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Some Fixed Point Results for (α, ψ, φ)- Geraghty Contraction Mappings in Bipolar Metric Space

Diksha[1], **Manoj Kumar**[2,*]

[1, 2]Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak-124021, Haryana, India
[1]Department of Mathematics, Government College, Matanhail, Jhajjar-124106, Haryana, India
[2]Department of Mathematics, Maharishi Markandeshwar (Deemed to be University), Mullana, Ambala-133207, Haryana, India
Email addresses- dikshaahlawat14@gmail.com, manojantil18@gmail.com
(*Corresponding Author)

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Abstract:
In this paper, we explore a generalization of (α, ψ)- Geraghty contractions and investigate the existence and uniqueness of fixed points for mappings satisfying this condition. The study extends, improves, and generalizes some earlier results in the literature on this topic. We consider the interplay of three parameters: α, ψ and φ, which play a crucial role in defining the contraction properties. Our goal is to establish fixed point theorems that encompass various scenarios within bipolar metric spaces.

Keywords: Fixed point, (α, ψ, φ)- Geraghty contraction mappings, covariant and contravariant mappings, bipolar metric space.

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1. Introduction

The concept of fixed points is fundamental in mathematics, and it arises in diverse fields such as differential equations and optimization. In 1922, Banach [2] introduced Banach contraction principle as the first constructive method to get a fixed point for a self map on a complete metric space. Continuation of this, in 1973, Geraghty [8] gives an extension of the Banach contraction mapping principle, provides a powerful tool for proving the existence of fixed points. Specifically, Geraghty's result ensures a unique fixed point under certain contractive conditions. Many authors generalized his work, see [1,4,6,7,13].

In 2012, Samet *et al*. [14] introduced the concepts of α-contractive and α-admissible mappings and proved various fixed point theorems of α- admissible contractive mappings in complete metric spaces. Recently, in 2015, Chandok [4] introduced the concept of (α, β)- admissible Geraghty type contractive mappings and proved some fixed point theorems of such kind of mappings in complete metric spaces. Some researcher extended their work in various spaces [9-16].

In 2019, Karapinar *et al*. [7] introduced the notion φ-Geraghty and Ciric type φ-Geraghty contractive mappings in complete metric space and proved some fixed points theorems and uniqueness of fixed points.

To get a new approach for fixed point results in 2016, Mutlu and Gürdal [10] introduced the concept of bipolar metric space. The major difference between the previously defined spaces and bipolar is of distance function. In bipolar metric space, the distance function is from the cartesian product of two different sets to non-negative real numbers. Since then, many authors have proved several fixed point results in bipolar metric space see [5], [9-11], [13].

Motivated by the work of Abduletif *et al.* [1], the main objective of this manuscript is to prove some fixed points results and their uniqueness for (α, ψ, φ)- Geraghty contraction mapping in complete bipolar metric spaces. Furthermore, we offer illustrations to support our essential findings.

2. Preliminaries

We need to introduce some new notations and terminology and provide some fundamental definitions which is used for the fixed point theorems for (α, ψ, φ)- Geraghty contraction mappings in bipolar metric spaces.

Definition 2.1. In 2016, Mutlu and Gürdal [10] introduced the concept of bipolar metric space.

Let X and Y are two non-empty sets and $d : X \times Y \to [0, \infty)$ be a function satisfying the following conditions:

(BP1) $d(x, y) = 0$ if and only if $x = y$, where $(x, y) \in X \times Y$,

(BP2) $d(x, y) = d(y, x)$ for all $x, y \in X \cap Y$,

(BP3) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Then d is called bipolar metric and (X, Y, d) is called bipolar metric space.

If $X \cap Y = \emptyset$, then space is called disjoint otherwise joint. The set X is called left pole and Y is called right pole of bipolar metric space (X, Y, d) and any element of left pole (X), right pole (Y) and $X \cap Y$ is called left element, right element and central element respectively.

Definition 2.2. Let (X, Y, d) be a bipolar metric space. Then any sequence $(x_n) \subseteq X$ is called left sequence and is said to be convergent to right element say 'y' if $d(x_n, y) \to 0$ as $n \to \infty$. Similarly, a right sequence $(y_n) \subseteq Y$ is said to be convergent to a left element say 'x' if $d(x, y_n) \to 0$ as $n \to \infty$.

Definition 2.3. Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces.

Let $T : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function such that

(i) If $T(X_1) \subseteq X_2$ and $T(Y_1) \subseteq Y_2$, then T is called covariant map and is denoted by $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.

(ii) If $T(X_1) \subseteq Y_2$ and $T(Y_1) \subseteq X_2$, then T is called contravariant map and is denoted by $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.

Definition 2.4. Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces.

(i) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left continuous at a point $x_0 \in X_1$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_2(Tx_0, Ty) < \varepsilon$ whenever $d_1(x_0, y) < \delta$.

(ii) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right continuous at a point $y_0 \in Y_1$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_2(Tx, Ty_0) < \varepsilon$ whenever $d_1(x, y_0) < \delta$.

(iii) A map T is called continuous, if it is left continuous at each $x_0 \in X_1$ and right continuous at each $y_0 \in Y_1$.

(iv) A contravariant map $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.

Definition 2.5. Let (X, Y, d) be a bipolar metric space.

(i) A sequence $\{(x_n, y_n)\}$ on the set $X \times Y$ is called a bisequence on (X, Y, d).

(ii) If both the sequences (x_n) and (y_n) converge, then bisequence $\{(x_n, y_n)\}$ is said to be convergent. If both the sequences (x_n) and (y_n) converge to same point v and $v \in X \cap Y$, then this bisequence is said to be biconvergent.

(iii) A bisequence $\{(x_n, y_n)\}$ on (X, Y, d) is said to be Cauchy bisequence, if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, y_m) < \epsilon$ for all $n, m \geq N$.

(iv) A bipolar metric space is said to be complete if every Cauchy bisequence is convergent in this space.

Definition 2.6. [11] Let X and Y be two non-empty sets. Let $T : (X, Y) \rightrightarrows (X, Y)$ and $\alpha : X \times Y \to [0, +\infty)$. Then T is called α-admissible (covariant) if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$$

for all $x \in X$ and $y \in Y$.

Definition 2.7. [11] Let X and Y be two non-empty sets. Let $T : (X, Y) \rightleftarrows (X, Y)$ and $\alpha : X \times Y \to [0, +\infty)$. Then T is called α-admissible (contravariant) if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Ty, Tx) \geq 1$$

for all $x \in X$ and $y \in Y$.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ which satisfying the following conditions:

(i) ψ is continuous,
(ii) ψ is strictly increasing,
(iii) $\psi(0) = 0$.

Consider Θ be the family of functions $\theta : [0, \infty) \to [0,1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\theta(t_n) \to 1$ implies that $t_n \to 0$, as $n \to \infty$.

Let Θ_t be the family of functions $\theta : [0, \infty) \to [0,1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\lim sup\, \theta(t_n) \to 1$ implies that $t_n \to 0$, as $n \to \infty$.

3. Main Results

In this section, we will introduce new notations for φ-Geraghty contraction mappings and prove various fixed point theorems for such type of mappings in complete bipolar metric spaces.

Definition 3.1. Let X and Y be two non-empty sets. Consider (X, Y, d) be a bipolar metric space, $T : (X, Y) \rightrightarrows (X, Y)$ is called Geraghty contraction if there exist a function $\theta \in \Theta$ which satisfies the following condition:

$$d(Tx, Ty) \leq \theta\big(d(x, y)\big)\, d(x, y) \text{ for all } x \in X \text{ and } y \in Y.$$

Definition 3.2. Let X and Y be two non-empty sets and (X, Y, d) be a bipolar metric space, $T : (X, Y) \rightrightarrows (X, Y)$. Suppose that $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is function and $\theta \in \Theta$ and T is called φ-Geraghty contraction if it satisfies the following condition:

(i)$\varphi(t) < t$ for any $t \in (0, \infty)$,
(ii)For any $\varepsilon > 0$, there exist $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta \Rightarrow \varphi(t) \leq \varepsilon$,
(iii)$d(Tx, Ty) \leq \theta\big(d(x, y)\big)\, \varphi(d(x, y))$ for all $x \in X$ and $y \in Y$.

Definition 3.3. Let X and Y be two non-empty sets and (X, Y, d) be a bipolar metric space, $T : (X, Y) \rightrightarrows (X, Y)$ is a self map and $\alpha : X \times Y \to [0, \infty)$. A mapping T is said to be (α, ψ, φ)-Geraghty contraction mapping if there exist $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi \in \Psi$ and $\theta \in \Theta$ satisfies the following condition:

(i)$\varphi(t) < t$ for any $t \in (0, \infty)$, (3.1)
(ii)For any $\varepsilon > 0$, there exist $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta \Rightarrow \varphi(t) \leq \varepsilon$, (3.2)
(iii)$\alpha(x, y)\psi(d(Tx, Ty)) \leq \theta\big(\psi(d(x, y))\big)\, \varphi(\psi(d(x, y)))$, for all $x \in X$ and $y \in Y$. (3.3)

Theorem 3.4. Let (X, Y, d) be a complete bipolar metric space, $T : (X, Y) \rightrightarrows (X, Y)$ is a covariant mapping and $\alpha : X \times Y \to [0, \infty)$. Suppose that the following conditions hold:

(i)T is α- admissible mapping,
(ii)T is an (α, ψ, φ)- Geraghty contraction mapping,
(iii) there exist $x_0 \in X$and $y_0 \in Y$ such that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, Ty_0) \geq 1$.

Then T has fixed point.

Proof: Let $x_0 \in X$and $y_0 \in Y$ such that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, Ty_0) \geq 1$. Now we define a bisequence $\{(x_n, y_n)\}$ in (X, Y) by

$$Tx_n = x_{n+1} \text{ and } Ty_n = y_{n+1} \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Since T is α- admissible mapping.

So, $\alpha(x_0, y_0) \geq 1 \Rightarrow \alpha(Tx_0, Ty_0) \geq 1$,

$$\alpha(x_0, y_1) = \alpha(x_0, Ty_0) \geq 1,$$

$$\alpha(x_1, y_1) = \alpha(Tx_0, Ty_0) \geq 1,$$

Using mathematical induction, we get

$$\alpha(x_n, y_{n+1}) \geq 1 \text{ and } \alpha(x_n, y_n) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

Putting $x = x_{n+1}$ and $y = y_{n+2}$ in equation (3.3), using equations (3.1), (3.2) and by the properties of ψ and θ, we have

$$\begin{aligned}\psi\big(d(x_{n+1}, y_{n+2})\big) &= \psi\big(d(Tx_n, Ty_{n+1})\big)\\ &\leq \alpha(x_n, y_{n+1})\psi\big(d(Tx_n, Ty_{n+1})\big)\\ &\leq \theta(\psi\big(d(x_n, y_{n+1})\big))\varphi(\psi\big(d(x_n, y_{n+1})\big))\end{aligned}$$

$$\leq \varphi(\psi(d(x_n, y_{n+1})))$$

$$< \psi(d(x_n, y_{n+1})) . \tag{3.5}$$

Hence, ψ is strictly increasing function so, we get

$$d(x_{n+1}, y_{n+2}) < d(x_n, y_{n+1}) \text{ for all } n \geq 0. \tag{3.6}$$

Similarly, putting $x = x_{n+1}$ and $y = y_{n+1}$ in equation (3.3), using equations (3.1), (3.2) and by the properties of ψ and θ, we have the following

$$\begin{aligned}\psi(d(x_{n+1}, y_{n+1})) &= \psi(d(Tx_n, Ty_n)) \\ &\leq \alpha(x_n, y_n)\psi(d(Tx_n, Ty_n)) \\ &\leq \theta(\psi(d(x_n, y_n)))\varphi(\psi(d(x_n, y_n))) \\ &\leq \varphi(\psi(d(x_n, y_n))) \\ &< \psi(d(x_n, y_n)) .\end{aligned} \tag{3.7}$$

Hence, ψ is strictly increasing function so, we obtain

$$d(x_{n+1}, y_{n+1}) < d(x_n, y_n) \text{ for all } n \geq 0. \tag{3.8}$$

From the above, we conclude that the sequences $\{d(x_n, y_{n+1})\}$ and $\{d(x_n, y_n)\}$ are monotonically decreasing and for the non-negative monotonically decreasing sequences $\{d(x_n, y_{n+1})\}$ and $\{d(x_n, y_n)\}$, there exist some $r_1 \geq 0$ and $r_2 \geq 0$, such that

$$d(x_n, y_{n+1}) \to r_1, d(x_n, y_n) \to r_2 \text{ as } n \to \infty . \tag{3.9}$$

We suppose on the contrary that $r_1 > 0$.

Hence, we have $0 < r_1 < d(x_n, y_{n+1})$ for all $n \geq 0$. Set $\varepsilon = r_1$.

From equation (3.2), there exist $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta \Rightarrow \varphi(t) \leq \varepsilon$.

On the other hand, by the definition of ε, we can choose $n_0 \in \mathbb{N}$ such that

$$\varepsilon < d(x_{n_0}, y_{n_0+1}) < \varepsilon + \delta$$

By the properties of ψ, θ,using equations (3.1), (3.2) and (3.3), we have

$$\psi(\varepsilon) < \psi(d(x_{n_0}, y_{n_0+1})) < \psi(\varepsilon + \delta) = \psi(\varepsilon) + \psi(\delta)$$

this implies that

$$\varphi(\psi(d(x_{n_0}, y_{n_0+1}))) \leq \psi(\varepsilon). \tag{3.10}$$

We have also

$$\varepsilon < d(x_{n_0+2}, y_{n_0+3}) < d(x_{n_0+1}, y_{n_0+2}) = d(Tx_{n_0}, Ty_{n_0+1}),$$

which implies that

$$\psi(\varepsilon) < \psi(d(x_{n_0+2}, y_{n_0+3}))$$

$$< \psi(d(x_{n_0+1}, y_{n_0+2})) = \psi(d(Tx_{n_0}, Ty_{n_0+1}))$$

$$\leq \alpha(x_{n_0}, y_{n_0+1})\, \psi(d(Tx_{n_0}, Ty_{n_0+1}))$$

$$\leq \theta(\psi\left(d(x_{n_0}, y_{n_0+1})\right))\, \varphi(\psi\left(d(x_{n_0}, y_{n_0+1})\right))$$

$$< \varphi(\psi\left(d(x_{n_0}, y_{n_0+1})\right))$$

$$\leq \psi(\varepsilon),$$

which is a contradiction. Hence

$$\lim_{n\to\infty} d(x_n, y_{n+1}) = r_1 = 0. \tag{3.11}$$

Similarly, $\lim_{n\to\infty} d(x_n, y_n) = r_2 = 0.$ (3.12)

Now, we shall prove that $\{(x_n, y_n)\}$ is a Cauchy bisequence.

We fix $\varepsilon_1 > 0$, then by (3.2) there exists $\delta_1 > 0$ such that

$$t < \varepsilon_1 + \delta_1 \Rightarrow \varphi(t) \leq \varepsilon_1. \tag{3.13}$$

Without loss of generality, we assume $\delta_1 < \varepsilon_1$. Due to (3.11), there exist $n_0 \in \mathbb{N}$ such that

$$d(x_n, y_{n+1}) < \delta_1, \text{ for all } n \geq n_0, \tag{3.14}$$

which implies that

$$\psi(d(x_n, y_{n+1})) < \psi(\delta_1).$$

By mathematical induction, we show that for any fixed $k \geq n_0$

$$d(x_k, y_{k+l}) < \varepsilon_1 + \delta_1, \text{ for all } l \in \mathbb{N}. \tag{3.15}$$

For $l = 1$, this inequality trivially holds by (3.14).

Now, assume that (3.15) is satisfied for some $j \in \mathbb{N}$ and we have to show that it holds for $l = j + 1$.

From the triangle inequality (BP3), properties of ψ, θ, equations (3.1), (3.2) and (3.3) $\psi(d(x_k, y_{k+j+1})) \leq \psi(d(x_k, y_{k+1}) + d(x_{k+1}, y_{k+1}) + d(x_{k+1}, y_{k+j+1}))$

$$\leq \psi(d(x_k, y_{k+1})) + \psi(d(x_{k+1}, y_{k+1})) + \psi(d(x_{k+1}, y_{k+j+1}))$$

$$\leq \psi(d(x_k, y_{k+1})) + \psi(d(x_{k+1}, y_{k+1})) + \psi(d(Tx_k, Ty_{k+j}))$$

$$\leq \psi(d(x_k, y_{k+1})) + \psi(d(x_{k+1}, y_{k+1}))$$

$$+\alpha(x_k, y_{k+j})\psi(d(Tx_k, Ty_{k+j})) \quad \leq \psi(d(x_k, y_{k+1})) + \psi(d(x_{k+1}, y_{k+1}))$$

$$+\theta(\psi(d(x_k, y_{k+j})))\varphi(\psi(d(x_k, y_{k+j})))$$

$$< \psi(d(x_k, y_{k+1})) + \psi(d(x_{k+1}, y_{k+1})) + \varphi(\psi(d(x_k, y_{k+j}))).$$

Using equations (3.14) and (3.15), we get

$$\psi(d(x_k, y_{k+j+1})) \leq \psi(\delta_1) + \psi(d(x_{k+1}, y_{k+1})) + \psi(\varepsilon_1).$$

By letting $k \to \infty$, using equation (3.12), we obtain

$$\begin{aligned}\psi(d(x_k, y_{k+j+1})) &\leq \psi(\delta_1) + \psi(\varepsilon_1)\\ &= \psi(\varepsilon_1 + \delta_1).\end{aligned}$$

By the property of ψ , we get

$$d(x_k, y_{k+j+1}) < \varepsilon_1 + \delta_1.$$

So, equation (3.15) is holds for $l = j + 1$.

Hence, by induction we prove that $d(x_k, y_{k+j+1}) < \varepsilon_1 + \delta_1$ for all $k \geq n_0$ and $l \geq 1$.

Since ε_1 is arbitrary, we conclude that

$$\lim_{m,n\to\infty} d(x_n, y_m) = 0.$$

Hence, $\{(x_n, y_n)\}$ is a Cauchy bisequence and (X, Y, d) is a complete bipolar metric space. So, $\{(x_n, y_n)\}$ is convergent and in fact biconvergent. So, there exists $u \in X \cap Y$ such that $(x_n) \to u$, $(y_n) \to u$ as $n \to \infty$.

We claim that $Tu = u$.

Let, if possible, $Tu \neq u$. Then there exist $r' > 0$ such that $d(u, Tu) = r' > 0$.

Since $(x_n) \to u$, $(y_n) \to u$ as $n \to \infty$, we can choose $n_0 \in \mathbb{N}$ such that

$d(x_n, u) < \frac{r'}{2}$, for all $n \geq n_0$ and $d(y_n, u) < \frac{r'}{2}$, for all $n \geq n_0$. (3.16)

From the triangle inequality (BP3), properties of ψ, θ, equations (3.1), (3.2) and (3.3),

$$\begin{aligned}\psi(r') &= \psi(d(u,Tu))\\ &\leq \psi(d(u, y_{n+1}) + d(x_{n+1}, y_{n+1}) + d(x_{n+1}, Tu))\\ &\leq \psi(d(u, y_{n+1})) + \psi(d(x_{n+1}, y_{n+1})) + \psi(d(x_{n+1}, Tu))\\ &\leq \psi(d(u, y_{n+1})) + \psi(d(x_{n+1}, y_{n+1})) + \psi(d(Tx_n, Tu))\\ &\leq \psi(d(u, y_{n+1})) + \psi(d(x_{n+1}, y_{n+1})) + \alpha(x_n, u)\, \psi(d(Tx_n, Tu))\\ &\leq \psi(d(u, y_{n+1})) + \psi(d(x_{n+1}, y_{n+1})) + \theta(\psi((x_n, u))\, \varphi(\psi(d(x_n, u)))\\ &< \psi(d(u, y_{n+1})) + \psi(d(x_{n+1}, y_{n+1})) + \varphi(\psi(d(x_n, u)))\\ &< \psi(d(u, y_{n+1})) + \psi(d(x_{n+1}, y_{n+1})) + \psi(d(x_n, u)).\end{aligned}$$

Using equations (3.12) and (3.16), we get

$$\psi(r') < \psi\left(\frac{r'}{2}\right) + \psi\left(\frac{r'}{2}\right)$$

$$= \psi\left(\frac{r'}{2}+\frac{r'}{2}\right) = \psi(r'),$$

which is a contradiction.

Thus $Tu = u$. i.e., u is the fixed point of T.

Example 3.5. Let $X = [0, +\infty)$ and $Y = [-1,1]$ and let $d : X \times Y \to [0, +\infty)$ be a function such that $d(x,y) = |x^2 - y^2|$ for all $(x,y) \in X \times Y$.

Then, clearly (X, Y, d) be a complete bipolar metric space.

Define $T : (X,Y) \rightrightarrows (X,Y)$ such that $Tx = \frac{x}{2}$ is a mapping and $\alpha : X \times Y \to [0,\infty)$ such that $\alpha(x,y) = \frac{3}{2}$ for $(x,y) \in X \times Y$.

Clearly, T is α-admissible mapping and there exist $(x_0, y_0) \in X \times Y$ such that $\alpha(x_0, Ty_0) \geq 1$, $X \cap Y = \{0\}$ and $T0 = 0$.

Taking $\psi(t) = \frac{t}{4}$, $\varphi(t) = \frac{t}{2}$ and $\theta(t) = \frac{3}{4}$.

Left hand side of equation (3.3) becomes

$\alpha(x,y)\, \psi\big(d(Tx,Ty)\big) = \frac{3}{2}\frac{|x^2-y^2|}{16}$.

Right hand side of equation (3.3) becomes

$\theta(\, \psi(d(x,y)))\, \varphi(\psi\big(d(x,y))\big) = \frac{3}{2}\frac{|x^2-y^2|}{16}$, for all $(x,y) \in X \times Y$,

which implies equation (3.3) holds.

Hence, T is an (α, ψ, φ)- Geraghty contraction mapping.

All the conditions of Theorem 3.4. are satisfied.

So, T has a fixed point and $x = 0$ is the fixed point of T.

Theorem 3.6. Let (X, Y, d) be a complete bipolar metric space, $T : (X,Y) \rightrightarrows (X,Y)$ is a covariant mapping and $\alpha : X \times Y \to [0, \infty)$. Suppose that the following conditions hold:

(i)T is α-admissible mapping,
(ii)T is an (α, ψ, φ)- Geraghty contraction mapping,
(iii)there exist $x_0 \in X$and $y_0 \in Y$ such that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, Ty_0) \geq 1$.

Then T has a unique fixed point.

Proof: Following the proof of Theorem 3.4. T has fixed point. To prove the uniqueness of fixed point of covariant mapping T in complete bipolar metric space, let us assume, if possible, u and v are two distinct fixed point of T. i.e. $Tu = u$ and $Tv = v$.

By using the properties of ψ, θ, equations (3.1), (3.2) and (3.3),

$$\psi\big(d(u,v)\big) = \psi\big(d(Tu,Tv)\big)$$

$$
\begin{aligned}
&\leq \alpha(u,v)\psi\big(d(Tu,Tv)\big)\\
&\leq \theta(\psi\big(d(u,v)\big))\varphi(\psi\big(d(u,v)\big))\\
&\leq \ \varphi(\psi\big(d(u,v)\big))\\
&< \ \psi\big(d(u,v)\big).
\end{aligned}
$$

This implies that $d(u,v) < d(u,v)$, which is a contradiction. Thus, u is the unique fixed point of T.

Example 3.7. In the Example 3.5, we can easily say that T satisfies all the conditions of Theorem 3.6. So, T has a unique fixed point.

Clearly, ‘0’ is unique fixed point of T.

Definition 3.8. Let X and Y be two non-empty sets. Consider (X, Y, d) be a bipolar metric space, a contravariant mapping $T : (X,Y) \rightleftarrows (X,Y)$ is called Geraghty contraction if there exist a function $\theta \in \Theta$ which satisfies the following condition:

$$d(Ty,Tx) \leq \theta\big(\, d(x,y)\big)\, d(x,y) \text{ for all } x \in X \text{ and } y \in Y.$$

Definition 3.9. Let (X, Y, d) be a bipolar metric space and $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant mapping where X and Y are two non-empty sets. Suppose that $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is function and $\theta \in \Theta$ and T is called φ- Geraghty contraction if it satisfies the following condition:

(i)$\varphi(t) < t$ for any $t \in (0,\infty)$,
(ii)For any $\varepsilon > 0$, there exist $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta \Rightarrow \varphi(t) \leq \varepsilon$,
(iii)$d(Ty,Tx) \leq \theta\big(\, d(x,y)\big)\, \varphi(d(x,y))$ for all $x \in X$ and $y \in Y$.

Definition 3.10. Let X and Y be two non-empty sets and (X, Y, d) be a bipolar metric space, $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant self map and $\alpha : X \times Y \to [0,\infty)$. A mapping T is said to be (α, ψ, φ)- Geraghty contraction mapping if there exist $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi \in \Psi$ and $\theta \in \Theta$ satisfies the following condition:

(i)$\varphi(t) < t$ for any $t \in (0,\infty)$,
(ii)For any $\varepsilon > 0$, there exist $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta \Rightarrow \varphi(t) \leq \varepsilon$,
(iii)$\alpha(x,y)\psi(d(Ty,Tx)) \leq \theta\big(\psi(\, d(x,y))\big)\, \varphi(\psi(d(x,y)))$,$\forall\, x \in X$ and $y \in Y$ (3.17)

Theorem 3.11. Let (X, Y, d) be a complete bipolar metric space, $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant mapping and $\alpha : X \times Y \to [0,\infty)$. Suppose that the following conditions hold:

(i)T is α- admissible mapping,
(ii)T is an (α, ψ, φ)- Geraghty contraction mapping,
(iii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Then T has fixed point.

Proof: Let $x_0 \in X$and $y_0 \in Y$ such that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. Now we define a bisequence $\{(x_n, y_n)\}$ in (X, Y) by $Tx_n = y_n$ and $Ty_n = x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is α-admissible mapping.

So, $\alpha(x_0, y_0) = \alpha(x_0, Tx_0) \geq 1,$

$\alpha(x_1, y_0) = \alpha(Ty_0, Tx_0) \geq 1,$

$\alpha(x_1, y_1) = \alpha(x_1, Tx_1) \geq 1.$

Using mathematical induction, we get

$\alpha(x_{n+1}, y_n) \geq 1$ and $\alpha(x_n, y_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. (3.18)

Putting $x = x_{n+1}$ and $y = y_n$ in equation (3.17), using equations (3.1), (3.2) and by the properties of ψ and θ, we have the following

$$\begin{aligned}\psi\big(d(x_{n+1}, y_n)\big) &= \psi\big(d(Ty_n, Tx_n)\big)\\ &\leq \alpha(x_n, y_n)\psi\big(d(Ty_n, Tx_n)\big)\\ &\leq \theta(\psi\big(d(x_n, y_n)\big))\varphi(\psi\big(d(x_n, y_n)\big))\\ &\leq \varphi(\psi\big(d(x_n, y_n)\big))\end{aligned}$$

$$\psi\big(d(x_{n+1}, y_n)\big) < \psi\big(d(x_n, y_n)\big)\,. \tag{3.19}$$

Hence, ψ is strictly increasing function so, we get

$$d(x_{n+1}, y_n) < d(x_n, y_n) \text{ for all } n \geq 0. \tag{3.20}$$

Similarly, putting $x = x_{n+1}$ and $y = y_{n+1}$ in equation (3.17), using equations (3.1), (3.2) and by the properties of ψ and θ, we have the following

$$\begin{aligned}\psi\big(d(x_{n+1}, y_{n+1})\big) &= \psi\big(d(Ty_n, Tx_{n+1})\big)\\ &\leq \alpha(x_{n+1}, y_n)\psi\big(d(Ty_n, Tx_{n+1})\big)\\ &\leq \theta(\psi\big(d(x_{n+1}, y_n)\big))\varphi(\psi\big(d(x_{n+1}, y_n)\big))\\ &\leq \varphi(\psi\big(d(x_{n+1}, y_n)\big))\\ &< \psi\big(d(x_{n+1}, y_n)\big)\,. \end{aligned} \tag{3.21}$$

Hence, ψ is strictly increasing function so, we obtain

$$d(x_{n+1}, y_{n+1}) < d(x_{n+1}, y_n) \text{ for all } n \geq 0. \tag{3.22}$$

From the above, we conclude that the sequences $\{d(x_{n+1}, y_n)\}$ and $\{d(x_{n+1}, y_{n+1})\}$ are monotonically decreasing and for the non-negative monotonically decreasing sequences $\{d(x_{n+1}, y_n)\}$ and $\{d(x_{n+1}, y_{n+1})\}$, there exist some $r_1 \geq 0$ and $r_2 \geq 0$, such that

$$d(x_{n+1}, y_n) \to r_1, d(x_{n+1}, y_{n+1}) \to r_2 \text{ as } n \to \infty. \tag{3.23}$$

We suppose on the contrary that $r_1 > 0$.

Hence, we have $0 < r_1 < d(x_{n+1}, y_n)$ for all $n \geq 0$. Set $\varepsilon = r_1$.

From equation (3.2), there exist $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta \Rightarrow \varphi(t) \leq \varepsilon$.

On the other hand, by the definition of ε, we can choose $n_0 \in \mathbb{N}$ such that

$$\varepsilon < d(x_{n_0+1}, y_{n_0}) < \varepsilon + \delta.$$

By the properties of ψ, θ,using equations (3.1), (3.2) and (3.17), we have

$$\psi(\varepsilon) < \psi(d(x_{n_0+1}, y_{n_0})) < \psi(\varepsilon + \delta) = \psi(\varepsilon) + \psi(\delta).$$

This implies that

$$\varphi(\psi(d(x_{n_0+1}, y_{n_0}))) \leq \psi(\varepsilon). \tag{3.24}$$

We have also

$$\varepsilon < d(x_{n_0+2}, y_{n_0+1}) < d(x_{n_0+1}, y_{n_0+1}) = d(Ty_{n_0}, Tx_{n_0+1}),$$

which implies that

$$\begin{aligned}\psi(\varepsilon) &< \psi(d(x_{n_0+2}, y_{n_0+1}))\\ &< \psi(d(x_{n_0+1}, y_{n_0+1})) = \psi(d(Ty_{n_0}, Tx_{n_0+1}))\\ &\leq \alpha(x_{n_0+1}, y_{n_0})\, \psi(d(Ty_{n_0}, Tx_{n_0+1}))\\ &\leq \theta(\psi\left(d(x_{n_0+1}, y_{n_0})\right))\, \varphi(\psi\left(d(x_{n_0+1}, y_{n_0})\right))\\ &< \varphi(\psi\left(d(x_{n_0+1}, y_{n_0})\right))\\ &\leq \psi(\varepsilon),\end{aligned}$$

which is a contradiction. Hence

$$\lim_{n\to\infty} d(x_{n+1}, y_n) = r_1 = 0. \tag{3.25}$$

Similarly, $$\lim_{n\to\infty} d(x_n, y_n) = r_2 = 0. \tag{3.26}$$

Now, we shall prove that $\{(x_n, y_n)\}$ is a Cauchy bisequence.

We fix $\varepsilon_1 > 0$, then by (3.2) there exists $\delta_1 > 0$ such that

$$t < \varepsilon_1 + \delta_1 \Rightarrow \varphi(t) \leq \varepsilon_1. \tag{3.27}$$

Without loss of generality, we assume $\delta_1 < \varepsilon_1$. Due to (3.25), there exist $n_0 \in \mathbb{N}$ such that

$$d(x_{n+1}, y_n) < \delta_1, \text{ for all } n \geq n_0, \tag{3.28}$$

which implies that

$$\psi(d(x_{n+1}, y_n)) < \psi(\delta_1).$$

By mathematical induction, we show that for any fixed $k \geq n_0$

$$d(x_{k+l}, y_k) < \varepsilon_1 + \delta_1, \text{ for all } l \in \mathbb{N}. \tag{3.29}$$

For $l = 1$, this inequality trivially holds by (3.28).

Let us assume that (3.29) is satisfied for some $j \in \mathbb{N}$ and we will show that it holds for $l = j + 1$.

From the triangle inequality (BP3), properties of ψ ,θ, equations (3.1), (3.2) and (3.3)

$$\begin{aligned}
\psi(\, d(x_{k+j+1}, y_k)) &\leq \psi(\, d(x_{k+j+1}, y_{k+1}) + \; d(x_{k+1}, y_{k+1}) + \; d(x_{k+1}, y_k)) \\
&\leq \psi(\, d(x_{k+j+1}, y_{k+1})) + \psi(\, d(x_{k+1}, y_{k+1})) + \psi(\, d(x_{k+1}, y_k)) \\
&\leq \psi(\, d(Ty_{k+j}, Tx_{k+1})) + \psi(\, d(x_{k+1}, y_{k+1})) + \psi(\, d(x_{k+1}, y_k)) \\
&\leq \psi(\, d(x_{k+1}, y_k)) + \psi(\, d(x_{k+1}, y_{k+1})) \\
&\quad + \alpha(x_{k+1}, y_{k+j})\psi(\, d(Ty_{k+j}, Tx_k)) \\
&\leq \psi(\, d(x_{k+1}, y_k)) + \psi(\, d(x_{k+1}, y_{k+1})) \\
&\quad + \theta(\psi(d(x_k, y_{k+j})))\varphi(\psi(\, d(x_k, y_{k+j}))) \\
&< \psi(\, d(x_{k+1}, y_k)) + \psi(\, d(x_{k+1}, y_{k+1})) + \varphi(\psi(\, d(x_k, y_{k+j}))).
\end{aligned}$$

Using equations (3.28) and (3.29), we get

$$\psi(\, d(x_{k+j+1}, y_k)) \leq \psi(\, \delta_1) + \psi(\, d(x_{k+1}, y_{k+1})) + \psi(\, \varepsilon_1).$$

By letting $k \to \infty$, using equation (3.26), we obtain

$$\begin{aligned}
\psi(\, d(x_{k+j+1}, y_k)) &\leq \psi(\, \delta_1) + \psi(\varepsilon_1), \\
&= \psi(\varepsilon_1 + \delta_1).
\end{aligned}$$

By the property of ψ, we get

$$d(x_{k+j+1}, y_k) < \varepsilon_1 + \delta_1.$$

So, equation (3.29) is holds for $l = j + 1$.

Hence, by induction we prove that $d(x_{k+j+1}, y_k) < \varepsilon_1 + \delta_1$ for all $k \geq n_0$ and $l \geq 1$.

Since ε_1 is arbitrary, we conclude that

$$\lim_{m,n\to\infty} d(x_n, y_m) = 0.$$

Hence, $\{(x_n, y_n)\}$ is a Cauchy bisequence and (X, Y, d) is a complete bipolar metric space. So, $\{(x_n, y_n)\}$ is convergent and in fact biconvergent. So, there exists $u \in X \cap Y$such that $(x_n) \to u$, $(y_n) \to u$ as $n \to \infty$.

We claim that $Tu = u$.

Let, if possible, $Tu \neq u$. Then there exist $r' > 0$ such that $d(Tu, u) = r' > 0$.

Since $(x_n) \to u$, $(y_n) \to u$ as $n \to \infty$, we can choose $n_0 \in \mathbb{N}$ such that

$d(x_n, u) < \frac{r'}{2}$, for all $n \geq n_0$ and $d(y_n, u) < \frac{r'}{2}$, for all $n \geq n_0$. (3.30)

From the triangle inequality (BP3), properties of ψ, θ, equations (3.1), (3.2) and (3.17)

$$
\begin{aligned}
\psi(r') &= \psi(d(Tu,u)),\\
&\le \psi(d(Tu,y_n) + d(x_{n+1},y_n) + d(x_{n+1},u))\\
&\le \psi(d(Tu,y_n)) + \psi(d(x_{n+1},y_n)) + \psi(d(x_{n+1},u))\\
&\le \psi(d(Tu,Tx_n)) + \psi(d(x_{n+1},y_{n+1})) + \psi(d(x_{n+1},u))\\
&\le \alpha(x_n,u)\,\psi(d(Tu,Tx_n)) + \psi(d(x_{n+1},y_{n+1})) + \psi(d(x_{n+1},u))\\
&\le \theta(\psi((x_n,u))\,\varphi(\psi(d(x_n,u))) + \psi(d(x_{n+1},y_{n+1})) + \psi(d(x_{n+1},u))\\
&< \varphi(\psi(d(x_n,u))) + \psi(d(x_{n+1},y_{n+1})) + \psi(d(x_{n+1},u))\\
&< \psi(d(x_n,u)) + \psi(d(x_{n+1},y_{n+1})) + \psi(d(x_{n+1},u)).
\end{aligned}
$$

Using equations (3.26) and (3.30), we get

$$
\begin{aligned}
\psi(r') &< \psi\left(\frac{r'}{2}\right) + \psi\left(\frac{r'}{2}\right)\\
&= \psi\left(\frac{r'}{2} + \frac{r'}{2}\right) = \psi(r'),
\end{aligned}
$$

which is a contradiction.

Thus $Tu = u$. i.e., u is the fixed point of T.

Example 3.12. Let $X = [0, +\infty)$ and $Y = [-1, 1]$ and let $d : X \times Y \to [0, +\infty)$ be a function such that $d(x,y) = |x^2 - y^2|$ for all $(x,y) \in X \times Y$.

Then, clearly (X, Y, d) be a complete bipolar metric space.

Define $T : (X,Y)$ ⤨ (X,Y) such that $Tx = \frac{-x}{2}$ is a mapping and $\alpha : X \times Y \to [0, \infty)$ such that $\alpha(x,y) = \frac{3}{2}$ for $(x,y) \in X \times Y$.

Clearly, T is α-admissible mapping and there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $X \cap Y = \{0\}$ and $T0 = 0$.

Taking $\psi(t) = \frac{t}{4}$, $\varphi(t) = \frac{t}{2}$ and $\theta(t) = \frac{3}{4}$.

Left hand side of equation (3.17) becomes

$$\alpha(x,y)\,\psi\big(d(Ty,Tx)\big) = \frac{3}{2}\frac{|x^2-y^2|}{16}.$$

Right hand side becomes

$$\theta(\psi(d(x,y)))\,\varphi(\psi\big(d(x,y)\big)) = \frac{3}{2}\frac{|x^2-y^2|}{16}, \text{ for all } (x,y) \in X \times Y,$$

which implies equation (3.17) holds.

Hence, T is an (α, ψ, φ)- Geraghty contraction mapping.

All the conditions of Theorem 3.11. are satisfied. So, T has a fixed point and $x = 0$ is the fixed point of T.

Theorem 3.13. Let (X, Y, d) be a complete bipolar metric space, $T : (X, Y) ⤨ (X, Y)$ is a contravariant mapping and $\alpha : X \times Y \to [0, \infty)$. Suppose that the following conditions hold:

(i) T is α- admissible mapping,
(ii) T is an (α, ψ, φ)- Geraghty contraction mapping,
(iii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Then T has a unique fixed point.

Proof: Following the proof of Theorem 3.13. T has fixed point. To prove the uniqueness of fixed point of contravariant mapping T in complete bipolar metric space, let if possible, u and v are two distinct fixed point of T. i.e., $Tu = u$ and $Tv = v$.

By using the properties of ψ ,θ, equations (3.1), (3.2) and (3.17),

$$\begin{aligned} \psi\big(d(u,v)\big) &= \psi\big(d(Tu,Tv)\big) \\ &\leq \alpha(u,v)\psi\big(d(Tu,Tv)\big) \\ &\leq \theta(\psi\big(d(u,v)\big))\varphi(\psi\big(d(u,v)\big)) \\ &\leq \varphi(\psi\big(d(u,v)\big)) \\ &< \psi\big(d(u,v)\big). \end{aligned}$$

This implies that $d(u,v) < d(u,v)$, which is a contradiction.

Hence, T has a unique fixed point.

Example 3.14. In the Example 3.12, we can easily say that T satisfies all the conditions of Theorem 3.13. So, T has a unique fixed point.

Clearly, '0' is unique fixed point of T.

References

[1] Abduletif M., Koyas K. and Gebregiorgis S., "Fixed point results for generalized (α, ψ, φ)- Geraghty contraction in b-metric spaces", *Int. J. Nonlinear Anal. Appl.*, **14**(1) (2023), 965-977.

[2] Banach S., "Sur les opérations dans les ensembles abstraits et leur application aux équations integrals", *Fundam. Math.*, **3**(1) (1922), 133-181.

[3] Border K.C., "Fixed Point Theorems with Applications to Economics and Game Theory", *Cambridge University Press*, Cambridge (1990).

[4] Chandok S., "Some fixed point theorems for $(\alpha, \beta) -$ admissible Geraghty type contractive mappings and related results", *Math. Sci.*, **9** (2015), 127-135.

[5] Gaba Y.U., Aphane A. and Aydi H., "Contractions in Bipolar Metric Spaces", *J. Math.*, **2021** (2021), 5562651.

[6] Karapinar E., "$\alpha - \psi$-Geraghty contraction type mappings and some related fixed point results", *Filomat*, **28**(1) (2014), 37-48.

[7] Karapinar E., Alqahtani and Fulga A., "On ciric type- ψ-geraghty contractions", *Thai J. Math.*, **17**(1) (2019), 205-216.

[8] Geraghty M., "On contractive mappings", *Proc. Am. Math. Soc.*, **40** (1973), 604-608.

[9] Mani G., Ramaswamy R., Gnanaprakasam A.J., Stojilkovic S., Fadail Z.M. and Radenovi´c S., “Application of fixed point results in the setting of F-contraction and simulation function in the setting of bipolar metric space”, *AIMS Math.*, **8** (2023), 3269–3285.

[10] Mutlu A. and Gürdal U., “Bipolar metric spaces and some fixed point theorems”, *J. Nonlinear Sci. Appl.*, **9**(9) (2016), 5362-5373.

[11] Mutlu A., Gürdal U. and Ozkan K., “Fixed point results for $\alpha - \psi -$contractive mappings in bipolar metric spaces”, *J. Inequalities Spec. Funct.*, **11**(1) (2020), 64-75.

[12] Ramaswamy R., Mani G., Gnanaprakasam A.J., Abdelnaby O.A.A., Stojiljkovic V., Radojevic S. and Radenovic S., “Fixed Points on Covariant and Contravariant Maps with an Application”, *Mathematics*, **10** (2022), 4385.

[13] Rao B.S., Kishore G.N.V. and Kumar G.K., “Geraghty type contraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy”, *Int. J. Math. Trends Technol.*, **63** (2018), 25–34.

[14] Samet B., Vetro C. and Vetro P., “Fixed point theorems for $\alpha - \psi -$ contractive type mappings”, *Nonlinear Anal. Theory Methods Appl.*, **75**(4) (2012), 2154-2165.

[15] Shahi P., Kaur J. and Bhatia S S, “Fixed point theorems for $(\xi, \alpha) -$ expansive mappings in complete metric space”, *Fixed Point Theory Appl.*, **2012** (2012), 157.

[16] Zeidler E., “Nonlinear Functinal Analysis and its Applications*”, Springer New York*, (1989).