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A Study of Bipolar Fuzzy Prime Ideals of a Lattice

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Abstract:

This study explores the investigation of Bipolar Fuzzy Prime Ideals (BFPI) in lattices. We provide a detailed exploration of their properties, characterizations, and associated homomorphisms.

Introduction: Fuzzy set theory, introduced by Zadeh L.A., is grounded in the concept of membership functions where each element in a set is assigned a membership degree ranging between 0 and 1. Although this model effectively combines supporting and opposing evidence for element membership, it lacks explicit representation of the uncertainty or dual nature of these evidence. To address this limitation, Gau and Buehrer introduced the concept of vague sets, characterized by two functions: one for membership and another for non-membership, where their sum does not exceed one.

Further contributions to fuzzy set theory came from Atanassov's intuitionistic fuzzy sets and Bustince and Burillo's work showing their mathematical equivalence to vague sets. The dual-function approach of vague sets has been applied extensively in decision-making, control systems, and fault diagnosis. Lattice theory has also benefited from these advancements, with Ajmal and Thomas pioneering fuzzy sublattice theory, and later works exploring intuitionistic fuzzy lattices and vague lattices. Bipolar fuzzy sets (BFS), introduced by Lee K.M., extended fuzzy sets by incorporating dual notions of positive and negative membership values within a range of [-1, 1]. This extension enables interpretations of bipolar information, making BFS a valuable tool in decision-making and information processing.

Objectives: Introduction of Bipolar fuzzy prime ideals of a Lattice, study of their characterizations and associated homomorphisms.

Keywords: Bipolar fuzzy ideal, Bipolar fuzzy prime ideal, Bipolar fuzzy homomorphism, Bipolar fuzzy magnified translation.

1. Introduction

Fuzzy set theory, introduced by Zadeh L.A. [1], is grounded in the concept of membership functions where each element in a set is assigned a membership degree ranging between 0 and 1. Although this model effectively combines supporting and opposing evidence for element membership, it lacks explicit representation of the uncertainty or dual nature of these evidences. To address this limitation, Gau and Buehrer [2] introduced the concept of vague sets, characterized by two functions: one for membership and another for non-membership, where their sum does not exceed one.

Further contributions to fuzzy set theory came from Atanassov's intuitionistic fuzzy sets [5]and Bustince and Burillo's [3]work showing their mathematical equivalence to vague sets. The dual-function approach of vague sets has been applied extensively in decision-making, control systems, and fault diagnosis. Lattice theory has also benefited from these advancements, with Ajmal and Thomas pioneering fuzzy sublattice theory, and later works exploring intuitionistic fuzzy lattices and vague lattices. Bipolar fuzzy sets (BFS), introduced by Lee K.M.[6], extended fuzzy sets by incorporating dual notions of positive and negative membership values within a range of [-1, 1]. This extension enables interpretations of bipolar information, making BFS a valuable tool in decision-making and information processing.

In particular, Ajmal. N and Thomas.K.V [7] both explored theory of (FL)fuzzy sublattice and introduced the idea of fuzzy sets to lattice theory. After then, in 2011, Thomas.K.V and Nair L.S.[4] presented idea of intuitionistic fuzzy lattices (IFLs). In 2017, Milles S [13] investigated the characterization of IFIs and IFFs based on lattice operations. Rao.R.P [15] later researched rough vague lattices in 2019. Nageswara Rao.B., RamaKrishna N and Eswarlal.T [14] introduced vague lattices(VL) in 2020. The principal IFI and IFF on a lattice were the subject of Boudaoud.S, Zedam.L and Milles S [10] study in 2020. Milles.S.[11] as well as studied the lattice of (IFT)intuitionistic fuzzy topologies produced by intuitionistic fuzzy relations in 2020. On residual lattices, Zhang H and Qingguo.Li [12] researched the (IFF)intuitionistic fuzzy filter theory. Bipolar vague cosets [9], Homomorphism on bipolar vague normal groups [8] were studied by Venkata Kalyani U in 2020. Later, Bipolar fuzzy sublattices were introduced in 2023 by Venkata Kalyani U and studied bipolar fuzzy ideals of a lattice [16]. Venkata Kalyani U studied the bipolar magnified fuzzy translation of a lattice [17] in 2024. Now, in this paper we introduce the theory of bipolar fuzzy prime ideals(BFPI) and their characterization using level sets and homomorphism of BFPI of a lattice.

2. Preliminaries

Definition 2.1[7]: "A poset $(\mathfrak{L}, \leq)$ is called a lattice if Sup$\{ p, q \}$ (also denoted by $p \vee q$) and Inf$\{p, q\}$ (also denoted by $p \wedge q$) exists for every pair of elements p, q in $\mathfrak{L}$."

Definition 2.2[1]: "Let $\mathcal{F}$ be any non-empty set. A mapping $\psi: \mathcal{F} \to [0,1]$ is called a fuzzy subset of $\mathcal{F}$."
Definition 2.3[1]: "Let $\psi: \mathcal{F} \to [0,1]$ be any FS. Then the set $\{\psi(p)/p \in \mathcal{F}\}$ is called the image of ψ and is denoted by Im(ψ). For $t \in [0,1], \psi_t = \{p \in \mathcal{F}/\psi(p) \geq t\}$ is called a level subset of ψ."
Definition 2.4[2]: "A vague set κ in the universe of discourse $\mathcal{F}$ is characterized by two M_{ship} functions given by
(i)A truth M_{ship} function $t_\kappa: \mathcal{F} \to [0,1]$ and
(ii) A false M_{ship} function $f_\kappa: \mathcal{F} \to [0,1]$,
where $t_\kappa(p)$ is a lower bound of the grade of M_{ship} of p derived from the evidence for p and $f_\kappa(p)$ is a lower bound on the negation of p derived from the evidence against p, with $t_\kappa(p) + f_\kappa(p) \leq 1$ ".

We give below a formation of the definition of vague set in the following way, which makes Atanassov, K.T.s intuitionistic fuzzy sets and Gau, W.L. and Buehrer, D.J.[2] vague sets in a mathematically equivalent form.

Definition 2.5[2]: "Let κ be a vague set of a universe F with true M_{ship} function t_κ and false M_{ship} function f_κ. For $\alpha, \Upsilon \in [0,1]$ with $\alpha \leq \Upsilon$, the (α, Υ)- cut or vague cut of a vague set κ is the crisp subset of F is given by $\kappa_{(\alpha,\Upsilon)} = \{p \in F / V_\kappa(p) \geq [\alpha, \Upsilon]\}$
i.e., $\kappa_{(\alpha,\Upsilon)} = \{p \in F / t_\kappa(p) \geq \alpha \text{ and } 1 - f_\kappa(p) \geq \Upsilon\}$."

Definition 2.6[2]: "The α-cut, κ_α of the vague set κ is the (α, α)-cut of κ and hence given by $\kappa_\alpha = \{p \in F / t_\kappa(p) \geq \alpha\}$."

Definition 2.7[6]: "Suppose F be a universal set. A (BFS) bipolar fuzzy set $\mathbb{B}$ in F is an object having the form $\mathbb{B} = \{< \hbar, \mathbb{B}^P(\hbar), \mathbb{B}^N(\hbar) > / \hbar \in F\}$ where $\mathbb{B}^P: F \to [0,1]$ and $\mathbb{B}^N: F \to [-1,0]$ are a positive and negative M_{ship} functions, respectively."

Definition 2.8[6]: " Let F be a nonempty set, and let $\mathbb{B}_\vartheta, \mathbb{B}_\omega \in BPFS(F)$.
(i) $\mathbb{B}_\vartheta$ is a subset of $\mathbb{B}_\omega$, denoted by $\mathbb{B}_\vartheta \subseteq \mathbb{B}_\omega$, if for each $\hbar \in F$, $\mathbb{B}^P_\vartheta(\hbar) \leq \mathbb{B}^P_\omega(\hbar)$ and $\mathbb{B}^N_\vartheta(\hbar) \geq \mathbb{B}^N_\omega(\hbar)$.
(ii) The complement of $\mathbb{B}_\vartheta$, denoted by $\mathbb{B}^c_\vartheta = ((\mathbb{B}^c_\vartheta)^N, (\mathbb{B}^c_\vartheta)^P)$, is a BFS in F defined as: for each $\hbar \in F$, $\mathbb{B}^c_\vartheta(\hbar) = \left(-1 - \mathbb{B}^N_\vartheta(\hbar), 1 - \mathbb{B}^P_\vartheta(\hbar)\right)$, i.e., $(\mathbb{B}^c_\vartheta)^P(\hbar) = 1 - \mathbb{B}^N_\vartheta(\hbar), (\mathbb{B}^c_\vartheta)^N(\hbar) = -1 - \mathbb{B}^N_\vartheta(\hbar)$.
(iii) The intersection of $\mathbb{B}_\vartheta$ and $\mathbb{B}_\omega$, denoted by $\mathbb{B}_\vartheta \cap \mathbb{B}_\omega$, is a BFS in F defined as: for each $\hbar \in F$, $(\mathbb{B}_\vartheta \cap \mathbb{B}_\omega)(\hbar) = \left(\mathbb{B}^N_\vartheta(\hbar) \vee \mathbb{B}^N_\omega(\hbar), \mathbb{B}^P_\vartheta(\hbar) \wedge \mathbb{B}^P_\omega(\hbar)\right)$.
(iv) The union of $\mathbb{B}_\vartheta$ and $\mathbb{B}_\omega$, denoted by $\mathbb{B}_\vartheta \cup \mathbb{B}_\omega$, is a BFS in F defined as: for each $\hbar \in F$, $(\mathbb{B}_\vartheta \cup \mathbb{B}_\omega)(\hbar) = \left(\mathbb{B}^N_\vartheta(\hbar) \wedge \mathbb{B}^N_\omega(\hbar), \mathbb{B}^P_\vartheta(\hbar) \vee \mathbb{B}^P_\omega(\hbar)\right)$."

Definition:2.9[17]: "Let $\mathfrak{B} =< \mathfrak{B}^P, \mathfrak{B}^N >$ be a BFS in F and $(\alpha, \omega) \in [0,1], (\theta, \vartheta) \in [\nabla, 0] \times [0, \triangle]$. By a BFMT of $B =< \mathfrak{B}^N, \mathfrak{B}^P >$, we mean a BFS $M = \left\{< r, \mathfrak{B}^P_{(\omega,\vartheta)}(r), \mathfrak{B}^N_{(\alpha,\theta)}(r) >: r \in F\right\}$ or simply as $M = \{< r, \mathfrak{B}^P_M(r), \mathfrak{B}^N_M(r) >: r \in F\}$, where $\mathfrak{B}^P_M(r) = \mathfrak{B}^P_{(\omega,\vartheta)}: F \to [0,1]$ and $\mathfrak{B}^N_M = \mathfrak{B}^N_{(\alpha,\theta)}: F \to [-1,0]$
and defined by $\mathfrak{B}^P_M(r) = \mathfrak{B}^P_{(\omega,\vartheta)}(r) = \omega\mathfrak{B}^P(r) + \vartheta$ for all $r \in F$ and $\mathfrak{B}^N_M(r) = \mathfrak{B}^N_{(\alpha,\theta)}(r) = \alpha\mathfrak{B}^N(r) + \theta$."

Definition:2.10[16]: "Suppose $\mathfrak{B} = (\mathfrak{B}^P, \mathfrak{B}^N)$ is a BFS in $\mathfrak{L}$ where $\mathfrak{B}^P: F \to [0,1]$ and $\mathfrak{B}^N: F \to [-1,0]$Then $\mathfrak{B}$ is known to be a BFL(Bipolar fuzzy sublattice) of $\mathfrak{L}$ when the following conditions are fulfilled for all $\hbar, m \in \mathfrak{L}$,

$(i)\mathfrak{B}^P(\hbar \vee m) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}$,
(ii) $\mathfrak{B}^P(\hbar \wedge m) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}$.
(iii) $\mathfrak{B}^N(\hbar \vee m) \leq \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}$,
(iv) $\mathfrak{B}^N(\hbar \wedge m) \leq \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}$."

Definition:2.11[16]: "Suppose $\mathfrak{B} = (\mathfrak{B}^P, \mathfrak{B}^N)$ is a BFS in $\mathfrak{L}$ where $\mathfrak{B}^P: F \to [0,1]$ and $\mathfrak{B}^N: F \to [-1,0]$Then $\mathfrak{B}$ is known to be a BFIL(Bipolar fuzzy ideal) of $\mathfrak{L}$ when the following conditions are fulfilled for all $\hbar, m \in \mathfrak{L}$,

$(i)\mathfrak{B}^P(\hbar \vee m) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}$,
(ii) $\mathfrak{B}^P(\hbar \wedge m) \geq \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}$.
(iii) $\mathfrak{B}^N(\hbar \vee m) \leq \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}$,
(iv) $\mathfrak{B}^N(\hbar \wedge m) \leq \min\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}$."

3. Bipolar Fuzzy Prime Ideals of a Lattice

In this section, we explore and study Bipolar Fuzzy prime ideals (BFPI) of $\mathfrak{L}$, their characterizations by using level subsets, homomorphism and anti-homomorphism of BFPIs.

Now, we introduce the following.
Definition 3.1: Suppose $\mathfrak{B} = (\mathfrak{B}^P, \mathfrak{B}^N)$ is a BFI in $\mathfrak{L}$. Then $\mathfrak{B}$ is known to be a BFPI of $\mathfrak{L}$ when the following conditions are fulfilled for all $\hbar, m \in \mathfrak{L}$,
(i) $\mathfrak{B}^P(\hbar \wedge m) \leq \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}$,
(ii) $\mathfrak{B}^N(\hbar \wedge m) \geq \min\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}$.

Example 3.2: Consider the lattice $\mathfrak{L}$ of" divisors of 6 ". We get $\mathfrak{L} = \{1,2,3,6\}$. Suppose $\mathfrak{B} = \{< 1,0.7,-0.3 >, < 2,0.4,-0.3 >, < 3,0.7,-0.1 >, < 6,0.4,-0.1 >\}$. We can easily prove that $\mathfrak{B}$ is a BFPI of $\mathfrak{L}$.

Theorem 3.3: Suppose $\mathfrak{B} = (\mathfrak{B}^P, \mathfrak{B}^N)$ be a BFS$(\mathfrak{L})$. Then it holds that $\mathfrak{B}$ is a BFPI on $\mathfrak{L}$ iff the following four statements hold:
(i) $\mathfrak{B}^P(\hbar \vee m) = \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}$,
(ii) $\mathfrak{B}^P(\hbar \wedge m) = \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}$,
(iii) $\mathfrak{B}^N(\hbar \vee m) = \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}$,
(iv) $\mathfrak{B}^N(\hbar \wedge m) = \min\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}$ for any $\hbar, m \in \mathfrak{L}$.

Proof: Proof is clear.

Theorem 3.4: If κ and $\mathfrak{B}$ are two BFPIs of a lattice $\mathfrak{L}$, then $\kappa \cap \mathfrak{B}$ is a BFPI of $\mathfrak{L}$.

Proof: Suppose $\kappa = (\kappa^P, \kappa^N)$ and $\mathfrak{B} = (\mathfrak{B}^P, \mathfrak{B}^N)$ be two BFPIs of $\mathfrak{L}$.
Now,

$$\begin{aligned}(\kappa \cap \mathfrak{B})^P(\hbar \wedge m) &= \min\{\kappa^P(\hbar \wedge m), \mathfrak{B}^P(\hbar \wedge m)\}\\ &\leq \min\{\max\{\kappa^P(\hbar), \kappa^P(m)\}, \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}\\ &= \max\{\min\{\kappa^P(\hbar), \mathfrak{B}^P(\hbar)\}, \min\{\kappa^P(m), \mathfrak{B}^P(m)\}\\ &= \max\{(\kappa \cap \mathfrak{B})^P(\hbar), (\kappa \cap \mathfrak{B})^P(m)\}.\end{aligned}$$

Thus

$$(\kappa \cap \mathfrak{B})^P(\hbar \wedge m) \leq \max\{\kappa \cap \mathfrak{B})^P(\hbar), \kappa \cap \mathfrak{B})^P(m)\} \text{ for all } \hbar, m \in \mathfrak{L}.$$

Now,

$$\begin{aligned}(\kappa \cap \mathfrak{B})^N(\hbar \wedge m) &= \max\{\kappa^N(\hbar \wedge m), \mathfrak{B}^N(\hbar \wedge m)\}\\ &\geq \max\{\min\{\kappa^N(\hbar), \kappa^N(m)\}, \min\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}\\ &= \min\{\max\{\kappa^N(\hbar), \mathfrak{B}^N(\hbar)\}, \max\{\kappa^N(m), \mathfrak{B}^N(m)\}\\ &= \min\{(\kappa \cap \mathfrak{B})^N(\hbar), (\kappa \cap \mathfrak{B})^N(m)\}.\end{aligned}$$

Thus

$$(\kappa \cap \mathfrak{B})^N(\hbar \wedge m) \geq \min\{\kappa \cap \mathfrak{B})^N(\hbar), \kappa \cap \mathfrak{B})^N(m)\} \text{ for all } \hbar, m \in \mathfrak{L}$$

Hence, $\kappa \cap \mathfrak{B}$ is a BFPI of $\mathfrak{L}$.

Remark 3.5: Union of two BFPIs of a lattice $\mathfrak{L}$ need not be a BFPI.

Since from [ ] union of two BFIs neednot be a BFI, hence, $\kappa \cup \mathfrak{B}$ need not be a BFPI of $\mathfrak{L}$.

Theorem 3.6: The arbitrary intersection of BFPIs of a complete lattice satisfying infinite meet distributive law $\mathfrak{L}$ is also a BFPI of $\mathfrak{L}$.

Proof: Proof is clear.

Theorem 3.7: Let $B =< B^P, B^N >\in BFS(\mathfrak{L})$. Then B is BFPI of $\mathfrak{L}$ if and only if the nonempty level subset $B_{(\alpha, \omega)}$ is a prime ideal of $\mathfrak{L}$ for each $\alpha \in [0, 1]$ and $\omega \in [-1, 0]$.

Proof:

Suppose that $B =< B^P, B^N >$ is BFPI($\mathfrak{L}$).

Let $\hbar$, m $\in B_{(\alpha, \omega)}$.

$\Rightarrow B^P(\hbar) \geq \alpha, B^P(m) \geq \alpha$ and

$B^N(\hbar) \leq \omega, B^N(m) \leq \omega$.

To show that $B_{(\alpha, \omega)}$ is a prime ideal of $\mathfrak{L}$,

we show that for each $\hbar$, m $\in \mathfrak{L}$ and $\hbar \wedge m \in B_{(\alpha, \omega)}$

then either $\hbar \in B_{(\alpha, \omega)}$ or $s \in B_{(\alpha, \omega)}$.

Suppose $\hbar$, m $\in \mathfrak{L}$ and $\hbar \wedge s \in B_{(\alpha, \omega)}$.

Then $B^P(\hbar \wedge m) \geq \alpha$, $B^N(\hbar \wedge m) \leq \omega$

$\Rightarrow \max\{B^P(\hbar), B^P(m)\} \geq \alpha$, $\min\{B^P(\hbar), B^P(m)\} \leq \omega$ (since B is a BFPI of $\mathfrak{L}$)

$\Rightarrow B^P(\hbar) \geq \alpha$ or $B^P(m) \geq \alpha$, $B^N(\hbar) \leq \omega$ or $B^N(m) \leq \omega$

$\Rightarrow B^P(\hbar) \geq \alpha$ and $B^N(\hbar) \leq \omega$ or $B^P(m) \geq \alpha$ and $B^N(m) \leq \omega$

Thus $\hbar \in B_{(\alpha, \omega)}$ or $s \in B_{(\alpha, \omega)}$

Hence, $B_{(\alpha, \omega)}$ is a prime ideal of $\mathfrak{L}$.

In converse assume that $B_{(\alpha, \omega)}$ is a prime ideal of $\mathfrak{L}$.

i.e To any $\hbar$, m $\in \mathfrak{L}$ and $\hbar \wedge s \in B_{(\alpha, \omega)}$

then either $\hbar \in B_{(\alpha, \omega)}$ or $s \in B_{(\alpha, \omega)}$.

We have to show that B is a BFPI of $\mathfrak{L}$.

Assume that B is not a BFPI of $\mathfrak{L}$.

Thus $B^P(\hbar \wedge m) > \max\{B^P(\hbar), B^P(m)\}$ and

$B^N(\hbar \wedge m) < \min\{B^N(\hbar), B^N(m)\}$

$\Rightarrow B^P(\hbar \wedge m) > B^P(\hbar)$ and $B^P(\hbar \wedge m) > B^P(m)$,

$B^N(\hbar \wedge m) < B^N(\hbar)$ and $B^N(\hbar \wedge m) < B^N(m)$.

Suppose that $B^P(\hbar \wedge m) = \alpha$ and $B^N(\hbar \wedge m) = \omega$

$\Rightarrow B^P(\hbar) < \alpha$ and $B^N(\hbar) > \omega$, $B^P(m) < \alpha$ and $B^N(m) > \omega$.

Hence $\hbar, m \notin B_{(\alpha, \omega)}$.

This is a contradiction to the fact that $B_{(\alpha, \omega)}$ is a prime ideal of $\mathfrak{L}$ for any $\alpha \in [0, 1]$ and $\omega \in [-1, 0]$. Hence B is a BFPI of $\mathfrak{L}$.

Theorem 3.8: Let $\mathfrak{L}$ be a lattice and $B \in BFS(\mathfrak{L})$. If B, BFPI of $\mathfrak{L}$ then we have Supp(B) forms a crisp prime ideal of $\mathfrak{L}$.

Proof:

Suppose $B = \{< \hbar, B^P(\hbar), B^N(\hbar) > / \hbar \in \mathfrak{L}\} \in BFS(\mathfrak{L})$.

Given B is a BFPI of $\mathfrak{L}$.

From [] we have Supp(B) is a crisp ideal in $\mathfrak{L}$. Now we prove that Supp(B) is prime.

Suppose $\hbar, m \in \mathfrak{L}$ so that $\hbar \wedge s \in$ Supp(B).

Thus $B^P(\hbar \wedge m) \neq 0$ or $B^N(\hbar \wedge m) \neq 0$.

But $B^P(\hbar \wedge m) = \max\{B^P(\hbar), B^P(m)\}$ (since B is BFPI of $\mathfrak{L}$).

$\Rightarrow$ either $B^P(\hbar) \neq 0$, or $B^P(m) \neq 0$

Likewise, we have either $B^N(\hbar) \neq 0$, or $B^N(m) \neq 0$

$\Rightarrow$ either $\hbar \in$ Supp(B) or $m \in$ Supp(B).

Hence Supp(B) is a prime ideal of $\mathfrak{L}$.

Remark 3.9: Converse part of the above theorem does not hold in general.

Suppose $B = \{< 1, 0.5, -0.1 >, < 2, 0.7, -0.2 >, < 3, 0.8, -0.05 >, < 6, 0.4, -0.01 >\}$ be a BF subset in $\mathfrak{L} = \{1, 2, 3, 6\}$ with divisors of 6.

$\text{Supp}(B) = \{1, 2, 3, 6\}$ is a prime ideal of $\mathfrak{L}$.

But $B^P(2 \wedge 3) = B^P(1) = 0.5 \geq \max\{B^P(2), B^P(8)\} = 0.8$

$\Rightarrow$ B is not a BFPI of $\mathfrak{L}$.

Theorem 3.10: Let B be a BFS of lattice $\mathfrak{L}$. Then, B is a BFPI of $\mathfrak{L}$ if and only if f the BFMT M of B form BF prime ideal of $\mathfrak{L}$.

Proof: Assume that $\mathfrak{B}$ is a BF primeideal of $\mathfrak{L}$ and M be a BFMT of $\mathfrak{B}$.
Now we have to show that M is a BFPI of $\mathfrak{L}$.
Let $\hbar, m \in \mathfrak{L}$.

Now, consider $\mathfrak{B}^P_{(\omega,\vartheta)}(\hbar \wedge m)$
$= \omega\mathfrak{B}^P(\hbar \wedge m) + \vartheta.$
$\leq \omega \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\} + \vartheta$ 。
$= \max\{\omega\mathfrak{B}^P(\hbar) + \vartheta, \omega\mathfrak{B}^P(m) + \vartheta\}$
$= \max\{\mathfrak{B}^P_{(\omega,\vartheta)}(\hbar), \mathfrak{B}^P_{(\omega,\vartheta)}(m)\}$
Thus $\mathfrak{B}^P_{(\omega,\vartheta)}(\hbar \wedge m) \leq \max\{\mathfrak{B}^P_{(\omega,\vartheta)}(\hbar), \mathfrak{B}^P_{(\omega,\vartheta)}(m)\}$
Now $\mathfrak{B}^N_{(\alpha,\theta)}(\hbar \wedge m) = \alpha\mathfrak{B}^N(\hbar \wedge m) + \theta.$
$\geq \alpha\min\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\} + \theta.$
$= \min\{\alpha\mathfrak{B}^N(\hbar) + \theta, \alpha\mathfrak{B}^N(m) + \theta\}$
$= \min\{\mathfrak{B}^N_{(\alpha,\theta)}(\hbar), \mathfrak{B}^N_{(\alpha,\theta)}(m)\}$
Thus $\mathfrak{B}^N_{(\alpha,\theta)}(\hbar \wedge m) \geq \min\{\mathfrak{B}^N_{(\alpha,\theta)}(\hbar), \mathfrak{B}^N_{(\alpha,\theta)}(m)\}$
Hence the BFMT M of $\mathfrak{B}$ is again a BFPI of $\mathfrak{L}$.
In converse assume that M a BFPI of $\mathfrak{L}$.
Now, consider $\mathfrak{B}^P(\hbar \wedge m)$
$= \frac{1}{\omega}\left(\mathfrak{B}^P_{(\omega,\vartheta)}(\hbar \wedge m) - \vartheta\right)$
$\leq \frac{1}{\omega}\left(\max\{\mathfrak{B}^P_{(\omega,\vartheta)}(\hbar), \mathfrak{B}^P_{(\omega,\vartheta)}(m)\} - \vartheta\right)$
$= \max\left\{\frac{1}{\omega}\left(\mathfrak{B}^P_{(\omega,\vartheta)}(\hbar) - \vartheta\right), \frac{1}{\omega}\left(\mathfrak{B}^P_{(\omega,\vartheta)}(m) - \vartheta\right)\right\}$
$= \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}.$
Thus $\mathfrak{B}^P(\hbar \wedge m) \leq \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(m)\}.$
Likewise, we can prove $\mathfrak{B}^N(\hbar \wedge m) \geq \min\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(m)\}.$
Thus M is a BF prime ideal of L. Hence the proof.

Theorem 3.11: Let $\varpi: \mathfrak{L} \to \mathfrak{L}^1$ be a lattice epimorphism.
If $\mathfrak{B}$ is a BFPI of $\mathfrak{L}$, then $\varpi(\mathfrak{B})$ is a BFPI of $\mathfrak{L}^1$.

Proof: Assume $\mathfrak{B} = (\mathfrak{B}^P, \mathfrak{B}^N)$ a BFPI of $\mathfrak{L}$.
Then by known Th we know that $\varpi(\mathfrak{B})$ is a BFI in $\mathfrak{L}^1$.
Now we show that $\varpi(\mathfrak{B})$ is prime.
Let $s, z \in \mathfrak{L}^1$.
Then

$$\begin{aligned}
\varpi(\mathfrak{B}^P)(s \wedge z) &= \sup\{\mathfrak{B}^P(\hbar) \mid \hbar \in \varpi^{-1}(s \wedge z)\} \\
&= \sup\{\mathfrak{B}^P(u \wedge \xi) \mid u \in \varpi^{-1}(m), \xi \in \varpi^{-1}(z)\} \\
&= \sup\{\max\{\mathfrak{B}^P(u), \mathfrak{B}^P(\xi)\} \mid u \in \varpi^{-1}(m), \xi \in \varpi^{-1}(z)\} \\
&= \max\left\{\sup\{\mathfrak{B}^P(u) \mid u \in \varpi^{-1}(m)\}, \sup\{\mathfrak{B}^P(\xi) \mid \xi \in \varpi^{-1}(z)\}\right\} \\
&= \max\{\varpi(\mathfrak{B}^P)(m), \varpi(\mathfrak{B}^P)(z)\}. \\
\varpi(\mathfrak{B}^N)(s \wedge z) &= \inf\{\mathfrak{B}^N(\hbar) \mid \hbar \in \varpi^{-1}(s \wedge z)\} \\
&= \inf\{\mathfrak{B}^N(u \wedge \xi) \mid u \in \varpi^{-1}(m), \xi \in \varpi^{-1}(z)\} \\
&= \inf\{\min\{\mathfrak{B}^N(u), \mathfrak{B}^N(\xi)\} \mid u \in \varpi^{-1}(m), \xi \in \varpi^{-1}(z)\} \\
&= \min\left\{\inf\{\mathfrak{B}^N(u) \mid u \in \varpi^{-1}(m)\}, \inf\{\mathfrak{B}^N(\xi) \mid \xi \in \varpi^{-1}(z)\}\right\} \\
&= \min\{\varpi(\mathfrak{B}^N)(m), \varpi(\mathfrak{B}^N)(z)\},
\end{aligned}$$

Hence, $\varpi(\mathfrak{B})$ is a BFPI of $\mathfrak{L}^1$.

Theorem 3.12: Let $\varpi: \mathfrak{L} \to \mathfrak{L}^1$ be a lattice homomorphism. If C is a BFPI of $\mathfrak{L}^1$, then $\varpi^{-1}(C)$ is a BFPI of $\mathfrak{L}$.

Proof: Suppose $C = (C^P, C^N)$ be a BFPI of $\mathfrak{L}^1$.
Then by known theorem[], we know that $\varpi^{-1}(C)$ is a BFI in $\mathfrak{L}$.
Now we show that $\varpi^{-1}(C)$ is prime.
Let $\hbar, m \in \mathfrak{L}$.
Then

$$\begin{aligned}
\varpi^{-1}(C^P)(\hbar \wedge m) &= C^P(\varpi(\hbar \wedge m)) \\
&= C^P\{(\varpi(\hbar) \wedge \varpi(m)\} \\
&= \max\{C^P(\varpi(\hbar)), C^P(\varpi(m))\} \\
&= \max\{\varpi^{-1}(C^P)(\hbar), \varpi^{-1}(C^P)(m)\} \\
\varpi^{-1}(C^N)(\hbar \wedge m) &= C^N(\varpi(\hbar \wedge m)) \\
&= C^N\{(\varpi(\hbar) \wedge \varpi(m)\} \\
&= \min\{C^N(\varpi(\hbar)), C^N(\varpi(m))\} \\
&= \min\{\varpi^{-1}(C^N)(\hbar), \varpi^{-1}(C^N)(m)\}
\end{aligned}$$

Hence, $\varpi^{-1}(C)$ is a BFPI of $\mathfrak{L}$.

4.Conclusion

This study explores the investigation of Bipolar Fuzzy Prime Ideals (BFPI) in lattices. We provide a detailed exploration of their properties, characterizations, and associated homomorphisms.

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