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Fixed Point Results in G Metric Space via Α-Series

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Abstract:

Introduction: Mustafa and sims [1] introduced the concept of G-metric space in 2005. Afterwards, Mustafa et al and many authors [3]-[19] obtained some common fixed-point theorems, coupled and tripled fixed point results for mappings satisfying different contractive conditions in G metric space. In this study we prove fixed point results in G metric space via α-series by using some conditions that are a sequence of a mappings and a self-mapping.

Objectives: To show tripled fixed-point theorems and common fixed point theorems by using sequence of mappings and self a self-mapping via α-series and shown an example which supports the main result.

Methods: In recent study the authors worked on fixed point results by using different contractions such as Suzuki type contraction, Rational type contraction, cyclic contraction, F-contraction, Mier keeler contraction, (ψ, ϕ)-weakly contractive mappings and Integral type contractions etc. in G-metric spaces. In this study we proved fixed point results in G metric space via α-series by using sequence of mappings.

Results: Obtained Unique common fixed point and tripled fixed point results in G metric spcace via α-series.

Conclusion: In this study we present unique tripled fixed-point and common fixed-point results for a sequence of mappings and a self-mapping in G metric space via α-series and discussed corollary with supporting example.

Keywords: G-metric space, tripled fixed point, α-series, compatible mapping, weakly reciprocally continuous mappings.

1.Introduction:

Mustafa and sims [1] introduced the concept of G-metric space in 2005. Afterwards, Mustafa et al and many authors [3]-[19] obtained some common fixed-point theorems, coupled and tripled fixed point results for mappings satisfying different contractive conditions in G metric space. In 2014, Sihag et al [21] proposed an α-series to find a common fixed point by utilizing the sequence of mappings and self-mappings. Chang and Ma [22] presented coupled fixed point in 1991.Later this concept has attracted numerous researchers [23]-[28] across various fields. The notation of tripled fixed point was initiated by Berinde and Borcut [32],[33] in partially ordered metric spaces and also presented the concept of tripled coincidence point and obtained tripled coincidence point outcomes. Later many authors obtained common tripled fixed-point results by using different contraction conditions in different metric spaces. In this paper we prove tripled fixed-point results in G metric space by utilizing the sequence of mappings via α- series and we provided illustrations to support our results.

2.Objectives: To show tripled fixed-point theorems and common fixed point theorems by using sequence of mappings and self a self-mapping via α-series and shown an example which supports the main result

3.Methodology:

Defnition 3.1 [1] Consider Ɱ be a non-void set and G: $Ɱ^3 \to [0,\infty)$ is a mapping such that

(G1) G (ƥ, ꝗ, 𝝕) = 0 if ƥ = ꝗ = 𝝕 for all ƥ, ꝗ, 𝝕 ϵ Ɱ

(G2) G (ƥ, ƥ, ꝗ) > 0 for ƥ, ꝗ ϵ Ɱ with ƥ≠ꝗ

(G3) G (ƥ, ƥ, ꝗ) ≤ G (ƥ, ꝗ, 𝝕) for all ƥ, ꝗ, 𝝕 ϵ Ɱ with ꝗ ≠ 𝝕

(G4) G (ƥ, ꝗ, 𝝕) = G (ꝗ, 𝝕, ƥ) = G (𝝕, ꝗ, ƥ) = _ _ _ _ _ (symmetry)

(G5) G (ƥ, ꝗ, 𝝕) ≤ G (ƥ, 𝝏, 𝝏) + G (𝝏, ꝗ, 𝝕) for all 𝝏, ƥ, ꝗ, 𝝕 ϵ Ɱ.

Then the function G is termed as G –metric on Ɱ and (Ɱ, G) is a G-metric space.

Definiton 3.2 [22]: Assume Ɱ ≠∅ and a mapping E: $Ɱ^2 \to$ Ɱ. An element (λ, μ) ϵ $Ɱ^2$ is a coupled fixed point of E if Ɱ (λ, μ) = λ & Ɱ (μ, λ)= μ.

Definition 3.3[32]: An element (ƥ, ꝗ, ν) ϵ $Ɱ^3$ be a tripled fixed point of mapping **γ**: $Ɱ^3 \to$ Ɱ if **γ** (ƥ, ꝗ, ν) = ƥ, **γ** (ꝗ, ƥ, ꝗ) = ꝗ and **γ** (ν, ꝗ, ƥ) = ν.

Definition 3.4 [33]: A trio (ϱ, ꝗ, Ꝛ) ϵ $\mathscr{F}^3$ is a tripled coincidence point of the mappings 𝕲: $\mathscr{F}^3 \to \mathscr{F}$ *and* ƒ: 𝓕→𝓕 *if* ƒ(ϱ)=𝓕(ϱ, ꝗ, Ꝛ), ƒ(ꝗ)=𝓕 (ꝗ, ϱ, ꝗ) and ƒ(Ꝛ)=𝓕 (Ꝛ, ꝗ, ϱ).

Definition 3.5[34]: Let (K, ≤) be a poset and 𝕻: $K^3 \to K$. If 𝕻 (λ, μ, ν) is monotone-non increasing in μ and monotone non-decreasing in λ & ν then 𝕻 has mixed monotone property. i.e, for any λ, μ, ν ϵ K.

$\lambda_1, \lambda_2,$ ϵ K, $\lambda_1 \le \lambda_2$ ⟹ 𝕻 $(\lambda_1, \mu, \nu) \le$ 𝕻 (λ_2, μ, ν)

μ_1, μ_2 ϵ K, $\mu_1 \le \mu_2$ ⟹ 𝕻 $(\lambda, \mu_1, \nu) \ge$ 𝕻 (λ, μ_2, ν)

ν_1, ν_2 ϵ K, $\nu_1 \le \nu_2$ ⟹ 𝕻 $(\lambda, \mu, \nu_1) \le$ 𝕻(λ, μ, ν_2)

Definition 3.6[34]: Let (E, ≤) be a partially ordered set and Ϋ: $E^3 \to E$ and g: E→E be two maps. If Ϋ (λ, μ, ν) is monotone non increasing in μ and Ϋ (λ, μ, ν) is monotone non decreasing in λ and ν then Ϋ has g-mixed monotone property. i.e, for any λ, μ, ν ϵ E.

$\lambda_1, \lambda_2,$ ϵ E, $g(\lambda_1) \le g(\lambda_2)$ ⟹ Ϋ $(\lambda_1, \mu, \nu) \le$ Ϋ (λ_2, μ, ν)

μ_1, μ_2 ϵ E, $g(\mu_1) \le g(\mu_2)$ ⟹ Ϋ $(\lambda, \mu_1, \nu) \ge$ Ϋ (λ, μ_2, ν)

ν_1, ν_2 ϵ E, $g(\nu_1) \le g(\nu_2)$ ⟹ Ϋ$(\lambda, \mu, \nu_1) \le$ Ϋ(λ, μ, ν_2)

Definition 3.7[24]: Let (U, d) be a metric space and mappings E and g where E: $U^3 \to U$ and g: U→U are compatible if

$$\lim_{n\to\infty} d\left(g\big(E(\lambda_n,\mu_n,\nu_n)\big), E(g\lambda_n, g\mu_n, g\nu_n)\right) = 0$$

$$\lim_{n\to\infty} d\left(g\big(E(\mu_n,\lambda_n,\mu_n)\big), E(g\mu_n, g\lambda_n, g\mu_n)\right) = 0$$

$$\lim_{n\to\infty} d\big(g(E(\nu_n,\mu_n,\lambda_n), E(g\nu_n, g\mu_n, g\nu_n)\big) = 0$$

Whenever $\{\lambda_n\},\{\mu_n\},\{\nu_n\}$ *are sequences in* X, *such that*

$\lim_{n\to\infty} E(\lambda_n, \mu_n, \nu_n) = \lim_{n\to\infty} g(\lambda_n) = \lambda$,

$\lim_{n\to\infty} E(\mu_n, \lambda_n, \mu_n) = \lim_{n\to\infty} g(\mu_n) = \mu$

$\lim_{n\to\infty} E(v_n, \mu_n, \lambda_n) = \lim_{n\to\infty} g(v_n) = v$

For all $\lambda, \mu, v, \epsilon U$

Definition 3.8[24]: The mapping E: $Ɱ^3 \to Ɱ$ and $\mathfrak{P}$: Ɱ→Ɱ are

(i) Reciprocally continuous if

$\lim_{n\to\infty} \mathfrak{P}\left(E\big(\lambda_n, \mu_n, v_n\big)\right) = \mathfrak{P}(\lambda)$ *and* $\lim_{n\to\infty} E(\mathfrak{P}(\lambda_n), \mathfrak{P}(\mu_n), \mathfrak{P}(v_n)) = E(\lambda, \mu, v)$

$\lim_{n\to\infty} \mathfrak{P}\left(E\big(\mu_n, \lambda_n, \mu_n\big)\right) = \mathfrak{P}(\mu)$ *and* $\lim_{n\to\infty} E(\mathfrak{P}(\mu_n), \mathfrak{P}(\lambda_n), \mathfrak{P}(\mu_n))$ = E (μ, λ, μ)

and $\lim_{n\to\infty} \mathfrak{P}\left(E\big(v_n, \mu_n, \lambda_n\big)\right) = \mathfrak{P}(v)$ *and* $\lim_{n\to\infty} E\big(\mathfrak{P}(v_n), \mathfrak{P}(\mu_n), \mathfrak{P}(\lambda_n)\big)$ = E(ν,μ,λ)

whenever $\{\lambda_n\},\{\mu_n\},\{v_n\}$ are sequences in Ɱ such that

$\lim_{n\to\infty} E\big(\lambda_n, \mu_n, v_n\big) = \lim_{n\to\infty} \mathfrak{P}(\lambda_n) = \lambda$

$\lim_{n\to\infty} E\big(\mu_n, \lambda_n, \mu_n\big) = \lim_{n\to\infty} \mathfrak{P}(\mu_n) = \mu$

$\lim_{n\to\infty} E\left(E\big(v_n, \mu_n, \lambda_n\big)\right) = \lim_{n\to\infty} \mathfrak{P}(v_n) = \nu$

(ii) Weakly reciprocally continuous if

$\lim_{n\to\infty} \mathfrak{P}(E(\lambda_n, \mu_n, v_n) = \mathfrak{P}(\lambda_n)$ or $\lim_{n\to\infty} E(\mathfrak{P}(\lambda_n), \mathfrak{P}(\mu_n), \mathfrak{P}(\nu_n)) = E(\lambda, \mu, \nu)$

$\lim_{n\to\infty} \mathfrak{P}(E(\mu_n, \lambda_n, \mu_n) = \mathfrak{P}(\mu)$ or $\lim_{n\to\infty} E(\mathfrak{P}(\mu_n), \mathfrak{P}(\lambda_n), \mathfrak{P}(v_n))$ = E (μ, λ, μ)

$\lim_{n\to\infty} \mathfrak{P}(E(v_n, \mu_n, \lambda_n) = \mathfrak{P}(\nu)$ or $\lim_{n\to\infty} E(\mathfrak{P}(v_n), \mathfrak{P}(\mu_n), \mathfrak{P}(\lambda_n))$ = E (ν, μ, λ)

Whenever $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ are sequences in Ɱ such that

$\lim_{n\to\infty} E(\lambda_n, \mu_n, \nu_n) = \lim_{n\to\infty} \mathfrak{P}(\lambda_n) = \lambda$

$\lim_{n\to\infty} E(\mu_n, \lambda_n, \mu_n) = \lim_{n\to\infty} \mathfrak{P}(\mu_n) = \mu$

$\lim_{n\to\infty} E(\nu_n, \mu_n, \lambda_n) = \lim_{n\to\infty} \mathfrak{P}(\nu_n) = \nu$

for some λ, μ, ν ∈ Ɱ.

Definition 3.9[21]: Assume $\langle a_n \rangle$ be a sequence of non-negative real numbers. If there exist $0 < \alpha < 1$ and $n_\alpha \in N$ such that $\sum_{i=1}^{k} a_i \le \alpha_k$ for each $k \ge n_\alpha$ then the series $\sum_{n=1}^{\infty} a_n$ is called an α-series.

Lemma 3.10 [34] Let $(K, \leq)$ be a poset and $g: K \to K$ and $\{T_n\}_{n\epsilon N}: K^3 \to K$ be a sequence of mappings such that $T_n(K^3) \subseteq g(K)$ then T_n has g-mixed monotone property if $g(\lambda) \leq g(\wp)$, $g(\varphi) \leq g(\mu)$ and $g(\nu) \leq g(\boldsymbol{\partial})$ for any $\lambda,\mu,\nu,\wp,\varphi,\boldsymbol{\partial} \in K$ imply

$$\left.\begin{aligned} T_n(\lambda, \mu, \nu) &\leq T_{n+1}(\wp, \varphi, \boldsymbol{\partial}) \\ T_{n+1}(\varphi, \wp, \varphi) &\leq T_n(\mu, \lambda, \mu) \\ T_n(\nu, \mu, \lambda) &\leq T_{n+1}(\boldsymbol{\partial}, \varphi, \wp) \end{aligned}\right\} \quad (1)$$

In our main proof we construct the sequence as follows
Assume $\lambda_0, \mu_0, \nu_0 \in K$ such that

$$g(\lambda_0) \leq T_0(\lambda_0, \mu_0, \nu_0),\ g(\mu_0) \geq T_0(\mu_0, \lambda_0, \mu_0),\ g(\nu_0) \leq T_0(\nu_0, \mu_0, \lambda_0) \quad (2)$$

Since $T_0(K^3) \subseteq g(K)$ we choose $\lambda_1, \mu_1, \nu_1 \in K$ such that

$$g(\lambda_1) = T_0(\lambda_0, \mu_0, \nu_0),\ g(\mu_1) = T_0(\mu_0, \lambda_0, \nu_0),\ g(\nu_1) = T_0(\nu_0, \mu_0, \lambda_0)$$

again, we choose $\lambda_2, \mu_2, \nu_2 \in K$ such that

$$g(\lambda_2) = T_1(\lambda_1, \mu_1, \nu_1), \qquad g(\mu_2) = T_1(\mu_1, \lambda_1, \mu_1), \qquad g(\nu_2) = T_1(\nu_1, \mu_1, \lambda_1)$$

continuing this process, we construct three sequences $\{\lambda_m\}$, $\{\mu_m\}$, $\{\nu_m\}$ such that

$$g(\lambda_{m+1}) = T_m(\lambda_m, \mu_m, \nu_m), \qquad g(\mu_{m+1}) = T_m(\mu_m, \lambda_m, \mu_m), \qquad g(\nu_{m+1}) = T_m(\nu_m, \mu_m, \lambda_m)$$

for all $m \geq 0$ (3)

By mathematical induction, we prove that

$g(\lambda_m) \leq g(\lambda_{m+1})$, $g(\mu_m) \geq g(\mu_{m+1})$ and $g(\nu_m) \leq g(\nu_{m+1})$ for all $m \geq 0$.
(4)

since condition 2 holds, in view of $g(\lambda_1) = T_0(\lambda_0, \mu_0, \nu_0)$, $g(\mu_1) = T_0(\mu_0, \lambda_0, \mu_0)$, $g(\nu_1) = T_0(\nu_0, \mu_0, \lambda_0)$

we obtain $g(\lambda_0) \leq g(\lambda_1)$, $g(\mu_0) \geq g(\mu_1)$, $g(\nu_0) \leq g(\nu_1)$.

That is condition … 4 is true for m=0. Now we consider condition 4 is true for some m>0

From (3) and (4) we deduce

$$g(\lambda_{m+1}) = T_m(\lambda_m, \mu_m, \nu_m) \leq T_{m+1}(\lambda_{m+1}, \mu_{m+1}, \nu_{m+1}) = g(\lambda_{m+2})$$

$$g(\mu_{m+2}) = T_{m+1}(\mu_{m+1}, \lambda_{m+1}, \mu_{m+1}) \leq T_m(\mu_m, \lambda_m, \mu_m) = g(\mu_{m+1})$$

$$g(\nu_{m+1}) = T_m(\mu_m, \lambda_m, \nu_m) \leq T_{m+1}(\nu_{m+1}, \mu_{m+1}, \lambda_{m+1}) = g(\nu_{m+2})$$

we conclude that 4 holds for all $m \geq 0$ by the mathematical induction

$\therefore$ we have $g(\lambda_0) \leq g(\lambda_1) \leq g(\lambda_2) \leq \ldots g(\lambda_{m+1}) \leq \ldots\ldots$

$g(\mu_0) \geq g(\mu_1) \geq g(\mu_2) \geq \ldots\ldots g(\mu_{m+1}) \geq \ldots\ldots$

$g(\nu_0) \leq g(\nu_1) \leq g(\nu_2) \leq \ldots\ldots g(\nu_{m+1}) \leq \ldots\ldots$

definitions 2.7 and 2.8 are revised in main results as follows by using above considerations.

4.Results and Discussion

Definition 4.1: Consider ($\mathcal{M}$, G) be a generalized metric space, $\{T_m\}_{m\in N}$: $\mathcal{M}^3 \to \mathcal{M}$ be a sequence of mappings and a self-mapping g: $\mathcal{M} \to \mathcal{M}$ are compatible if

$$\lim_{m\to\infty} G(g(T_m(\lambda_m, \mu_m, \nu_m)), T_m(g\lambda_m, g\mu_m, g\nu_m), T_m(g\lambda_m, g\mu_m, g\nu_m)) = 0$$

$$\lim_{m\to\infty} G(g(T_m(\mu_m, \lambda_m, \mu_m)), T_m(g\mu_m, g\lambda_m, g\mu_m), T_m(g\mu_m, g\lambda_m, g\mu_m)) = 0$$

$$\lim_{m\to\infty} G(g(T_m(\nu_m, \mu_m, \lambda_m)), T_m(g\nu_m, g\mu_m, g\lambda_m), T_m(g\nu_m, g\mu_m, g\lambda_m)) = 0$$

Whenever $\{\lambda_m\}$, $\{\mu_m$, $\{\nu_m\}$ are sequences in $\mathcal{M}$ such that

$$\lim_{m\to\infty} T_m(\lambda_m, \mu_m, \nu_m) = \lim_{m\to\infty} g(\lambda_{m+1}) = \lambda$$

$$\lim_{m\to\infty} T_m(\mu_m, \lambda_m, \mu_m) = \lim_{m\to\infty} g(\mu_{m+1}) = \mu$$

$$\lim_{m\to\infty} T_m(\nu_m, \mu_m, \lambda_m) = \lim_{m\to\infty} g(\nu_{m+1}) = \nu$$

for some $\lambda, \mu, \nu \in \mathcal{M}$.

Definition 4.2: Assume ($\mathcal{Q}$, G) be a G- metric space, the mappings $\{T_m\}_{m\in N}$: $\mathcal{Q}^3 \to \mathcal{Q}$ and g: $\mathcal{Q} \to \mathcal{Q}$ are weakly reciprocally continuous if

$$\lim_{m\to\infty} g\big(T_m(\xi_m, \eta_m, \theta_m)\big) = g(\xi)$$
$$\lim_{m\to\infty} g\big(T_m(\eta_m, \xi_m, \eta_m)\big) = g(\eta)$$
$$\lim_{m\to\infty} g\big(T_m(\theta_m, \eta_m, \xi_m)\big) = g(\theta)$$

whenever $\{\xi_m\}$, $\{\eta_m\}$, $\{\theta_m\}$ are sequences in $\mathcal{Q}$ such that

$$\lim_{m\to\infty} T_m(\xi_m, \eta_m, \theta_m)) = \lim_{m\to\infty} g(\xi_{m+1}) = \xi$$

$$\lim_{m\to\infty} T_m(\eta_m, \xi_m, \eta_m) = \lim_{m\to\infty} g(\eta_{m+1}) = \eta$$

$$\lim_{m\to\infty} T_m(\theta_m, \eta_m, \xi_m) = \lim_{m\to\infty} g(\theta_{m+1}) = \theta$$

for some $\xi, \eta, \theta \in \mathcal{Q}$.

Theorem 4.3 : Assume ($\mho$,G) be a partially ordered complete G-metric space . Let a sequence of mappings $T_n : \mho^3 \to \mho$ and a self-mapping g:$\mho \to \mho$ such that $T_n(\mho^3) \subseteq g(\mho)$ and $g(\mho)$ is closed. $\{T_n\}_{n\in N}$ & g has g-mixed monotone property and both are continuous, compatible, weakly reciprocally continuous and satisfying the below conditions.

1. There exists $\lambda_0, \mu_0, \nu_0 \in$ X such that

   $g(\lambda_0) \le T_0(\lambda_0, \mu_0, \nu_0)$, $g(\mu_0) \ge T_0(\mu_0, \lambda_0, \mu_0)$, $g(\nu_0) \le T_0(\nu_0, \mu_0, \lambda_0)$ holds.

2. $\{T_n\}_{n\in N}$ and g satisfies the condition

$$G(T_n(\lambda, \mu, \nu), T_j(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\varkappa}), T_j(\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau})) \le \gamma_{n,j}[G(g_\lambda, T_n(\lambda, \mu, \nu), T_n(\lambda, \mu, \nu)) + G(g_\zeta, T_j(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\varkappa}), T_j(\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}))] + \delta_{n,j} G(g\lambda, g\boldsymbol{\zeta}, g\boldsymbol{\rho}) \quad (5)$$

for $\lambda, \mu, \nu, \boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\varkappa}, \boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mho$ and $0 \leq \gamma_{n,j}, \delta_{n,j} < 1$, $n \neq j = 1,2,\ldots$ i.e $n, j \in N$

if $\sum_{n=1}^{\infty}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)$ is an α-series then g and $\{T_n\}_{n\in N}$ have a tripled coincidence point.

3. If $\{T_n\}_{n\in N}$ and g have tripled coincidence point comparable with respect to g then $\{T_n\}_{n\in N}$ and g have a unique tripled fixed point.

Proof: For any $\lambda_0, \mu_0, \nu_0 \in \mho$. We consider three sequences $\{\lambda_m\},\{\mu_m\}, \{\nu_m\}$ constructed above by taking

$g\lambda_{m+1}=T_m(\lambda_m, \mu_m, \nu_m)$, $g\mu_{m+1}=T_m(\mu_m, \lambda_m, \boldsymbol{\mu}_m)$, $g\nu_{m+1}=T_m(\nu_m, \mu_m, \lambda_m)$

By condition (5)

$G(g\lambda_1, g\lambda_2, g\lambda_2) = G(T_0(\lambda_0, \mu_0, \nu_0), T_1(\lambda_1, \mu_1, \nu_1), T_1(\lambda_1, \mu_1, \nu_1))$

$\leq \gamma_{0,1}[G(g\lambda_0, T_0(\lambda_0, \mu_0, \nu_0), T_0(\lambda_0, \mu_0, \nu_0)) + G((g\lambda_1, T_1(\lambda_1, \mu_1, \nu_1), T_1(\lambda_1, \mu_1, \nu_1))] + \delta_{0,1}G(g\lambda_0, g\lambda_1, g\lambda_1)$

$\Rightarrow G(g\lambda_1, g\lambda_2, g\lambda_2) \leq \gamma_{0,1}[G(g\lambda_0, g\lambda_1, g\lambda_1) + G(g\lambda_1, g\lambda_2, g\lambda_2)] + \delta_{0,1} G(g\lambda_0, g\lambda_1, g\lambda_1)$

$$\leq \frac{(\gamma_{0,1}+\delta_{0,1})}{(1-\gamma_{0,1})} G(g\lambda_0, g\lambda_1, g\lambda_1)$$

similarly $G(g\lambda_2, g\lambda_3, g\lambda_3) \leq \frac{(\gamma_{1,2}+\delta_{1,2})}{(1-\gamma_{1,2})} G(g\lambda_1, g\lambda_2, g\lambda_2)$

$$\leq \left(\frac{\gamma_{1,2}+\delta_{1,2}}{1-\gamma_{1,2}}\right)\left(\frac{\gamma_{0,1}+\delta_{0,1}}{1-\gamma_{0,1}}\right) G(g\lambda_0, g\lambda_1, g\lambda_1)$$

Repeating the above, we obtain

$G(g\lambda_m, g\lambda_{m+1}, g\lambda_{m+1}) \leq \prod_{n=0}^{m-1}\left(\frac{\boldsymbol{\gamma_{n,n+1}+\delta_{n,n+1}}}{\boldsymbol{1-\gamma_{n,n+1}}}\right) G(g\lambda_0, g\lambda_1, g\lambda_1)$ ---------(6)

Using similar procedure, we can also prove that

$G(g\mu_m, g\mu_{m+1}, g\mu_{m+1}) \leq \prod_{n=0}^{m-1}\left(\frac{\boldsymbol{\gamma_{n,n+1}+\delta_{n,n+1}}}{\boldsymbol{1-\gamma_{n,n+1}}}\right) G(g\mu_0, g\mu_1, g\mu_1)$ ------(7)

and $G(\nu_m, \nu_{m+1}, \nu_{m+1}) \leq \prod_{n=0}^{m-1}\left(\frac{\boldsymbol{\gamma_{n,n+1}+\delta_{n,n+1}}}{\boldsymbol{1-\gamma_{n,n+1}}}\right) G(g\nu_0, g\nu_1, g\nu_1)$ -------(8)

adding (6), (7), (8) we get

$G(g\lambda_m, g\lambda_{m+1}, g\lambda_{m+1}) + G(g\mu_m, g\mu_{m+1}, g\mu_{m+1}) + G(\nu_m, \nu_{m+1}, \nu_{m+1}) \leq \prod_{n=0}^{\square-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)[G(g\lambda_0, g\lambda_1, g\lambda_1) + G(g\mu_0, g\mu_1, g\mu_1) + G(g\nu_0, g\nu_1, g\nu_1)]$

Let $\beta_m \leq \prod_{n=0}^{m-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\beta_0$

where $\beta_m = G(g\lambda_m, g\lambda_{m+1}, g\lambda_{m+1}) + G(g\mu_m, g\mu_{m+1}, g\mu_{m+1}) + G(\nu_m, \nu_{m+1}, \nu_{m+1})$.

For p > 0 and by using (G5), we have

$G(g\lambda_m, g\lambda_{m+p}, g\lambda_{m+p}) + G(g\mu_m, g\mu_{m+p}, g\mu_{m+p}) + G(\nu_m, \nu_{m+p}, \nu_{m+p}) \leq G(g\lambda_m, g\lambda_{m+1}, g\lambda_{m+1}) + G(g\mu_m, g\mu_{m+1}, g\mu_{m+1}) + G(\nu_m, \nu_{m+1}, \nu_{m+1}) + G(g\lambda_{m+1}, g\lambda_{m+2}, g\lambda_{m+2}) + G(g\mu_{m+1}, g\mu_{m+2}, g\mu_{m+2}) + G(\nu_{m+1}, \nu_{m+2}, \nu_{m+2}) + \text{----} + G(g\lambda_{m+p-1}, g\lambda_{m+p}, g\lambda_{m+p}) + G(g\mu_{m+p-1}, g\mu_{m+p}, g\mu_{m+p}) + G(\nu_{m+p-1}, \nu_{m+p}, \nu_{m+p})$

$$\leq \prod_{n=0}^{\square-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\beta_0 + \prod_{n=0}^{\square}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\beta_0 + \text{----} + \prod_{n=0}^{\square+\square-2}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\beta_0$$

$$\leq \sum_{\square=0}^{\square-1}\prod_{\square=0}^{\square+\square-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\beta_0$$

$$\leq \sum_{\square=\square}^{\square+\square-1}\prod_{\square=0}^{\square-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\beta_0$$

$$G(g\lambda_m, g\lambda_{m+p}, g\lambda_{m+p}) + G(g\mu_m, g\mu_{m+p}, g\mu_{m+p}) + G(\nu_m, \nu_{m+p}, \nu_{m+p}) \leq \sum_{k=m}^{m+p-1}\prod_{n=0}^{k-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\beta_0 \quad \ldots\ldots(9)$$

By the fact that the arithmetic mean is greater than or equal to geometric mean for non-negative real numbers and α, n_α are like in definition 2.9 for $m \geq n_\alpha$ follows as below

$$\sum_{k=m}^{m+p-1}\prod_{n=0}^{k-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right) \leq \sum_{k=m}^{m+p-1}\left[\frac{1}{k}\sum_{n=0}^{k-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\right]^K \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

From (9) and (10)

$$\therefore G(g\lambda_m, g\lambda_{m+p}, g\lambda_{m+p}) + G(g\mu_m, g\mu_{m+p}, g\mu_{m+p}) + G(\nu_m, \nu_{m+p}, \nu_{m+p}) \leq \sum_{k=m}^{m+p-1}\left[\frac{1}{k}\sum_{n=0}^{k-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)\right]^K \beta_0$$

$$\leq \left(\sum_{k=m}^{m+p-1}\alpha^k\right)\beta \quad \text{Where} \quad \alpha = \frac{1}{k}\sum_{n=0}^{k-1}\left(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}}\right)$$

$$\leq (\alpha^m + \alpha^{m+1} + \alpha^{m+2} + \text{------} + \alpha^{m+p-1})\beta_0$$

$$\leq \alpha^m(1 + \alpha^1 + \alpha^2 + \text{-------} + \alpha^{p-1})\beta_0$$

$$\leq \left(\frac{\alpha^m}{1-\alpha}\right)\beta_0$$

$G(g\lambda_m, g\lambda_{m+p}, g\lambda_{m+p}) + G(g\mu_m, g\mu_{m+p}, g\mu_{m+p}) + G(g\nu_m, g\nu_{m+p}, g\nu_{m+p}) + G(g\nu_m, g\nu_{m+p}, g\nu_{m+p}) \leq \left(\frac{\alpha^m}{1-\alpha}\right)\beta_0$

Now letting the limit as $m, p \to \infty$, we obtain

$\lim_{m\to\infty} G(g\lambda_m, g\lambda_{m+p}, g_{\lambda m+p}) + G((g\mu_m, g\mu_{m+p}, g\mu_{m+p}) + G(g\nu_m, g\nu_{m+p}, g\nu_{m+p}) = 0$ since $0 < \alpha < 1$

Further which implies that

$$\left.\begin{aligned} &\lim_{m\to\infty} G(g\lambda_m, g\lambda_{m+p}, g\lambda_{m+p}) = 0 \\ &\lim_{m\to\infty} G(g\mu_m, g\mu_{m+p}, g\mu_{m+p}) = 0 \\ &\lim_{m\to\infty} G(g\nu_m, g\nu_{m+p}, g\nu_{m+p}) = 0 \end{aligned}\right\} \quad (11)$$

Thus, $\{g\lambda_m\}$, $\{g\mu_m\}$, $\{g\nu_m\}$ are Cauchey sequences in Ʊ. Since g(Ʊ) is closed and by the completeness of Ʊ there exists (r, s, t) ϵ $Ʊ^3$ with $\lim_{m\to\infty}\{g\lambda_m\}=g(r)=\lambda$,

$\lim_{m\to\infty}\{g\mu_m\}=g(s)=\mu, \lim_{n\to\infty}\{g\nu_m\}=g(t)=\nu$

By construction $\lim_{m\to\infty} g(\lambda_{m+1}) = \lim_{m\to\infty} T_m(\lambda_m, \mu_m, \nu_m)=\lambda$

$$\lim_{m\to\infty} g(\mu_{m+1}) = \lim_{m\to\infty} T_m(\mu_m, \lambda_m, \mu_m)=\mu \text{ and}$$

$$\lim_{m\to\infty} g(\nu_{m+1}) = \lim_{m\to\infty} T_m(\nu_m, \mu_m, \lambda_m)=\nu.$$

Now by compatibility and weakly reciprocally continuous of g and $\{T_n\}_{n\epsilon N}$ we have

$$\left.\begin{aligned} \lim_{m\to\infty} T_m(g\lambda_m, g\mu_m, g\nu_m) &= g(\lambda) \\ \lim_{m\to\infty} T_m(g\mu_m, g\lambda_m, g\mu_m) &= g(\mu) \\ \lim_{m\to\infty} T_m(g\nu_m, g\mu_m, g\lambda_m) &= g(\nu). \end{aligned}\right\} \quad (12)$$

Suppose $\{T_n\}_{n\epsilon N}$ is continuous & using G(5), we have

$G(T_n(\lambda, \mu, \nu), T_m(g\lambda_m, g\mu_m, g\nu_m), T_m(g\lambda_m, g\mu_m, g\nu_m)) \leq G(T_n(\lambda, \mu, \nu), g(T_m(\lambda_m, \mu_m, \nu_m)), g(T_m(\lambda_m, \mu_m, \nu_m))) + G(g(T_m(\lambda_m, \mu_m, \nu_m)), T_m(g\lambda_m, g\mu_m, g\nu_m), T_m(g\lambda_m, g\mu_m, g\nu_m))$

Similarly

$G(T_n(\mu, \lambda, \mu), T_m(g\mu_m, g\lambda_m, g\mu_m), T_m(g\mu_m, g\lambda_m, g\mu_m)) \leq G(T_n(\mu, \lambda, \mu), g(T_m(\mu_m, \lambda_m, \mu_m)), g(T_m(\mu_m, \lambda_m, \mu_m))) + G(g(T_m(\mu_m, \lambda_m, \mu_m)), T_m(g\mu_m, g\lambda_m, g\mu_m), T_m(g\mu_m, g\lambda_m, g\mu_m))$

And

$G(T_n(\nu, \mu, \lambda), T_m(g\nu_m, g\mu_m, g\lambda_m), T_m(g\nu_m, g\mu_m, g\lambda_m)) \leq G(T_n(\nu, \mu, \lambda), g(T_m(\nu_m, \mu_m, \lambda_m)), g(T_m(\nu_m, \mu_m, \lambda_m))) + G(g(T_m(\nu_m, \mu_m, \lambda_m)), T_m(g\nu_m, g\mu_m, g\lambda_m), T_m(g\nu_m, g\mu_m, g\lambda_m))$

Taking limit as $m\to\infty$, and weakly reciprocally continues we get

$G(T_n(\lambda, \mu, \nu), g\lambda, g\lambda) \leq G(T_n(\lambda, \mu, \nu), g\lambda, g\lambda) + G(g\lambda, g\lambda, g\lambda)$

$G(T_n(\lambda, \mu, \nu), g\lambda, g\lambda)) = 0.$

Similarly, $G(T_n(\mu, \lambda, \mu), g\mu, g\mu) = 0$

and $G(T_n(\nu, \mu, \lambda), g\nu, g\nu) = 0$

i.e., $T_n(\lambda, \mu, \nu) = g\lambda$

$T_n(\mu, \lambda, \mu) = g\mu$

$T_n(\nu, \mu, \lambda) = g\nu$

Thus (λ, μ, ν) is tripled coincidence point of $\{T_n\}$ and g.

Now we prove it is unique.

Assume (ꝑ, ꝗ, r) is another tripled coincidence points.

i.e, g(ꝑ)=T_n(ꝑ, ꝗ,r), g(ꝗ)=T_n(ꝗ,ꝑ,ꝗ) and g(r)=T_n(r,ꝗ,ꝑ)

then we prove that g(λ)= g(ꝑ), g(μ)=g(ꝗ) and g(v)=g(r)

since the set of tripled coincidence points are comparable, applying condition (5), we obtain

$G(g\lambda, gꝑ, gꝑ) = G(T_n(\lambda,\mu,\nu),T_m(ꝑ,ꝗ,r),T_m(ꝑ,ꝗ,r))$

$\leq \gamma_{n,m}[G(g\lambda,T_n(\lambda,\mu,\nu),T_n(\lambda,\mu,\nu))+G(gp,T_m(p,q,r),T_m(p,q,r))]+\delta_{n,m}G(g\lambda,gp,gp)$

$G(g\lambda, gp, gp) \leq \gamma_{n,m}[G(g\lambda, g\lambda, g\lambda)+G(gp,gp,gp)]+\delta_{n,m}(g\lambda,gp,gp)$

$G(g\lambda, gp, gp) = 0$ as $\delta_{n,m} < 1$

∴ $g\lambda = gp$

In the same way we can prove g(μ)=g(q) and g(v) =g(r)

Hence g and $\{T_n\}_{n\epsilon N}$ have a unique tripled

point of coincidence. i.e, (gλ, gμ, gν).

$\{T_n\}_{n\epsilon N}$ and g are weakly compatible since they are compatible, i.e, they commute at their coincidence points, i.e, $\lambda=g\lambda=T_n(\lambda, \mu, \nu)$, $\mu=g\mu=T_n(\mu, \lambda, \mu)$, $\nu=g\nu=T_n(\nu, \mu, \lambda)$.

Thus, $\{T_n\}_{n\epsilon N}$ and g have unique tripled common fixed point whenever they are weakly compatible.

COROLLARY 4.4: Assume ($\mathscr{B}$, G, ≤) is a complete partially ordered G – metric space. Let $\{T_n\}_{n\epsilon N}$: $\mathscr{B}^3 \rightarrow \mathscr{B}$ be a sequence of mappings and g is an identity mapping for $\rho, \boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\varkappa}, \boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathscr{B}$, $\{T_n\}_{n\epsilon N}$ with $\boldsymbol{\rho} \leq \boldsymbol{\varkappa}$, $\boldsymbol{\eta} \leq \boldsymbol{\sigma}$, $\boldsymbol{\tau} \leq \boldsymbol{\zeta}$ or $\boldsymbol{\varkappa} \leq \boldsymbol{\rho}$, $\boldsymbol{\sigma} \leq \boldsymbol{\eta}$, $\boldsymbol{\zeta} \leq \boldsymbol{\tau}$, satifies the following conditions.

(i) $T_m(\boldsymbol{\rho},\boldsymbol{\sigma},\boldsymbol{\tau}) \leq T_{m+1}(\boldsymbol{\varkappa},\boldsymbol{\eta},\boldsymbol{\zeta})$

(ii) $G(T_n(\boldsymbol{\rho},\boldsymbol{\sigma},\boldsymbol{\tau}), T_m(\boldsymbol{\varkappa},\boldsymbol{\eta},\boldsymbol{\zeta}),T_m(\boldsymbol{\varsigma},\boldsymbol{\alpha},\boldsymbol{\partial})) \leq \gamma_{n,m}[(\rho, T_n(\boldsymbol{\rho},\boldsymbol{\sigma},\boldsymbol{\tau}),T_n(\boldsymbol{\rho},\boldsymbol{\sigma},\boldsymbol{\tau}))+G(\boldsymbol{\varkappa},T_m(\boldsymbol{\varkappa},\boldsymbol{\eta},\boldsymbol{\zeta}),T_m(\boldsymbol{\varsigma},\boldsymbol{\alpha},\boldsymbol{\partial}))]+\delta_{n,m} G(\boldsymbol{\rho},\boldsymbol{\varkappa},\boldsymbol{\varsigma})$ with $0 \leq \gamma_{n,m}, \delta_{n,m} < 1$ and n, m ∈N

If $\sum_{n=1}^{\infty}(\frac{\gamma_{n,n+1}+\delta_{n,n+1}}{1-\gamma_{n,n+1}})$ is an α- series and $\mathscr{B}$ is regular ,then there exists a tripled fixed point of $\{T_n\}_{n\in N}$. i.e, ∃ $(\boldsymbol{\rho},\boldsymbol{\sigma},\boldsymbol{\tau}) \in \mathscr{B}^3$ such that $\boldsymbol{\rho}=T_n(\boldsymbol{\rho},\boldsymbol{\sigma},\boldsymbol{\tau})$, $\boldsymbol{\sigma}= T_n(\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{\sigma})$,$\boldsymbol{\tau}=T_n(\boldsymbol{\tau},\boldsymbol{\sigma},\boldsymbol{\rho})$ for n∈N.

Theorem 4.5: Consider ($\mathscr{B}$, G, ≤) be a partially ordered complete G-metric space and is regular.Let g and $\{T_m\}_{m\in N}$ is same as in theorem 3.3 and $\lim_{n\rightarrow\infty} \sup \beta_{m,n} < 1, 0 \leq \beta_{i,j}, \gamma_{i,j} < 1$ *and let* *g and* $\{T_m\}$ *satifies conditions* (2), (3) *and* (5) *then g and* $\{T_m\}_{m\in N}$ have tripled coincidence point.

Proof: from the theorem ,sequences $\{g\lambda_m\}, \{g\mu_m\}$ *and* $\{g\upsilon_m\}$ *are cauchy sequences in* $G(\mathscr{B})$.Since $\{g\lambda_m\}$ *and* $\{g\upsilon_m\}$ *are increasing sequences and* $\{g\mu_m\}$ *is decreasing sequences respectively*: *by using the regularity of* ($\mathscr{B}$, G, ≤), we have $g\alpha_m \leq \boldsymbol{\alpha}$, $\rho \leq g\rho_m$, $g\sigma_m \leq \boldsymbol{\sigma}$ for all m ≥ 0 then by (5), we obtain $G(T_m(g\alpha_m, g\rho_m, g\sigma_m), T_n(\alpha,\rho,\sigma), T_n(\alpha,\rho,\sigma)) \leq$

$\beta_{m,n}[G(g(g\alpha_m), T_m(g\alpha_m, g\rho_m, g\sigma_m), T_m(g\alpha_m, g\rho_m, g\sigma_m)) + G(g\alpha, T_n(\alpha,\rho,\sigma), T_n(\alpha,\rho,\sigma)) + \gamma_{m,n}(G(g(g\alpha_m), g\alpha, g\alpha)$

Taking as m→∞, we obtain $T_m(\alpha,\rho,\sigma) = g\alpha \; as \; \beta_{m,n} < 1,$

Similarly we can prove $T_m(\rho,\alpha,\rho) = g\boldsymbol{\rho}$

And $T_m(\sigma,\rho,\alpha) = g\boldsymbol{\sigma}$

Thus (α,ρ,σ) *is tripled coincidence point of* $\{T_m\}_{m\in N}$ *and* g

Example 4.6: Consider E = [0,1] and G$(\alpha,\rho,\sigma) = \max\{|\alpha-\rho|, |\rho-\sigma|, |\sigma-\alpha|\}$.

Clearly (E, G) is complete G metric space

Define $\gamma_{i,j} = \frac{1}{4^{2i+1}}, \delta_{i,j} = \frac{1}{4^i}$ *for all* $i,j = 1,2,\dots\dots\dots$

Consider the mapping T_n: E^3→E and g: E→E by $T_i(\boldsymbol{\alpha}, \boldsymbol{\rho}, \boldsymbol{\sigma}) = \frac{\alpha+\rho+\sigma}{3^i}$, g($\boldsymbol{\alpha}$)=2$\boldsymbol{\alpha}$ for all

$\boldsymbol{\alpha}, \boldsymbol{\rho}, \boldsymbol{\sigma} \in$ E, n=1,2,3------

Assume i<j and for all $\alpha, \rho, \sigma, \mathfrak{m}, \mathfrak{n}, \mathfrak{o}, \mathscr{f}, \mathscr{g}, \mathscr{h} \in$ E with $\alpha > \mathfrak{m} \geq \mathscr{f}$, $\rho < \mathfrak{n} \leq \mathscr{g}$,

$$\sigma > \mathfrak{o} \geq \mathscr{h}$$

$G(T_i(\boldsymbol{\alpha}, \boldsymbol{\rho}, \boldsymbol{\sigma}), T_j(\mathfrak{m},\mathfrak{n},\mathfrak{o}), T_j(\mathscr{f}, \mathscr{g}, \mathscr{h})) = \left|\frac{\alpha+\rho+\sigma}{3^i} - \frac{\mathscr{f}+\mathscr{g}+\mathscr{h}}{3^j}\right|$ and

$G(g\alpha, T_i(\alpha,\rho,\sigma), T_i(\alpha,\rho,\sigma)) + G(g\mathfrak{m}, T_j(\mathfrak{m},\mathfrak{n},\mathfrak{o}), T_j(\mathscr{f},\mathscr{g},\mathscr{h}) = \left|2\alpha - \frac{\alpha+\rho+\sigma}{3^i}\right| + \left|2\mathfrak{m} - \frac{\mathscr{f}+\mathscr{g}+c}{3^j}\right|$

$$G(g\boldsymbol{\alpha}, \boldsymbol{g\mathfrak{m}}, \boldsymbol{g\mathscr{f}}) = |\boldsymbol{6\alpha - 6\mathscr{f}}|$$

Since $\gamma_{i,j}, \delta_{i,j} < 1$, condition (5) satisfied for all $\alpha, \rho, \sigma, \mathfrak{m}, \mathfrak{n}, \mathfrak{o}, \mathscr{f}, \mathscr{g}, \mathscr{h} \in$ E with

$$\alpha > \mathfrak{m} \geq \mathscr{f}, \qquad \rho < \mathfrak{n} \leq \mathscr{g}, \qquad \sigma > \mathfrak{o} \geq \mathscr{h}$$

Moreover, the series

$\sum_{i=1}^{\infty}\left(\frac{\gamma_{i,i+1}+\delta_{i,i+1}}{1-\gamma_{i,i+1}}\right) = \sum_{i=1}^{\infty}\frac{\frac{1}{4^{2i+1}}+\frac{1}{4^i}}{1-\frac{1}{4^{2i+1}}} = \sum_{i=1}^{\infty}\frac{4^{i+1}+1}{4^{2i+1}-1}$ is an α-series with $\alpha = \frac{1}{4}$.

Unique common tripled fixed point for g and T_n is (0,0,0)

Theorem 4.7: Let ($\mathscr{H}$, G) be complete G metric space and $\{T_n\}$: $\mathscr{H} \to \mathscr{H}$ be a sequence of self-mappings such that

$$G(T_j(\wp), T_k(\varphi), T_k(\upsilon)) \leq \beta_{j,k}[G(\wp, T_j(\wp), T_j(\wp)) + G(\varphi, T_k(\varphi), T_k(\varphi)) + G(\upsilon, T_k(\upsilon), T_k(\upsilon))] + \gamma_{j\,k}\, G(\wp, \varphi, \upsilon) \qquad \text{(A)}$$

for $\wp, \varphi, \upsilon \in \mathscr{H}$ with $\wp \neq \varphi$, $0 \leq \beta_{j,k}, \gamma_{j,k} < \frac{1}{2}$, j,k=1,2,----

If $\sum_{j=1}^{\infty}\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}$ is an α-series then $\{T_n\}$ has unique common fixed point in $\mathscr{H}$.

Proof: For any $\psi_0 \in \mathscr{H}$, we can consider the sequence $\psi_n = T_n(\psi_{n-1})$, n=1,2----

From (A) we have

$$G(\psi_1,\psi_2,\psi_2)= G(T_1\psi_0, T_2\psi_1, T_2\psi_1)$$
$$\leq \beta_{1,2}[G(\psi_0, T_1\psi_0, T_1\psi_0)+ G(\psi_1, T_2\psi_1, T_1\psi_1)+G(\psi_1,T_2\psi_1, T_2\psi_1)]+ \gamma_{1,2}G(\psi_0, \psi_1, \psi_1)$$
$$\leq \beta_{1,2}[G(\psi_0, \psi_1, \psi_1)+G(\psi_1, \psi_2, \psi_2)+G(\psi_1, \psi_2, \psi_2)]+ \gamma_{1,2}G(\psi_0, \psi_1, \psi_1)$$
$$\leq \beta_{1,2}[G(\psi_0, \psi_1, \psi_1)+2G(\psi_1, \psi_2, \psi_2)]+ \gamma_{1,2}G(\psi_0, \psi_1, \psi_1)$$

$$G(\psi_1,\psi_2, \psi_2) \leq \left(\frac{\beta_{1,2}+\gamma_{1,2}}{1-2\beta_{1,2}}\right) G(\psi_0, \psi_1, \psi_1).$$

Also we get $G(\psi_2, \psi_3, \psi_3)] = G(T_2\psi_1, T_3\psi_2, T_3\psi_2)$

$$\leq \left(\frac{\beta_{2,3}+\gamma_{2,3}}{1-2\beta_{2,3}}\right)G(\psi_1, \psi_2, \psi_2)$$
$$\leq \left(\frac{\beta_{2,3}+\gamma_{2,3}}{1-2\beta_{2,3}}\right)\left(\frac{\beta_{1,2}+\gamma_{1,2}}{1-2\beta_{1,2}}\right) G(\psi_0, \psi_1, \psi_1).$$

Repeating the above reasoning we obtain

$$G(\psi_n, \psi_{n+1},\psi_{n+1}) \leq \prod_{j=1}^{n}\left(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}\right) G(\psi_0, \psi_1, \psi_1).$$

For p>0 and by G_5

$$G(\psi_n, \psi_{n+p}, \psi_{n+p}) \leq G(\psi_n, \psi_{n+1},\psi_{n+1})+ G(\psi_{n+1}, \psi_{n+2},\psi_{n+2})+ G(\psi_{n+2}, \psi_{n+3},\psi_{n+3})+\text{---}$$
$$+ G(\psi_{n+p-1}, \psi_{n+p}, \psi_{n+p})$$
$$\leq \prod_{j=1}^{n}\left(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}\right)+\prod_{j=1}^{n+1}\left(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}\right) +\text{----}$$
$$+\prod_{j=1}^{n+p-1}\left(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}\right)]\ G(\psi_0, \psi_1, \psi_1)).$$
$$\leq \sum_{k=0}^{p-1}\prod_{j=1}^{n+k}\left(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}\right) G(\psi_0, \psi_1, \psi_1).$$
$$\leq \sum_{k=n}^{n+p-1}\prod_{j=1}^{k}\left(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}\right) G(\psi_0, \psi_1, \psi_1)$$

By using the fact that the Geometric mean is less than or equal to arithmetic mean for non-negative real numbers and let α and n_α for $m \geq n_\alpha$ as in definition 2.9 follows as below

$$\leq \sum_{k=n}^{n+p-1}\left[\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}\right)\right]^{k} G(\psi_0, \psi_1, \psi_1).$$

$$\therefore\ G(\psi_n, \psi_{n+p}, \psi_{n+p}) \leq \sum_{k=n}^{n+p-1}\left[\frac{1}{k}\sum_{j=1}^{k}\left(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}}\right)\right]^{k} G(\psi_0, \psi_1, \psi_1).$$
$$\leq \left(\sum_{k=n}^{n+p-1}\alpha^{k}\right)G(\psi_0, \psi_1, \psi_1).$$
$$\leq (\alpha^{n} + \alpha^{n+1} + \alpha^{n+2} +\text{---}+\alpha^{n+p-1})G(\psi_0, \psi_1, \psi_1).$$

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$\leq \alpha^n(1+\alpha+\alpha^2+\alpha^3+\text{------}+\alpha^{p-1})\ G(\wp_0, \wp_1, \wp_1)$.

$\leq \frac{\alpha^n}{1-\alpha} G(\wp_0, \wp_1, \wp_1)$.

Let n, p $\to\infty$, $G(\wp_n, \wp_{n+p}, \wp_{n+p})\to 0$ since $0 < \alpha < 1$.

Thus, $\{\wp_n\}$ is a Cauchy sequence in $\mathscr{H}$.Due to completeness of $\mathscr{H}$ it converges to ς in $\mathscr{H}$.

For m>0, we have

$G(\wp_n,T_m(\varsigma),T_m(\varsigma)) = G(T_n(\wp_{n-1}),T_m(\varsigma), T_m(\varsigma))$

$\leq \beta_{n,m}[G(\wp_{n-1}, \wp_n, \wp_n)+ G(\varsigma, T_m\varsigma, T_m\varsigma)+ G(\varsigma, T_m\varsigma, T_m\varsigma)]+$

$\gamma_{n,m} (\wp_{n-1}, \varsigma, \varsigma)$

Letting $n\to\infty$, as $\wp_n\to\varsigma$ we obtain

$G(\varsigma, T_m\varsigma, T_m\varsigma) \leq \beta_{n,m}[G(\varsigma,\varsigma,\varsigma)+2G(\varsigma, T_m\varsigma, T_m\varsigma)+\gamma_{n,m}G(\varsigma, \varsigma, \varsigma)$

$G(\varsigma, T_m\varsigma, T_m\varsigma) \leq 2\beta_{n,m}\ G(\varsigma, T_m\varsigma, T_m\varsigma)$

Since $\beta_{n,m}<\frac{1}{2}$, $\quad G(\varsigma, T_m\varsigma, T_m\varsigma) = 0$

$\therefore\ T_m\varsigma = \varsigma$

ς is fixed point of $\{T_m\}$.

Assume ϑ is another fixed point of $\{T_m\}$, i.e., $T_m(\vartheta)=\vartheta$. Such that $\varsigma\neq\vartheta$.

$G(\varsigma, \vartheta, \vartheta) = G(T_m\varsigma, T_m\vartheta, T_m\vartheta)$

$\leq \beta_{n,m}[G(\varsigma, T_m\varsigma, T_m\varsigma)+ G(\vartheta, T_m\vartheta, T_m\vartheta)+ G(\vartheta, T_m\vartheta, T_m\vartheta)]+\ \gamma_{n,m}G(\varsigma, \vartheta, \vartheta)$

$G(\varsigma, \vartheta, \vartheta) \leq \beta_{n,m}[G(\varsigma,\varsigma,\varsigma)+G(\vartheta, \vartheta, \vartheta)+ G(\vartheta, \vartheta, \vartheta)]+ \gamma_{n,m}G(\varsigma, \vartheta, \vartheta)$.

$G(u,\vartheta,\vartheta) \quad \leq \gamma_{n,m}G(u, \vartheta, \vartheta)$.

, which is a contradiction since $\gamma_{n,m}< \frac{1}{2}$

$\therefore \varsigma = \vartheta$

$\therefore$ ς is a unique fixed point of $\{T_m\}$.

Corollary 4.8: Consider a sequence of self mappings be $\{T_n\}$ and ($\mathscr{H}$, G) be a complete G- metric space satisfies the following contraction such that

$G(T_j\wp, T_k\varphi, T_k\upsilon) \leq \beta_{j,k}[G(\wp, T_j\wp,T_j\wp)+G(\varphi,T_k\varphi,T_k\varphi)+ G(\upsilon,T_k\upsilon,T_k\upsilon)$ (B)

for $\wp, \varphi,\upsilon \in \mathscr{H}$ with $\wp \neq \varphi$, $0\leq\beta_{j,k}<\frac{1}{2}$ j,k=1,2,3----

then $\{T_n\}$ has a unique fixed point in $\mathscr{H}$ if $\sum_{j=1}^{\infty}(\frac{\beta_{j,j+1}}{1-2\beta_{j,j+1}})$ is an α-series

Example 4.9. Let $\mathscr{H}$= [0,1] and $G(\wp,\varphi,\upsilon) = max\{|\wp - \varphi|, |\varphi - \upsilon|, |\upsilon - \wp|\}$.

Clearly ($\mathscr{H}$, G) is a complete Generalized metric space. Define $\beta_{j,k}=\frac{1}{2+3^k}$ for all j,k = 1,2,3,….

and $T_j(\wp) = \frac{\wp}{3^j}$ for all $\wp \in \mathscr{H}$ and j = 1,2,….

Assume j < k and $\wp > \varphi \geq \upsilon$ so we have

$$G(T_j\wp, T_k\varphi, T_k\upsilon) = max\{|\frac{\wp}{3^j} - \frac{\varphi}{3^k}|,|\frac{\varphi}{3^k} - \frac{\upsilon}{3^k}|,|\frac{\upsilon}{3^k} - \frac{\wp}{3^j}|\}$$

$$= |\frac{\wp}{3^j} - \frac{\varphi}{3^k}|$$

$G(\wp, T_j\wp, T_j\wp)+G(\varphi, T_k\varphi, T_k\varphi)+ G(\upsilon, T_k\upsilon, T_k\upsilon) = |\wp - \frac{\wp}{3^j}| + |\varphi - \frac{\varphi}{3^k}| + |\upsilon - \frac{\upsilon}{3^k}|$

Clearly $G(T_j\wp, T_k\varphi, T_k\upsilon) \leq G(\wp, T_j\wp, T_j\wp)+G(\varphi, T_k\varphi, T_k\varphi)+ G(\upsilon, T_k\upsilon, T_k\upsilon)$.

∴ condition (B) satisfied for all $\wp, \varphi, \upsilon \in \mathscr{H}$ with $\wp \neq \varphi$

$$\sum_{j=1}^{\infty}(\frac{\beta_{j,j+1}}{1-2\beta_{j,j+1}}) = \sum_{j=1}^{\infty}\frac{\frac{1}{2+3^j}}{1-2\frac{1}{2+3^j}}$$

$$= \sum_{j=1}^{\infty}\frac{\frac{1}{2+3^j}}{\frac{2+3^j-2}{2+3^j}}$$

$$= \sum_{j=1}^{\infty}\frac{1}{3^j} \text{ is an } \alpha \text{ series with } \alpha = \frac{1}{3}.$$

By corollary 3.8, $\{T_n\}$ has a unique fixed point $0 \in \mathscr{H}$.

Theorem 4.10: Assume ($\mathscr{H}$,G) be a complete G metric space. $\{T_n\}$ be a sequence of self mappings on $\mathscr{H}$ such that

$$G(T_j^p(\wp), T_k^p(\varphi), T_k^p(\upsilon)) \leq \beta_{j,k}[G(\wp, T_j^p(\wp), T_j^p(\wp))+G(\varphi, T_k^p(\varphi), T_k^p(\varphi))+ G(\upsilon, T_k^p(\upsilon), T_k^p(\upsilon))+ \gamma_{j,k}G(\wp,\varphi,\upsilon).$$

For $\wp,\varphi,\upsilon \in \mathscr{H}$, $\wp\neq\varphi$, $0\leq\beta_{j,k}, \gamma_{j,k}<\frac{1}{2}$ j,k=1,2,3-----

If $\sum_{j=1}^{\infty}(\frac{\beta_{j,j+1}+\gamma_{j,j+1}}{1-2\beta_{j,j+1}})$ is an α-series, then $\{T_n\}$ has unique common fixed point in $\mathscr{H}$.

5.CONCLUSION

In this study we present unique tripled fixed-point and common fixed-point results for a sequence of mappings and a self-mapping in G metric space via α-series with supporting examples.

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