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Approximation of Fixed Points via Picard-Abbas Hybrid Iteration Scheme for ρ-Quasi-Nonexpansive Multivalued Mappings

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Abstract:

We present results on stability and convergence for the Picard-Abbas iteration scheme for ρ-quasi-nonexpansive multivalued mappings within modular function spaces. Furthermore, we demonstrate an application of the iteration scheme in differential equations.

Keywords: Picard-Abbas hybrid iteration; convergence; stability; quasi-nonexpansive mappings; modular function space.

1. Introduction

Nakano [28] generalized the ordered spaces theory to modular spaces. Later, Musielak and Orlicz [26] further extended and generalized this theory. In modular spaces, it was Khamsi, Kozlowski, and Reich [9] who first studied the theory of fixed points. But, it was Khan and Abbas [10] who first studied the fixed point approximation for ρ-nonexpansive multivalued mappings in modular function spaces, utilizing Mann iteration scheme. Later, by utilizing a three-step iteration scheme, Khan et al. [11] gave fixed point approximation results for ρ-quasi-nonexpansive multivalued mappings. Okeke et al. [29] also provided significant results for these mappings, by utilizing the hybrid Picard-Krasnoselski iteration scheme. These spaces have rich structural properties, and are equipped with modular equivalents of metric and norm concepts. The modular type conditions, offer a more intuitive and verifiable framework compared to the traditional norm-based assumptions, thus contributing to their increasing popularity.

This paper purposes to further the existing work by presenting new results on convergence and stability for ρ-quasi-nonexpansive mappings within modular function spaces, utilizing the Picard Abbas-type hybrid iterative approach. In doing so, we contribute to the on-going discourse in fixed point theory, enhancing our understanding of the interplay between modular structures and iterative methods.

2. Preliminaries

Consider a nontrivial σ-algebra Σ on $\Omega \neq \emptyset$, and $\mathcal{P}$ a δ-ring of subsets of Ω, with $E \cap A = \mathcal{P}$ $\forall E \in \mathcal{P}, A \in \Sigma$. Let $K_n \in \mathcal{P}$ be an increasing sequence of sets with $\Omega = \cup K_n$. Let $\mathcal{M}_\infty$ be the space of all extended measurable functions. Let $\mathcal{E}$ be the vector space of simple functions with support being contained within $\mathcal{P}$ and 1_A be the characteristic function of A in Ω.

We define

$$\mathcal{M}(\Omega,\Sigma,\mathcal{P},\rho)=\{f\in\mathcal{M}_\infty:\ |f(\omega)|<\infty\ \rho\text{- a.e.}\},$$

For simplicity, we write $\mathcal{M}$ instead of $\mathcal{M}(\Omega,\Sigma,\mathcal{P},\rho)$.

Definition 2.1. [12] A functional ρ of a vector space X ($\mathbb{R}$ or $\mathbb{C}$) is a modular if for arbitrary $f,g\in X$, the statements below hold:

(i) $\rho(f)=0 \iff f=0$

(ii) $\rho(\alpha f)=\rho(f)$ whenever $|\alpha|=1$

(iii) $\rho(\alpha f+\beta g)\leqslant\rho(f)+\rho(g)$ whenever $\alpha,\beta\geqslant 0,\ \alpha+\beta=1$.

If we replace (iii) by

(iv) $\rho(\alpha f+\beta g)\leqslant\alpha\rho(f)+\beta\rho(g)$ whenever $\alpha,\beta\geqslant 0,\ \alpha+\beta=1$.

then the modular ρ is convex.

Definition 2.2. [12] The following set is called a modular function space, for a convex modular ρ in X: $\quad L_\rho=\{f\in\mathcal{M}:\ \rho(\lambda f)\to 0 \text{ as } \lambda\to 0\}$

In general, ρ is not sub-additive. We can equip the L_ρ with the following F-norm:

$$\|f\|_\rho=\inf\left\{\alpha>0:\ \rho\left(\frac{f}{\alpha}\right)\leqslant\alpha\right\}.$$

If ρ is a convex, then the norm

$$\|f\|_\rho=\inf\left\{\alpha>0:\ \rho\left(\frac{f}{\alpha}\right)\leqslant\alpha\right\}$$

is called the Luxemburg Norm on the modular space L_ρ.

Definition 2.3. [12] The nontrivial, even and convex function $\rho:\mathcal{M}_\infty\to[0,\infty]$ is called a regular convex function pseudomodular if

(i) $\rho(0)=0$;

(ii) $|f(\omega)|\leqslant|g(\omega)|$ for any $\omega\in\Omega$ implies $\rho(f)\leqslant\rho(g)$, where $f,g\in\mathcal{M}_\infty$. Thai is, ρ is monotone.

(iii) $\rho(f1_{A\cup B})\leqslant\rho(f1_A)+\rho(f1_B)$ for any $A,B\in\Sigma$ such that $A\cap B\neq\emptyset$, $f\in\mathcal{M}_\infty$. That is, ρ is orthogonally sub-additive.

(iv) $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$. That is, ρ has Fatou property.

(v) $g_n \in \mathcal{E}$, and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$. That is ρ is order continuous in $\mathcal{E}$.

A set $A \in \Sigma$ is ρ- null if $\rho(g 1_A) = 0 \ \forall g \in \mathcal{E}$. A property $\rho(\omega)$ is ρ-almost everywhere (ρ- a.e.) if $\{\omega \in \Omega \colon \rho(\omega) \text{ does not hold}\}$ is ρ- null.

Definition 2.4. [12] A regular function pseudomodular ρ is called a regular convex function modular if $\rho(f) = 0 \Rightarrow f = 0$ ρ- a.e.

Let $\mathfrak{R}$ be the class of all non-zero regular convex function modular on Ω.

For $\rho \in \mathfrak{R}$, define $L_\rho^0 = \{f \in L_\rho \colon \rho(f, \cdot) \textit{ is order continuous}\}$ and
$E_\rho = \{f \in L_\rho \colon \lambda f \in L_\rho^0 \textit{ for every } \lambda > 0\}$. Note that $L_\rho \supset L_\rho^0 \supset E_\rho$.

Definition 2.5. [11] A $\rho \in \mathfrak{R}$ satisfy Δ_2-condition if $\sup_{n \geq 1} \rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{D_k\}$ decreases to ϕ and $\sup_{n \geq 1} \rho(f_n, D_k) \to 0$ as $k \to \infty$.

$L_\rho = E_\rho$ holds true if ρ is convex and satisfies Δ_2-condition.

Let $\rho \in \mathfrak{R}$. Let $r > 0$, $\epsilon > 0$. Define

$$D_1(r, \epsilon) = \{(f, g) \colon f, g \in L_\rho, \ \rho(f) \leq r, \ \rho(g) \leq r\}, \ \rho(f - g) \geq \epsilon r.$$

Let $\delta_1(r, \epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) \colon (f, g) \in D_1(r, \epsilon)\right\}$ if $D_1(r, \epsilon) \neq \emptyset$ and $\delta_1(r, \epsilon) = 1$ if $D_1(r, \epsilon) = \emptyset$.

Definition 2.6. [8] A $\rho \in \mathfrak{R}$ satisfy (UC1) if $\forall r > 0$, $\epsilon > 0$, we have $\delta_1(r, \epsilon) > 0$. Note that $\forall r > 0$, $D_1(r, \epsilon) \neq \emptyset$ for $\epsilon > 0$ small enough.

Definition 2.7. [8] A $\rho \in \mathfrak{R}$ satisfy (UUC1) if $\forall s \geq 0$, $\epsilon > 0$, $\exists \eta_1(s, \epsilon) > 0$ depending only upon s and ϵ such that $\delta_1(r, \epsilon) > \eta_1(s, \epsilon) > 0$ for any $r > s$.

Definition 2.8. [12] Let $\rho \in \mathfrak{R}$ and $D \subset L_\rho$.

(i) $\{f_n\} \subset L_\rho$ is ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(ii) $\{f_n\} \subset L_\rho$ is ρ-Cauchy, if $\rho(f_n - f_m) \to 0$ as $m, n \to \infty$.

(iii) D is ρ-closed if for $\{f_n\} \subset D$, $f_n \to f$ implies $f \in D$.

(iv) D is ρ-compact if for $\{f_n\}\subset D$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f\in D$ such that $\rho(f_n-f)\to 0$ as $k\to\infty$.

(v) D is ρ- a.e. closed if for $\{f_n\}\subset D$ which is ρ- a.e. convergent, $f_n\to f$ as $n\to\infty$ implies $f\in D$.

(vi) D is ρ- a.e. compact if for $\{f_n\}\subset D$, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f\in D$ exist such that $\rho(f_n-f)\to 0$ as $k\to\infty$.

(vii) D is ρ-bounded if the ρ-diameter of D is finite, that is

$$\mathrm{diam}_\rho(D)=\sup\{\rho(f-g):\ f,g\in D\}<\infty.$$

ρ-convergence does not imply ρ-Cauchy. If ρ satisfy the Δ_2-condition, then this will be true. The ρ-distance from $f\in L_\rho$ to $D\subset L_\rho$ is defined by

$$\mathrm{dist}_\rho(f,D)=\inf\{\rho(f-h):\ h\in D\}.$$

Definition 2.9. [10] A set $D\subset L_\rho$ is called ρ-proximinal if $\forall f\in L_\rho, \exists\, g\in D$ such that $\rho(f-g)=\mathrm{dist}_\rho(f,D)$.

Let $P_\rho(D)$ be the family of non-empty, ρ-bounded, ρ-proximinal subsets of D and $C_\rho(D)$ be the family of non-empty, ρ-closed, ρ-bounded subsets of D.

We define ρ-Hausdorff distance $H_\rho(\cdot\,,\cdot)$ on $C_\rho(L_\rho)$ as

$$H_\rho(A,B)=\max\left\{\sup_{f\in A}\ \mathrm{dist}_\rho(f,B),\sup_{g\in B}\ \mathrm{dist}_\rho(g,A)\right\},\ A,B\in C_\rho(L_\rho).$$

Definition 2.10. [10] A multivalued mapping $T{:}D\to C_\rho(D)$ is

(i) ρ-nonexpansive if $H_\rho(Tf,Tg)\leqslant\rho(f-g)\ \ \forall f,g\in D$.

(ii) ρ-quasi-nonexpansive if $H_\rho(Tf,p)\leqslant\rho(f-p)\ \ \forall f\in D$ and $p\in F_\rho(T)$.

Theorem 2.11. [6] *Let $\rho\in\Re$ satisfy the Δ_2-condition. Consider the sequences $\{f_n\}$ and $\{g_n\}$ in L_ρ. Then*

$$\lim_{n\to\infty}\rho(g_n)=0 \text{ implies } \limsup_{n\to\infty}\ \rho(f_n+g_n)=\limsup_{n\to\infty}\ \rho(f_n)$$

and

$$\lim_{n\to\infty}\rho(g_n)=0 \text{ implies } \liminf_{n\to\infty}\ \rho(f_n+g_n)=\liminf_{n\to\infty}\ \rho(f_n)$$

A sequence $\{t_n\} \subset (0,1)$ is bounded away from 0 if $\exists\, a > 0$ such that $t_n \geqslant a \ \forall n \in \mathbb{N}$, and is bounded away from 1 if $\exists\, b < 1$ such that $t_n \leq b \ \forall n \in \mathbb{N}$.

Lemma 2.12. [3, 8] *Let (UUC1) be satisfied by $\rho \in \Re$, and $\{t_k\} \subset (0,1)$ be bounded away from 0 and 1. If $\exists\, R > 0$ such that*

$$\limsup_{n \to \infty} \rho(f_n) \leqslant R, \ \limsup_{n \to \infty} \rho(g_n) \leqslant R \ \textit{and} \ \lim_{n \to \infty} \rho(t_n f_n + (1 - t_n) g_n) = R,$$

then

$$\lim_{n \to \infty} \rho(f_n - g_n) = 0.$$

Definition 2.13. $f \in L_\rho$ is a fixed point of $T : L_\rho \to P_\rho(D)$ if $f \in Tf$.

Let $F_\rho(T)$ be the set of all fixed points of T.

Definition 2.14. [11] A multivalued mapping $T : D \to P_\rho(D)$ satisfy $\text{Condition}(\text{I})$ if there exists a continuous non-decreasing function $l : [0,\infty) \to [0,\infty)$ with $l(0) = 0$, $l(r) > 0 \ \forall r \in (0,\infty)$ such that $\text{dist}_\rho(f, Tf) \geqslant l(\text{dist}_\rho(f, F_\rho(T))) \ \forall f \in D$.

Lemma 2.15. [12] *For a multivalued mapping $T : D \to P_\rho(D)$ with*

$$P_\rho^T(f) = \{g \in Tf : \rho(f - g) = \text{dist}_\rho(f, Tf)\},$$

the following statements are equivalent:

(i) $f \in F_\rho(T)$, *i.e.,* $f \in Tf$.

(ii) $P_\rho^T(f) = \{f\}$, *i.e,* $f = g$ *for each* $g \in P_\rho^T(f)$.

(iii) $f \in F(P_\rho^T(f))$, *i.e,* $f \in P_\rho^T(f)$. *Also,* $F_\rho(T) = F(P_\rho^T(T))$, *with* $F(P_\rho^T(f))$ *being the set of fixed points of* $P_\rho^T(f)$.

Definition 2.16. [8] A set $C \subset L_\rho$ is said to possess the Vitali property if $C \subset E_\rho$, and for any $g \in L_\rho$ and $g_n \in C$ with $\rho(g_n - g) \to 0$, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that for every $\alpha > 0$ the subadditive measures $\rho(\alpha g_{n_k}, \cdot)$ are order equicontinuous.

Definition 2.17. [8] The function modular $\rho \in \Re$ is called separable if $\|f 1_{(\cdot)}\|_\rho$ is a separable set function for each $f \in \mathcal{E}$, which means that there exists a countable $\mathcal{A} \in \mathcal{P}$ such that to every $A \in \mathcal{P}$ there corresponds a sequence $\{A_k\}$ of elements of $\mathcal{A}$ with $\rho(\alpha f, A \Delta A_k) \to 0$ for every $\alpha > 0$, where Δ denotes the symmetric difference.

3. Convergence Analysis

The Picard-Abbas iteration for singlevalued mappings was introduced by Chyne and Kumar [11]. We define the Picard-Abbas iteration for multivalued mapping as follows:

$$\begin{aligned} f_{n+1} &\in P_T(h_n) \\ h_n &= (1-\alpha_n)w_n + \alpha_n v_n \\ g_n &= (1-\beta_n)u_n + \beta_n v_n \\ e_n &= (1-\gamma_n)f_n + \gamma_n u_n \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$, and $u_n \in P_T(f_n),\ v_n \in P_T(e_n), w_n \in P_T(g_n)$.

Theorem 3.1. *Let Δ_2-condition and (UUC1) be satisfied by $\rho \in \Re$. Let $D \neq \emptyset$ be ρ-bounded, ρ-closed, and convex in L_ρ. Consider a multivalued mapping $T: D \to P_\rho(D)$ with P_ρ^T being a ρ-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$. If $\{f_n\} \subset D$ be defined by (3.1), then $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F_\rho(T)$.*

Proof. Let $p \in F_\rho(T)$. Using Lemma 2.15, we get

$$P_\rho^T(p) = p$$

$$\rho(f_{n+1} - p) \leqslant H_\rho(P_\rho^T(h_n), P_\rho^T(p)) \leqslant \rho(h_n - p) \tag{3.2}$$

$$\begin{aligned} \rho(h_n - p) &= \rho((1-\alpha_n)w_n + \alpha_n v_n - p) \\ &\leqslant (1-\alpha_n)\rho(w_n - p) + \alpha_n(v_n - p) \quad [\text{using convexity of } \rho] \\ &\leqslant (1-\alpha_n)H_\rho(P_\rho^T(g_n), P_\rho^T(p)) + \alpha_n H_\rho(P_\rho^T(e_n), P_\rho^T(p)) \\ &\leqslant (1-\alpha_n)\rho(g_n - p) + \alpha_n \rho(e_n - p) \end{aligned} \tag{3.3}$$

$$\begin{aligned} \rho(g_n - p) &= \rho((1-\beta_n)u_n + \beta_n v_n - p) \\ &\leqslant (1-\beta_n)\rho(u_n - p) + \beta_n(v_n - p) \quad [\text{using convexity of } \rho] \\ &\leqslant (1-\beta_n)H_\rho(P_\rho^T(f_n), P_\rho^T(p)) + \beta_n H_\rho(P_\rho^T(e_n), P_\rho^T(p)) \\ &\leqslant (1-\beta_n)\rho(f_n - p) + \beta_n \rho(e_n - p) \end{aligned} \tag{3.4}$$

$$\begin{aligned} \rho(e_n - p) &= \rho((1-\gamma_n)f_n + \gamma_n u_n - p) \\ &\leqslant (1-\gamma_n)\rho(f_n - p) + \gamma_n(u_n - p) \quad [\text{using convexity of } \rho] \\ &\leqslant (1-\gamma_n)\rho(f_n - p) + \gamma_n H_\rho(P_\rho^T(f_n), P_\rho^T(p)) \\ &\leqslant (1-\gamma_n)\rho(f_n - p) + \gamma_n \rho(f_n - p) \\ &\leqslant \rho(f_n - p) \end{aligned} \tag{3.5}$$

Using (3.5) in (3.4), we get $$\rho(g_n - p) \leqslant \rho(f_n - p) \tag{3.6}$$

Using (3.5) and (3.6) in (3.3), we get

$$\rho(h_n - p) \leqslant \rho(f_n - p) \tag{3.7}$$

From (3.2) and (3.7), we get

$$\rho(f_{n+1} - p) \leqslant \rho(f_n - p) \tag{3.8}$$

Therefore, the sequence $\{\rho(f_n - p)\}$ is decreasing.

Thus, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F_\rho(T)$.

Theorem 3.2. *Let Δ_2-condition and (UUC1) be satisfied by $\rho \in \Re$. Let $D \neq \emptyset$ be ρ-bounded, ρ-closed, and convex in L_ρ. Consider a multivalued mapping $T{:}D \to P_\rho(D)$ with P_ρ^T being a ρ-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$. If $\{f_n\} \subset D$ be defined by (3.1, then* $\lim_{n\to\infty} \operatorname{dist}_\rho(f_n, P_\rho^T(f_n)) = 0$.

Proof. By Theorem 3.1, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F_\rho(T)$. Let

$$\lim_{n\to\infty} \rho(f_n - p) = L \tag{3.7}$$

Since $\operatorname{dist}_\rho(f_n, P_\rho^T(f_n)) \leqslant \rho(f_n - u_n)$, it suffices to show that $\lim_{n\to\infty} \rho(f_n - u_n) = 0$.

Now

$$\rho(u_n - p) \leqslant H_\rho(P_\rho^T(f_n), P_\rho^T(p)) \leqslant \rho(f_n - p)$$

implies

$$\limsup_{n\to\infty} \rho(u_n - p) \leqslant \limsup_{n\to\infty} \rho(f_n - p)$$

So, (3.7) gives

$$\limsup_{n\to\infty} \rho(u_n - p) \leqslant L \tag{3.8}$$

Also, from (3.5), we get

$$\limsup_{n\to\infty} \rho(e_n - p) \leqslant \limsup_{n\to\infty} \rho(f_n - p)$$

So,

$$\limsup_{n\to\infty} \rho(e_n - p) \leqslant L \tag{3.9}$$

Similarly, we can show that

$$\limsup_{n\to\infty} \rho(g_n - p) \leqslant L \tag{3.10}$$

and

$$\limsup_{n\to\infty} \rho(h_n - p) \leqslant L \tag{3.11}$$

Now

$$\rho(w_n - p) \leqslant H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \leqslant \rho(g_n - p) \leqslant \rho(f_n - p)$$

So,

$$\limsup_{n\to\infty} \rho(w_n - p) \leqslant \limsup_{n\to\infty} \rho(f_n - p) \leqslant L \tag{3.12}$$

Similarly, we can show that

$$\limsup_{n\to\infty} \rho(v_n - p) \leqslant L \tag{3.13}$$

Again

$$\begin{aligned}\lim_{n\to\infty} \rho(f_{n+1} - p) &= \lim_{n\to\infty} \rho((1-\alpha_n)w_n + \alpha_n v_n - p)\\ &= \lim_{n\to\infty} \rho((1-\alpha_n)(w_n - p) + \alpha_n(v_n - p))\\ &= L \end{aligned}\tag{3.14}$$

Therefore, (3.12), (3,13), (3.14) and Lemma 2.12 gives

$$\lim_{n\to\infty} \rho(w_n - v_n) = 0 \tag{3.15}$$

Let $\varepsilon > 0$ be given. There exists $n_0 \in \mathbb{N}$ such that $\rho(w_n - v_n) < \varepsilon$ for all $n \geqslant n_0$.

Since $\rho(\alpha_n(w_n - v_n)) \leqslant \omega_\rho(\alpha_n)\rho(w_n - v_n) < \varepsilon$, we have

$$\lim_{n\to\infty} \rho(\alpha_n(w_n - v_n)) = 0.$$

Also,

$$\rho(f_{n+1} - p) = \rho((1-\alpha_n)w_n + \alpha_n v_n - p) = \rho((w_n - p) + \alpha_n(v_n - w_n))$$

From Theorem 2.11, we get

$$\liminf_{n\to\infty} \rho((w_n - p) + \alpha_n(v_n - w_n)) = \liminf_{n\to\infty} \rho(w_n - p).$$

So,

$$\liminf_{n\to\infty} \rho(w_n - p) = L \tag{3.16}$$

From (3.12) and (3.16), we get

$$\lim_{n\to\infty} \rho(w_n - p) = L \tag{3.17}$$

Using (3.15) and Theorem 2.11, we get

$$L = \liminf_{n\to\infty} \rho(w_n - p) = \liminf_{n\to\infty} \rho((w_n - v_n) + (v_n - p)) = \liminf_{n\to\infty} \rho(v_n - p).$$

But

$$\rho(v_n - p) \leqslant H_\rho(P_\rho^T(e_n), P_\rho^T(p)) \leqslant \rho(e_n - p).$$

Therefore,

$$L \leqslant \liminf_{n\to\infty} \rho(e_n - p) \tag{3.18}$$

From (3.9) and (3.18) we get

$$\lim_{n\to\infty} \rho(e_n - p) = L$$

That is,

$$\lim_{n\to\infty} \rho((1-\gamma_n)(f_n - p) + \gamma(u_n - p)) = L.$$

From (3.7), (3.8) and Lemma 2.12, we get

$$\lim_{n\to\infty} \rho(f_n - u_n) = 0.$$

Hence, $\lim_{n\to\infty} \mathrm{dist}_\rho(f_n, P_\rho^T(f_n)) = 0$.

Theorem 3.3. *Let Δ_2-condition and (UUC1) be satisfied by $\rho \in \Re$. Let $D \neq \emptyset$ be ρ-bounded, ρ-closed, and convex in L_ρ. Consider a multivalued mapping $T: D \to P_\rho(D)$ with P_ρ^T being a ρ-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$. If $\{f_n\} \subset D$ be defined by (3.1), then $\{f_n\}$ is ρ-convergent to a fixed point of T.*

Proof. By D being compact, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ exists, such that for some $q \in D$ $\lim_{k\to\infty} \rho(f_{n_k} - p) = 0$. We show that $q \in F_\rho(T)$.

Let g be arbitrary chosen from $P_\rho^T(q)$ and f from $P_\rho^T(f_{n_k})$.

Now,

$$\rho\left(\frac{q-g}{3}\right) \leqslant \rho\left(\frac{q-f_{n_k}}{3}\right) + \rho\left(\frac{f_{n_k}-f}{3}\right) + \rho\left(\frac{f-g}{3}\right)$$

$$\leqslant \frac{1}{3}\rho(q - f_{n_k}) + \frac{1}{3}\rho(f_{n_k} - f) + \frac{1}{3}\rho(f - g)$$

$$\leqslant \rho(q - f_{n_k}) + \mathrm{dist}_\rho(f_{n_k}, P_\rho^T(f_{n_k})) + \mathrm{dist}_\rho(P_\rho^T(f_{n_k}), g)$$

$$\leqslant \rho(q - f_{n_k}) + \mathrm{dist}_\rho(f_{n_k}, P_\rho^T(f_{n_k})) + H_\rho(P_\rho^T(f_{n_k}), P_\rho^T(q))$$

$$\leqslant \rho(q - f_{n_k}) + \mathrm{dist}_\rho(f_{n_k}, P_\rho^T(f_{n_k})) + \rho(q - f_{n_k})$$

Using theorems (3.1) and (3.2), we get

$$\rho(q - g) = 0.$$

Thus, $q \in F(P_\rho^T) = F_\rho(T)$. That is, $\{f_n\}$ is ρ-convergent to a fixed point of T.

Theorem 3.4. *Let Δ_2-condition and (UUC1) be satisfied by $\rho \in \Re$. Let $D \neq \emptyset$ be ρ-bounded, ρ-closed, and convex in L_ρ. Consider a multivalued mapping $T: D \to P_\rho(D)$ satisfying Condition (I)* *with P_ρ^T being a ρ-quasi-nonexpansive mapping, and $F_\rho(T)\neq\emptyset$. If $\{f_n\}\subset D$ be defined by (3.1), then $\{f_n\}$ is ρ-convergent to a fixed point of T.*

Proof.

We have shown in Theorem 3.1 that $\lim\limits_{n\to\infty}\rho(f_n-p)$ exists for all $p\in F(P_\rho^T)=F_\rho(T)$.

If $\lim\limits_{n\to\infty}\rho(f_n-p)=0$, there is nothing to prove. We assume $\lim\limits_{n\to\infty}\rho(f_n-p)=L>0$.

Again, from Theorem 3.1, we have $\rho(f_{n+1}-p)\leqslant\rho(f_n-p)$. So,

$$\mathrm{dist}_\rho(f_{n+1},F_\rho(T))\leqslant\mathrm{dist}_\rho(f_n,F_\rho(T)).$$

Hence, $\lim\limits_{n\to\infty}\mathrm{dist}_\rho(f_n,F_\rho(T))$ exists. We show that $\lim\limits_{n\to\infty}\mathrm{dist}_\rho(f_n,F_\rho(T))=0$.

Using Theorem 3.1 and Condition (I), we get

$\lim\limits_{n\to\infty}l\big(\mathrm{dist}_\rho(f_n,F_\rho(T))\big)\leqslant\lim\limits_{n\to\infty}\mathrm{dist}_\rho(f_n,F_\rho(T))=0$.

That is, $\lim\limits_{n\to\infty}l\big(\mathrm{dist}_\rho(f_n,F_\rho(T))\big)=0$.

Since l is nondecreasing function and $l(0)=0$, we have $\lim\limits_{n\to\infty}\big(\mathrm{dist}_\rho(f_n,F_\rho(T))\big)=0$.

Next, we show that $\{f_n\}$ is a ρ-Cauchy sequence in D.

Let $\varepsilon>0$. Since $\lim\limits_{n\to\infty}\mathrm{dist}_\rho(f_n,F_\rho(T))=0$, there is a constant $n_0\in\mathbb{N}$ such that $\forall n\geqslant n_0$, we have

$$\mathrm{dist}_\rho(f_n,F_\rho(T))<\frac{\varepsilon}{2}.$$

In particular, $\inf\{\rho(f_{n_0})-p\colon p\in F_\rho(T)\}<\frac{\varepsilon}{2}$. The must exist $p^*\in F_\rho(T)$ such that

$$\rho\big(f_{n_0}-p^*\big)<\varepsilon.$$

For $m,n\geqslant n_0$, we have

$$\rho\left(\frac{f_{n+m}-f_n}{2}\right)\leqslant\frac{1}{2}\rho(f_{n+m}-p^*)+\frac{1}{2}\rho(f_n-p^*)\leqslant\rho(f_{n_0}-p^*)<\varepsilon.$$

Therefore, $\{f_n\}$ is a ρ-Cauchy in $D\subset L_\rho$. Thus, it is convergent in D.

Let $\lim\limits_{n\to\infty}f_n=q$. Using Theorem 3.3, we get $q\in F_\rho(T)$.

Hence, $\{f_n\}$ is ρ-convergent to a fixed point of T.

4. Stability Analysis

Here we give a stability result for the Picard-Abbas iteration. We begin by stating the stability definition as follows.

Definition 4.1. [29] Let $D \subset L_\rho$ $(D \neq \emptyset)$ and operator $T: D \to D$. For a fixed $x_1 \in D$, let $x_{n+1} = f(T, x_n)$ be an iteration generating a sequence $\{x_n\}_{n=1}^{\infty} \subset D$. Let $\{x_n\}_{n=1}^{\infty}$ be strongly convergent to $p \in F_\rho(T) \neq \emptyset$. Let the sequence $\{y_n\}_{n=1}^{\infty}$ be bounded in D and $\varepsilon_n = \rho(y_{n+1} - f(T, y_n))$.

(i) $\{x_n\}_{n=1}^{\infty}$ is T-stable on D if $\lim\limits_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim\limits_{n\to\infty} y_n = p$

(ii) $\{x_n\}_{n=1}^{\infty}$ is almost T-stable on D if $\sum\limits_{n=1}^{\infty} \varepsilon_n < \infty \Rightarrow \lim\limits_{n\to\infty} y_n = p$.

Theorem 4.2. *Let $D \subset L_\rho$ $(D \neq \emptyset)$ be convex and bounded. If $T: D \to P_\rho^T(D)$ be a multivalued mapping with P_ρ^T being a ρ-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$, then the iteration (3.1) is T-stable.*

Proof. Let $\{y_n\} \subset D$. Define $\varepsilon_n = \rho(y_{n+1} - f(T, y_n))$. Let $p \in F_\rho(T)$ be unique.

Suppose $\lim\limits_{n\to\infty} y_n = p$. Using (3.1) and the convexity of ρ, we have

$$\begin{aligned} \varepsilon_n &= \mathrm{dist}_\rho(P_\rho^T(y_{n+1}), P_\rho^T(h_n)) \\ &\leqslant H_\rho(P_\rho^T(y_{n+1}), P_\rho^T(h_n)) \\ &\leqslant \rho(y_{n+1} - h_n) \\ &\leqslant \rho(y_{n+1} - p) + \rho(p - h_n) \\ &\leqslant \rho(y_{n+1} - p) + \rho(y_n - p) \end{aligned} \tag{4.1}$$

Thus, $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Conversely, let $\lim\limits_{n\to\infty} \varepsilon_n = 0$. By (4.1), we get $\lim\limits_{n\to\infty} y_n = p$.

Therefore, $\lim\limits_{n\to\infty} y_n = p$ if and only if $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Hence the proof.

5. Applications to differential equations

Let $\rho \in \Re$. For an unknown function $u: [0, A] \to C$, with $C \subset E_\rho$, consider the initial value problem (IVP)

$$\begin{cases} u(0)=f \\ u'(t)+(I-T)u(t), \end{cases} \tag{5.1}$$

for fixed $f \in C$, $A > 0$ and $T:C \to C$ with P_ρ^T being ρ-quasi-nonexpansive mapping. For any $t > 0$ define

$$K(t)=1-e^{-t}=\int_0^t e^{s-t}ds. \tag{5.2}$$

For a function $v:[0,A] \to L_\rho$, $A > 0$, $t \in [0,A]$, define

$$S(v)\,(t)=\int_0^t e^{s-t}v(s)ds. \tag{5.3}$$

Let us denote

$$S_\tau(v)\,(t)=\sum_{t=0}^{n-1}(t_{i+1}-t_i)e^{t_i-t}v(t_i), \tag{5.4}$$

for any subdivision $\tau=\{t_{0,}t_1,...,t_n\}$ of $[0,A]$.

Lemma 5.1. [8] *Consider a separable* $\rho \in \Re$. *Let* $x,y:[0,A] \to L_\rho$ *be two Bochner-integrable* $\|\cdot\|_\rho$*-bounded functions, with* $A > 0$. *Then* $\forall t \in [0,A]$, *we have*

$$\rho\left(e^{-t}y(t)+\int_0^t e^{s-t}x(s)ds\right) \leqslant e^{-t}\pi(y(t))+K(t)\sup_{s\in[0,t]}\rho(x(s)). \tag{5.5}$$

We now prove our Theorem.

Theorem 5.2. *Consider a separable* $\rho \in \Re$. *Let* $D \subset E_\rho$ *be non-empty,* ρ*-closed,* ρ*-bounded, convex set having Vitali property, and a multivalued mapping* $T:D \to P_\rho(D)$ *with* P_ρ^T *being a* ρ*-quasi-nonexpansive mapping. For fixed* $f \in C, A > 0$, *we define* $u_n:[0,A] \to C$

by
$$\begin{cases} u_0(t)=f \\ u_{n+1}(t)=e^{-t}f+\int_0^t e^{s-t}T(u_n(s))ds. \end{cases} \tag{5.6}$$

Then $\forall t \in [0,A]$ $\exists\, u(t) \in C$ *such that*

$$\rho(u_n(t)-u(t)) \to 0 \tag{5.7}$$

and $u:[0,A] \to C$ *defined by (5.7) is a solution of the IVP (5.1).*

Moreover,

$$\rho(f-u_n(t)) \leqslant K^{n+1}(A)\delta_\rho(C). \tag{5.8}$$

Proof. The proof follows that of ([15], Theorem 5.28), since P_ρ^T being a ρ-quasi-nonexpansive mapping.

Corollary 5.3. *Consider a separable $\rho \in \Re$. Let $D \subset E_\rho$ be non-empty, ρ-closed, ρ-bounded, convex set having Vitali property, and a multivalued mapping $T: D \to P_\rho(D)$ with P_ρ^T being a ρ-nonexpansive mapping. For fixed $f \in C, A > 0$, we define $u_n: [0, A] \to C$ by*

$$\begin{cases} u_0(t) = f \\ u_{n+1}(t) = e^{-t} f + \int_0^t e^{s-t} T(u_n(s)) ds. \end{cases} \tag{5.9}$$

Then $\forall t \in [0, A]$ $\exists u(t) \in C$ such that

$$\rho(u_n(t) - u(t)) \to 0 \tag{5.10}$$

and $u: [0, A] \to C$ defined by (5.10) is a solution of the IVP (5.1).

Moreover,

$$\rho(f - u_n(t)) \leqslant K^{n+1}(A) \delta_\rho(C). \tag{5.11}$$

Corollary 5.4. *Consider a separable $\rho \in \Re$. Let $D \subset E_\rho$ be non-empty, ρ-closed, ρ-bounded, convex set having Vitali property, and a multivalued mapping $T: D \to P_\rho(D)$ with P_ρ^T being a ρ-contraction mapping. For fixed $f \in C, A > 0$, we define $u_n: [0, A] \to C$ by*

$$\begin{cases} u_0(t) = f \\ u_{n+1}(t) = e^{-t} f + \int_0^t e^{s-t} T(u_n(s)) ds. \end{cases} \tag{5.12}$$

Then $\forall t \in [0, A]$ $\exists u(t) \in C$ such that

$$\rho(u_n(t) - u(t)) \to 0 \tag{5.13}$$

and $u: [0, A] \to C$ defined by (5.13) is a solution of the IVP (5.1).

Moreover,

$\rho(f - u_n(t)) \leqslant K^{n+1}(A) \delta_\rho(C).$ (5.14).

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