\{d(u, v), d(u, Tu), d(v, Tv), \frac{1}{2v}[d(u, Tv) + d(v, Tu)]\}$$ then T has a unique fixed point.

In 1975, Dass and Gupta [6] extended the BCP type of rational terms.

Theorem 1.8. ([3]). Let (H, d) be a CMS and H : K → K be a mapping such that there exist α, β ≥ 0 with α + β < 1 satisfying

$d(Hu, Hv) \leq \alpha \frac{d(v,Hv)[1+d(u,Hv)]}{1+d(u,v)} + \beta d(u,v)$ for all $u, v \in H$.

Then H has a unique fixed point.

Lemma [9] Let (M, d,v) be a metric space with $v \geq 1$ and let $T: M \to M$ be a self mapping . Let $x_0 \in M$ be given and $\{x_n\}$ be a sequence in M such that $x_n = Tx_{n-1}$ for all n in N, the sequence defined by $A_n = \max\{d(x_p, x_q) | 0 \leq p, q \leq n \text{ and } p, q \in N_0\}$, *for* $n \in N_0$. If T satisfies the contractivity condition in (1), then $\{A_n\}$ is bounded .

In 2024, kalo.et.al [1] proved fixed point theorems in Geraghty-ciric-type contraction mapping in b-metric spaces

Theorem 1.10[1] *Let* (M, d, v) *be a complete b-metric space with* $t \geq 1$ *and let* $T: M \to M$ *be a self- mapping* Ciric -type Geraghty contraction (1), Then T has a unique fixed point x* in K.

This Lemma can use to prove results.

Lemma [1.11]. Let (M, d,v) *be a b-metric space with* $t \geq 1$ and let $\{a_n\}$ and $\{b_n\}$ be b-convergent to x, y in M, then we have

$$\frac{1}{v^2} d(a,b) \leq \liminf_{n\to\infty} d(a_n, b_n) \leq \limsup_{n\to\infty} d(a_n, by_n) \leq v^2\, d(a,b)$$

In particular, if a=b, we have $\lim_{n\to\infty} d(a_n, b_n) = 0$.

and for any c in M, $\frac{1}{v} d(a,c) \leq \liminf_{n\to\infty} d(a_n, c) \leq \limsup_{n\to\infty} d(a_n, cz) \leq v\, d(a,c)$.

MAIN RESULTS:

Now, we define ciric type Geraghty contraction with rational type of expressions in b metric spaces.

Definition 2.1: A mapping $H: K \to K$ on a b- metric space (K, d, t) with $t \geq 1$ is called a– Ciric -type Geraghty contraction with rational mapping , if there exists $\beta \in S$ such that

$d(Hu, Hv) \leq M(u,v), for\ all\ u, v \in K$,……2.1.1

Where $M(u,v) =$
$\max\{\beta\left(\frac{d(u,Hu)[1+d(Hv,Hu)]}{1+d(v,Hv)}\right)(\frac{d(u,Hu)[1+d(Hv,Hu)]}{1+d(v,Hv)}), \beta\left(\frac{d(v,Hu)[1+d(v,Hv)]}{1+d(u,Hv)}\right)(\frac{d(v,Hu)[1+d(v,Hv)]}{1+d(u,Hv)}),$
$\beta\left(\frac{d(v,Hv)[1+d(v,Hu)]}{1+d(u,Hv)}\right)(\frac{d(v,Hv)[1+d(v,Hu)]}{1+d(u,Hv)})\}$

First, we prove that sequence is $\{B_n\}$ *is bounded.*

Theorem 2. 2: Let (K, d,t) be a b-metric space with t ≥ 1 and let H : K → K be a self- mapping. Let $s_0 \in K$ be given and $\{s_n\}$ be a sequence in K , $s_n = H s_{n-1}$ for all n ∈ N.

The sequence $B_n = \max\{d(s_p, s_q) | 0 \leq p, q \leq n \text{ and } p, q \in N_0\}$(2.2.1) for n $\in N_0$. If H satisfies the contractivity condition in (2.1.1), then $\{B_n\}$ is bounded $\{d(s_p, s_q)/ 0 \leq p, q \leq n \text{ and } p, q \in N_0\}$ for n $\in N_0$. If K satisfies condition of (2.1.1) , then $\{B_n\}$ is bounded.

Proof: Let n $\in$ N. Then for any p, q $\in$ N with $1 \leq p, q \leq n$, from condition (2.1.1), we have

$$d(s_p, s_q) = d(Hs_{p-1}, Hs_{q-1})$$

$$\leq M(s_{p-1}, s_{q-1})$$

$$= \max\{\beta\left(\frac{d(s_{p-1},Hs_{p-1})[1+d(Hs_{q-1},Hs_{p-1})]}{1+d(s_{q-1},Hs_{q-1})}\right)(\frac{d(s_{p-1},Hs_{p-1})[1+d(Hs_{q-1},Hs_{p-1})]}{1+d(s_{q-1},Hs_{q-1})}),$$

$$\beta\left(\frac{d(s_{q-1},Hs_{p-1})[1+d(s_{q-1},Hs_{q-1})]}{1+d(s_{p-1},Hs_{q-1})}\right)(\frac{d(s_{q-1},Hs_{p-1})[1+d(s_{q-1},Hs_{q-1})]}{1+d(s_{p-1},Hs_{q-1})}),$$

$$\beta\left(\frac{d(s_{q-1},Hs_{q-1})[1+d(s_{q-1},Hs_{p-1})]}{1+d(s_{p-1},Hs_{q-1})}\right)(\frac{d(s_{q-1},Hs_{q-1})[1+d(s_{q-1},Hs_{p-1})]}{1+d(s_{p-1},Hs_{q-1})})\}$$

$$< \frac{1}{t}\max\{\left(\frac{d(s_{p-1},Hs_{p-1})[1+d(Hs_{q-1},Hs_{p-1})]}{1+d(s_{q-1},Hs_{q-1})}\right),$$

$$\left(\frac{d(s_{q-1},Hs_{p-1})[1+d(s_{q-1},Hs_{q-1})]}{1+d(s_{p-1},Hs_{q-1})}\right),\frac{d(s_{q-1},Hs_{q-1})[1+d(s_{q-1},Hs_{p-1})]}{1+d(s_{p-1},Hs_{q-1})}\}$$

$\leq B_n$, so that $\max\{d(s_p, s_q)|\, 0 \leq p, q \leq n \text{ and } p, q \in N_0\} < B_n$.

Consequently, there is $w_n \in N, with\ 1 \leq w_n \leq n$ such that $B_n = d(s_p, s_{wn}\}$

Here, we can see that $0 \leq B_n \leq B_{n+1}$ for all $n \in N$.

Now we have to prove sequence $\{B_n\}$ is bounded.

On the contrary, we assume that $\{B_n\}$ is not bounded. Since $\{B_n\}$ is non decreasing sequence of non negative reals, we have $\lim_{n\to\infty} B_n = \infty$.

Now, by using b- triangular inequality on $d(s_p, s_{wn}\}$

and using the inequality (1) ,

$B_n = d(s_p, s_{wn}\} \leq t[d(s_0, s_{wn}) \leq v[d(s_0, s_1) + d(s_1, s_{wn})]$ ---(i)

$= t[d(s_0, s_1) + vM(s_0, s_{wn-1})]$

Where , $M(s_0, s_{wn-1}) = \max\{\beta\left(\frac{d(s_0,Hs_0)[1+d(Hs_{q-1},Hs_0)]}{1+d(s_{q-1},Hs_{q-1})}\right)(\frac{d(s_0,Hs_0)[1+d(Hs_{q-1},Hs_0)]}{1+d(s_{q-1},Hs_{q-1})}),$

$$\beta\left(\frac{d(s_{wn-1},Hs_0)[1+d(s_{wn-1},Hs_{wn-1})]}{1+d(s_0,Hs_{wn-1})}\right)(\frac{d(s_{wn-1},Hs_0)[1+d(s_{wn-1},Hs_{wn-1})]}{1+d(s_0,Hs_{wn-1})}),$$

$$\beta\left(\frac{d(s_{wn-1},Hs_{wn-1})[1+d(s_{wn-1},Hs_0)]}{1+d(s_0,Hs_{wn-1})}\right)(\frac{d(s_{wn-1},Hs_{wn-1})[1+d(s_{wn-1},Hs_0)]}{1+d(s_0,Hs_{wn-1})})\}$$

Here observe that the sequences $\left\{\beta\left(\frac{d(s_0,Hs_0)[1+d(Hs_{q-1},Hs_0)]}{1+d(s_{wn-1},Hs_{wn-1})}\right)\right\},$ $\beta\left(\frac{d(s_{wn-1},Hs_0)[1+d(s_{wn-1},Hs_{wn-1})]}{1+d(s_0,Hs_{wn-1})}\right)\}$, $\{\beta\left(\frac{d(s_{wn-1},Hs_{wn-1})[1+d(s_{wn-1},Hs_0)]}{1+d(s_0,Hs_{wn-1})}\right)\}$ are the sequences of real

numbers and the sub sequences $\left\{\beta\left(\frac{d(s_0,Hs_0)\left[1+d\left(Hs_{q_k-1},Hs_0\right)\right]}{1+d(v,Hv)}\right)\right\}$,
$\beta\left(\frac{d\left(s_{wn_k-1},Hs_0\right)\left[1+d\left(s_{wn_k-1},Hs_{wn_k-1}\right)\right]}{1+d\left(s_0,Hs_{wn_k-1}\right)}\right)\}$, $\{\ \beta\left(\frac{d\left(s_{wn_k-1},Hs_{wn_k-1}\right)\left[1+d\left(s_{wn_k-1},Hs_0\right)\right]}{1+d\left(s_0,Hs_{wn_k-1}\right)}\right)\}$

Case (i):

Suppose that $M(s_0,s_{wn-1}) = \left\{\beta\left(\frac{d(s_0,Hs_0)\left[1+d\left(Hs_{q_k-1},Hs_0\right)\right]}{1+d(v,Hv)}\right)\right\}$ ------(ii)

then from (i) and (ii) , we get

$B_{n_k} \le d\left(s_p , s_{wn}\right\} \le t[d(s_0,s_{wn}) \le v[d(s_0,s_1) + d(s_1,s_{wn})]$ ---(i)

$\le t[d(s_0,s_1) + vM(s_0,s_{wn-1})]$

$\le t\left[d(s_0,s_1) + v\left\{\beta\left(\frac{d(s_0,Hs_0)\left[1+d\left(Hs_{q_k-1},Hs_0\right)\right]}{1+d(v,Hv)}\right)\right\}B_{nk}\right]$----(iii)

so that,

$\frac{1}{t} - \frac{d(s_0,s_1)}{B_{n_k}} \le \beta\left(\frac{d(s_0,Hs_0)\left[1+d\left(Hs_{q_k-1},Hs_0\right)\right]}{1+d(v,Hv)}\right) < \frac{1}{t}$

Since as $k \to \infty, B_{nk} \to \infty$,

so that $\lim_{n\to\infty}(\frac{1}{t} - \frac{d(s_0,s_1)}{B_{n_k}}) = \frac{1}{t}$. Therefore $\lim_{\sup\ k\to\infty} \beta\left(\frac{d(s_0,Hs_0)\left[1+d\left(Hs_{q_k-1},Hs_0\right)\right]}{1+d(v,Hv)}\right) = \frac{1}{t}$.

Hence $\lim_{n\to\infty} d\left(s_0,s_{wn_k-1}\right) = 0$.

Since β is the class of functions S, taking limit on (iii), we get

$$\lim_{k\to\infty} \mathrm{B_{nk}} \le t\left[d(s_0,s_1) + t\left\{\beta\left(\frac{d(s_0,Hs_0)\left[1+d\left(Hs_{q_k-1},Hs_0\right)\right]}{1+d(v,Hv)}\right)\right\}\right]$$

$t.d(s_0,s_1)$, contradiction, $B_{nk} \to \infty$, as $k \to \infty$.

Case (ii) Suppose that $M(s_0,s_{wn-1}) = \left\{\beta\left(\frac{d(s_{wn-1},Hs_0)[1+d(s_{wn-1},Hs_{wn-1})]}{1+d(s_0,Hs_{wn-1})}\right)\right\}$ ------(v)

then from (i) and (v) , we get

$$B_{n_k} \le t\left[d(s_0,s_1) + t\left\{\beta\left(\frac{d(s_0,Hs_0)\left[1+d\left(Hs_{q_k-1},Hs_0\right)\right]}{1+d(v,Hv)}\right)\right\}B_{nk}\right]\text{----(iii)}$$

< (t+1) $d(s_0,s_1)$, which is contradiction to , $B_{nk} \to \infty$, as $k \to \infty$.

Case (iii)

Suppose that $M(s_0,s_{wn-1}) = \beta\left(\frac{d(s_{wn-1},Hs_{wn-1})[1+d(s_{wn-1},Hs_0)]}{1+d(s_0,Hs_{wn-1})}\right)$ ------(vi)

then from (i) and (vi) , we get

$$B_{n_k} \leq t\left[d(s_0, s_1) + t\left\{\beta\left(\frac{d(s_{wn-1}, Hs_{wn-1})[1+d(s_{wn-1}, Hs_0)]}{1+d(s_0, Hs_{wn-1})}\right)\right\} B_{nk}\right]$$

< (t+1) $d(s_0, s_1)$, which is contradiction to , $B_{nk} \to \infty$, as $k \to \infty$.

so that we are getting contradictions in all the cases and hence the sequence $\{B_n\}$ not bounded.

Now we prove the existence of the fixed point with unique in – Ciric -type Geraghty contraction with rational terms in b –metric spaces.

Theorem 2.3: Let (K, d, t) be a CbMS (Complete b- metric space) space with

$t \geq 1$ and let $H : K \to K$ be a map – Ciric -type Geraghty contraction with rational terms (2.11), Then H has a unique fixed point s in K.

Proof: Let$s_0 \in K$.

Now consider a sequence $\{s_n\}$ in K by defining

$s_n = Ks_{n-1} = K^n s_0 \ \forall\, n \in N$.

Now we show that $\{s_n\}_n \in N$ is a b- Cauchy sequence in K.

We take a sequence as $B_n = \ max\{d(s_p, s_q) | 0 \leq p, q \leq n, p, q \in N\}$

By using the above theorem 2. 2 , ∃M >0 such that $B_n \leq M\ , for\ all\ n \in N$.

As $\{B_n\}$ is an increasing sequence, we have $\lim_{n\to\infty} B_n \leq M$.

We take a sequence $\{\lambda_n\}$ on a b- metric space (K, d,t) by

$\{\lambda_n\} = \sup\{\ d(s_p, s_q) | p, q \geq n, p, q \in N\}$, then we have

$0 \leq \lambda_n \ \leq \lambda_{n-1} \leq \lambda_{n-2} \leq \cdots \leq \lambda_0 = \lim_{n\to\infty} B_n \leq M \ \forall\, n \in N$.

The sequence $\{\lambda_n\}$ is a decreasing and bounded sequence of non negative real numbers, and hence it is converges to some l ≥0, that is $\lim_{n\to\infty} \lambda_n = l$.

Then there exist two sub sequences { s_{p_k}} and { s_{q_k}} of { s_n} with $q_k > p_k \geq k$ for $k \in N$ such that $d(s_{p_k}, s_{q_k}) \to l\ as\ k \to \infty$. 2.3.1

Now we have to prove that$l = 0$.

On the contrary, assume that $l > 0$.

Put $u = s_{pk-1}, and\ v = s_{qk-1}$ in the inequality.

We have $d(s_{pk}, s_{qk}) = d(Hs_{pk-1}, Hs_{qk-1}) \leq M(s_{pk-1}, s_{qk-1})$ 2.3.2

Where $(s_{pk-1}, s_{qk-1}) =$
max{ $\beta\left(\frac{d(s_{pk-1}, Hs_{pk-1})[1+d(Hs_{qk-1}, Hs_{pk-1})]}{1+d(s_{qk-1}, Hs_{qk-1})}\right)\left(\frac{d(s_{pk-1}, Hs_{pk-1})[1+d(Hs_{qk-1}, Hs_{pk-1})]}{1+d(s_{qk-1}, Hs_{qk-1})}\right)$

$$\beta\left(\frac{d\left(s_{qk-1}, Hs_{pk-1,}\right)\left[1+d\left(s_{qk-1}, Hs_{qk-1}\right)\right]}{1+d\left(s_{pk-1,}, Hs_{qk-1}\right)}\right)\left(\frac{d\left(s_{qk-1}, Hs_{pk-1,}\right)\left[1+d\left(s_{qk-1}, Hs_{qk-1}\right)\right]}{1+d\left(s_{pk-1,}, Hs_{qk-1}\right)}\right)$$

$$\beta\left(\frac{d\left(s_{qk-1}, Hs_{qk-1}\right)\left[1+d\left(s_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{pk-1,}, Hs_{qk-1}\right)}\right)\left(\frac{d\left(s_{qk-1}, Hs_{qk-1}\right)\left[1+d\left(s_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{pk-1,}, Hs_{qk-1}\right)}\right)\}$$

Now the maximum is one of the terms of R.H.S of $M\left(s_{pk-1,}s_{qk-1}\right)$

Now we consider three as cases as the possibility of each one term of R H S.

Suppose that

$$M\left(s_{pk-1,}s_{qk-1}\right)=\beta\left(\frac{d\left(s_{pk-1,}, Hs_{pk-1,}\right)\left[1+d\left(Hs_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{qk-1}, Hs_{qk-1}\right)}\right)\left(\frac{d\left(s_{pk-1,}, Hs_{pk-1,}\right)\left[1+d\left(Hs_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{qk-1}, Hs_{qk-1}\right)}\right)$$

for all k in N.

Therefore , from 2.32$d\left(s_{pk}, s_{qk}\right) \leq$

$$\beta\left(\frac{d\left(s_{pk-1,}, Hs_{pk-1,}\right)\left[1+d\left(Hs_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{qk-1}, Hs_{qk-1}\right)}\right)\left(\frac{d\left(s_{pk-1,}, Hs_{pk-1,}\right)\left[1+d\left(Hs_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{qk-1}, Hs_{qk-1}\right)}\right)$$

$$\leq \beta\left(\frac{d\left(s_{pk-1,}, Hs_{pk-1,}\right)\left[1+d\left(Hs_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{qk-1}, Hs_{qk-1}\right)}\right)\lambda_{k-1} \quad \text{.....2.3.3}$$

Taking limit suprimum $as\ k \to \infty$ on both sides of (2.3.2), we get

$$\lim_{k\to\infty} sup\ d\left(s_{pk}, s_{qk}\right) \leq \lim_{k\to\infty} sup\ \beta\left(\frac{d\left(s_{pk-1,}, Hs_{pk-1,}\right)\left[1+d\left(Hs_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{qk-1}, Hs_{qk-1}\right)}\right)\lambda_{k-1}$$

From 2.3.1 , we get

$$l \leq \beta\left(\frac{d\left(s_{pk-1,}, Hs_{pk-1,}\right)\left[1+d\left(Hs_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{qk-1}, Hs_{qk-1}\right)}\right)l,$$

$$\frac{1}{t} \leq 1 \leq \lim_{k\to\infty} sup\ \beta\left(\frac{d\left(s_{pk-1,}, Hs_{pk-1,}\right)\left[1+d\left(Hs_{qk-1}, Hs_{pk-1,}\right)\right]}{1+d\left(s_{qk-1}, Hs_{qk-1}\right)}\right) < \frac{1}{t}$$

Since β is the class of functions S have $\lim_{k\to\infty} d\left(s_{pk-1}, s_{qk-1}\right) = 0$

Using 2.3.1 and 2.3.3 we get $l = \lim_{k\to\infty} d\left(s_{pk}, s_{qk}\right) = 0,$

Which is contradiction to our assumption $l > 0$.

Hence, $\lim_{n\to\infty} \lambda_n = l = 0$.

Similarly, we can show that in the remaining two cases $l = \lim_{n\to\infty} \lambda_n = 0$.

Now, let $m, n \in N\ , with\ m > n,$

We get $\lim_{n\to\infty} d(s_n, s_m) \leq \lim_{n\to\infty} \lambda_n = 0$.

Hence, the sequence $\{s_n\}$ is a b- Cauchy sequence in K. Since K is complete, the sequence $\{s_n\}$ is convergent to some s* in M.

Now we prove that s* is a fixed point of H.

Assume that Hs* ≠s*, $(Hs^*, s^*) > 0$, taking u= s_n, v = s^*

$d(s_{n+1}, Hs^*) = d(Hs_n, Hs^*) \leq M(s_n, s^*)$……2.3.4

Where $M(s_n, s^*) =$
$max\{\beta\left(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)}\right)(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)}),\beta\left(\frac{d(s^*,Hs_n)[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}\right)(\frac{d(s^*,Hs_n)[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}),$
$\beta\left(\frac{d(s^*,Hs^*)[1+d(s^*,Hs_n)]}{1+d(s_n,Hs^*)}\right)(\frac{d(s^*,Hs^*)[1+d(s^*,Hs_n)]}{1+d(s_n,Hs^*)})\}$

$=$
$max\{\beta\left(\frac{d(s_n,s_{n+1})[1+d(Hs^*,s_{n+1})]}{1+d(s^*,Hs^*)}\right)(\frac{d(s_n,s_{n+1})[1+d(Hs^*,s_{n+1})]}{1+d(s^*,Hs^*)}),\beta\left(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}\right)(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}),$
$\beta\left(\frac{d(s^*,Hs^*)[1+d(s^*,s_{n+1})]}{1+d(s_n,Hs^*)}\right)(\frac{d(s^*,Hs^*)[1+d(s^*,s_{n+1})]}{1+d(s_n,Hs^*)})\}$

Again, three cases will arise :

Case (i) : Suppose that $M(s_n, s^*) = \beta\left(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)}\right)(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)})$

then from 2.3.4, we have

$$d(s_{n+1}, Hs^*) = \beta\left(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)}\right)(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)})$$

as limit $n \to \infty$ on both sides of the above inequality and from the theorem (2.2.1) ,

we get $\lim_{n\to\infty}\left(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)}\right) = 0$, so that

$\frac{1}{t}\ d(s^*, Hs^*) \leq \lim_{n\to\infty}\sup d(s_{n+1}, Hs^*) \leq$
$\lim_{n\to\infty} sup\ \beta\left(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)}\right)\lim_{n\to\infty} sup(\frac{d(s_n,Hs_n)[1+d(Hs^*,Hs_n)]}{1+d(s^*,Hs^*)}) = 0$

And hence consequently, we get $d(s^*, Hs^*) = 0$,

which contradicts our assumption $d(Hs^*, s^*) > 0$.

Case (ii) :

Suppose that $M(s_n, s^*) = \beta\left(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}\right)(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)})$

then from 2.3.4, we have

$$d(s_{n+1}, Hs^*) = \beta\left(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}\right)(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)})$$

as limit $n \to \infty$ on both sides of the above inequality and from the theorem (i) , we get
$\lim_{n\to\infty}(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}) = 0$, so that

$\frac{1}{t}\ d(s^*, Hs^*) \leq \lim_{n\to\infty}\sup d(s_{n+1}, Hs^*) \leq \lim_{n\to\infty} sup\ \beta\left(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}\right)(\frac{d(s^*,s_{n+1})[1+d(s^*,Hs^*)]}{1+d(s_n,Hs^*)}) = 0$

And hence consequently, we get $d(s^*, Hs^*) = 0$,

which contradicts our assumption $d(\mathrm{Hs}^*, \mathrm{s}^*) > 0$.

Case (iii) : Suppose that $\mathrm{M}(\mathrm{s_n}, \mathrm{s}^*) = \beta\left(\frac{d(\mathrm{s}^*,H\mathrm{s}^*)[1+d(\mathrm{s}^*,H\mathrm{s_n})]}{1+d(\mathrm{s_n},H\mathrm{s}^*)}\right)\left(\frac{d(\mathrm{s}^*,H\mathrm{s}^*)[1+d(\mathrm{s}^*,H\mathrm{s_n})]}{1+d(\mathrm{s_n},H\mathrm{s}^*)}\right)$

then from 2.3.4, we have

$$d(s_{n+1}, \mathrm{Hs}^*) = \beta\left(\frac{d(\mathrm{s}^*, H\mathrm{s}^*)[1 + d(\mathrm{s}^*, H\mathrm{s_n})]}{1 + d(\mathrm{s_n}, H\mathrm{s}^*)}\right)\left(\frac{d(\mathrm{s}^*, H\mathrm{s}^*)[1 + d(\mathrm{s}^*, H\mathrm{s_n})]}{1 + d(\mathrm{s_n}, H\mathrm{s}^*)}\right)$$

as limit $n \to \infty$ in the above and from the theorem (i) ,

we get $\lim_{n\to\infty}\left(\frac{d(\mathrm{s}^*,H\mathrm{s}^*)[1+d(\mathrm{s}^*,H\mathrm{s_n})]}{1+d(\mathrm{s_n},H\mathrm{s}^*)}\right) = 0$, so that

$\frac{1}{t}\ d(s^*, Hs^*) \le \lim_{n\to\infty}\sup d(s_{n+1}, \mathrm{Hs}^*) \le \lim_{n\to\infty} sup\ \beta\left(\frac{d(\mathrm{s}^*,H\mathrm{s}^*)[1+d(\mathrm{s}^*,H\mathrm{s_n})]}{1+d(\mathrm{s_n},H\mathrm{s}^*)}\right)\left(\frac{d(\mathrm{s}^*,H\mathrm{s}^*)[1+d(\mathrm{s}^*,H\mathrm{s_n})]}{1+d(\mathrm{s_n},H\mathrm{s}^*)}\right) = 0$

And hence consequently, we get $d(s^*, Hs^*) = 0$,

This is contradiction to our assumption, $d(\mathrm{Hs}^*, \mathrm{s}^*) > 0$.

Therefore from all these cases , we can get $s^* = Hs^*$.

Hence s^* is a fixed point of H.

Now, our aim is to prove that the uniqueness of the fixed point.

Assume that $r \in H$ is other fixed point of H such that$s^* \neq r$, $d(s^*, r) > 0$.

From (2.3.1)

$d(s^*, r) = d(Hs^*, r) \le M(s^*, r) =$

$max\ \{\beta\left(\frac{d(s^*,Hs^*)[1+d(Hr,Hs^*)]}{1+d(r,Hr)}\right)\left(\frac{d(s^*,Hs^*)[1+d(Hr,Hs^*)]}{1+d(r,Hr)}\right), \beta\left(\frac{d(r,Hs^*)[1+d(r,Hr)]}{1+d(s^*,Hr)}\right)\left(\frac{d(r,Hs^*)[1+d(r,Hr)]}{1+d(s^*,Hr)}\right),$
$\beta\left(\frac{d(r,Hr)[1+d(r,Hs^*)]}{1+d(s^*,Hr)}\right)\left(\frac{d(r,Hr)[1+d(r,Hs^*)]}{1+d(s^*,Hr)}\right)\}$

$max\ \{\beta\left(\frac{d(s^*,s^*)[1+d(Hr,Hs^*)]}{1+d(r,r)}\right)\left(\frac{d(s^*,s^*)[1+d(Hr,Hs^*)]}{1+d(r,r)}\right), \beta\left(\frac{d(r,Hs^*)[1+d(r,r)]}{1+d(s^*,Hr)}\right)\left(\frac{d(r,Hs^*)[1+d(r,r)]}{1+d(s^*,Hr)}\right),$
$\beta\left(\frac{d(r,r)[1+d(r,Hs^*)]}{1+d(s^*,Hr)}\right)\left(\frac{d(r,r)[1+d(r,Hs^*)]}{1+d(s^*,Hr)}\right)\}$

$$= \beta\left(\frac{d(r, Hs^*)[1 + d(r, r)]}{1 + d(s^*, Hr)}\right)\left(\frac{d(r, Hs^*)[1 + d(r, r)]}{1 + d(s^*, Hr)}\right)$$

$\le \left(\frac{d(r,Hs^*)[1+d(r,r)]}{1+d(s^*,Hr)}\right) < \frac{1}{t} d(s^*, r)$, which is contradiction.

So that $s^* = r$ hence s^* is the only fixed point of H in M.

We derive corollaries from Theorem2.3

Corollaries:

Corollary 2.4. Let (K, d, t) be a CbMS with $t \ge 1$. Suppose that $H: K \to K$ be a self map and β is the class of Geraghty functions S then for any $u, v \in K$

$$d(Hu,Hv) \leq \beta(\ N(u,v))N(u,v)$$

Where $N(u,v)= max\ \{(\frac{d(u,Hu)[1+d(Hv,Hu)]}{1+d(v,Hv)}), (\frac{d(v,Hu)[1+d(v,Hv)]}{1+d(u,Hv)}),(\frac{d(v,Hv)[1+d(v,Hu)]}{1+d(u,Hv)})\}$.

Then H has a unique fixed point $u^* \in K.$

$Proof$: For *any u, v in K, N(u,v)* one of the term of $\frac{d(u,Hu)[1+d(Hv,Hu)]}{1+d(v,Hv)}), (\frac{d(v,Hu)[1+d(v,Hv)]}{1+d(u,Hv)}),\left(\frac{d(v,Hv)[1+d(v,Hu)]}{1+d(u,Hv)}\right)$

Then it follows that

$d(Hu,Hv) \leq \beta(\ N(u,v))N(u,v)$

$\leq \beta(\ N(u,v))N(u,v)$

$\leq$

$max\ \{\beta\left(\frac{d(u,Hu)[1+d(Hv,Hu)]}{1+d(v,Hv)}\right)(\frac{d(u,Hu)[1+d(Hv,Hu)]}{1+d(v,Hv)}), \beta\left(\frac{d(v,Hu)[1+d(v,Hv)]}{1+d(u,Hv)}\right)(\frac{d(v,Hu)[1+d(v,Hv)]}{1+d(u,Hv)}),$
$\beta\left(\frac{d(v,Hv)[1+d(v,Hu)]}{1+d(u,Hv)}\right)(\frac{d(v,Hv)[1+d(v,Hu)]}{1+d(u,Hv)})\}$

= M(u, v), and observe that all the hypothesis of Theorem 2 are satisfies, and hence we can have that H has unique fixed point.

The following is an example in support our main result.

Examples.

Example 2.5: Let $K = \left\{1,\frac{1}{2},\frac{1}{4}\right\}U\{0\}$ and define the function from $d: KXK \rightarrow R^+$ by $d(x,y) = (x-y)^2, H(x) = \frac{1}{t^2+1}, if\ x = \frac{1}{t}$, 0 if *x=0.*

We define$\beta(\alpha) = e^{-\alpha}, \alpha > 0, \beta(0) = 0$, all the conditions holds and '0' fixed point and is unique.

Conclusions: In this we proved the existence and uniqueness of the fixed points generalized ciric type Geraghty rational contractions in b-metric spaces, our results extend some of the known theorems, kalo.et.al [1] proved fixed point theorems in Geraghty-ciric-type contraction mapping in b- metric spaces. We derived some corollaries and given examples in support our main result.

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