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Some Fixed Point Theorems Using Φ_p Operator

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Abstract:

This article's goal is to use the Φp operator to prove a few fixed point theorems in complete metric space. Furthermore, we investigate whether fixed points for self mappings that meet the requirements of rational expression exist and are unique in a complete metric space. Our findings expand upon and generalize a great deal of previously published research.

Keywords: Rational Expression, Complete Metric Spaces, Fixed Point, Φp operator, Self mapping

1. Introduction

In 1889, H. Poincare the French mathematician, introduced the fixed points in different version The origin of fixed point theory, in the 19th century was notorious by mathematicians like Cauchy, Fredholm, Caristi, Liouville, Lipschitz, Peano and Picard. Banach's contribution to metric fixed point theory was not recognized until F. Brouwer's work and contribution to the development of the non-linear functional analysis as an active and vital branch of mathematics. In this paper, we investigate some Fixed point theorems of Complete Metric Sapces using Φ_p operator.

Definition 1.1: Let X be a none-empty set, a function $d: X \times X \to R$ is called a metric on X, if it satisfies the following conditions,

(i) $d(\varpi, \zeta) \geq 0$ and $d(\varpi, \zeta) = 0$ if and only if $\varpi = \zeta, \forall \varpi, \zeta \in X$

(ii) $d(\varpi, \zeta) = d(\zeta, \varpi) \forall\ \varpi, \zeta \in X$

(iii) $d(\varpi, \zeta) \leq d(\varpi, z) + d(z, \zeta)\ \forall\ \varpi, \zeta, z \in X$

Then (X, d) is called metric space.

Definition 1.2: A sequence $\{\varpi_n\}$ is said to be a Cauchy sequence if give $\varepsilon > 0$, there exists a positive integer m such that $|\varpi_n - \varpi_m| < \varepsilon$ whenever $n \geq m$.

Definition 1.3: A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X

Definition 1.4: Let (X, d) be a complete metric space $\Phi_p: X \to X$ is an increasing and positive mapping.

If $X = R$, then $\Phi_p: R \to R$ is a p-Lapalcian operator,

$\Phi_p(\varpi) = |\varpi|^{p-2}\varpi$ for some $p > 1$

Lemma 1.5: Show that the operator $\Phi_p: X \to X$ holds the following properties

(i) If $\varpi \leq \zeta$ then, $\Phi_p(\varpi) \leq \Phi_p(\zeta)\ \ \forall \varpi, \zeta \in X$

(ii) Φ_p is continuous bijection and its inverse mapping is also continuous.

That is Φ_p is homeomorphishm

(iii) $\Phi_p(\varpi\zeta) = \Phi_p(\varpi)\Phi_p(\zeta)\ \forall\, \varpi, \zeta \in X$

(iv) $\Phi_p\,(\varpi + \zeta) \leq \Phi_p(\varpi) + \Phi_p(\zeta)\ \ \forall\, \varpi, \zeta \in X$

2. Main result

Theorem 2.1: Let T be continuous self map, defined on a complete metric space X and $\Phi_p: X \to X$ Further T satisfies the following conditions

$$\Phi_p\big(d(T\varpi, T\zeta)\big) \leq \lambda_1 \Phi_p\big(d(\varpi, \zeta)\big) + \lambda_2 \Phi_p[d(T\varpi, \varpi) + d(T\zeta, \zeta)]$$

$$+\lambda_3 \Phi_p[d(T\zeta, \varpi) + d(T\varpi, \zeta)] + \lambda_4 \Phi_p\left[\frac{d(\varpi, T\varpi)d(\zeta, T\zeta)}{d(\varpi, \zeta)}\right]$$

$$+\lambda_5 \Phi_p\left[\frac{d(\varpi, T\zeta)d(\zeta, T\varpi)}{d(\varpi, \zeta)}\right]$$

$$+\lambda_6 \Phi_p\left[\frac{d(\varpi, T\varpi)d(\zeta, T\zeta) + d(\varpi, T\zeta)d(\zeta, T\varpi)}{d(\varpi, \zeta)}\right] + \lambda_7 \Phi_p\left[\frac{d(\varpi, T\varpi)d(\zeta, T\zeta) + d(\varpi, T\zeta)d(\zeta, T\varpi)}{d(\varpi, \zeta)}\right]$$

$$+\lambda_8 \Phi_p\left[\frac{d(\varpi, T\zeta)[d(\varpi, T\varpi) + d(\zeta, T\zeta) + d(\varpi, T\zeta) + d(\zeta, T\varpi)]}{d(\varpi, T\varpi) + d(\zeta, T\zeta) + d(\varpi, T\zeta) + d(\zeta, T\varpi)}\right]$$

For all $\varpi, \zeta \in X, \varpi \neq \zeta$ and

$\Phi_p(\lambda_1) + 2\Phi_p(\lambda_2) + 2\Phi_p(\lambda_3) + \Phi_p(\lambda_4) + \Phi_p(\lambda_6) + \Phi_p(\lambda_7) < 1$ then T has unique fixed point in T.

Proof:

Let ϖ_0 be an arbitrary point in X, and we define a sequence $\{\varpi_n\}$ by means of iterates of T.

By setting $T^n\varpi_0 = \varpi_n$, where n is a positive integers. If $\varpi_n = \varpi_{n+1}$, for some n, then we have $T\varpi_n = \varpi_n$, then ϖ_n is a fixed point of T taking $\varpi_n \neq \varpi_{n+1}$ for all n

$$\Phi_p\big(d(\varpi_{n+1}, \varpi_n)\big) = \Phi_p\big(d(T\varpi_n, T\varpi_{n-1})\big)$$

$$\Phi_p\big(d(T\varpi_n, T\varpi_{n-1})\big) \leq \lambda_1 \Phi_p(d(\varpi_n, \varpi_{n-1})) + \lambda_2 \Phi_p([d(T\varpi_n, \varpi_n) + d(T\varpi_{n-1}, \varpi_{n-1})])$$

$$+\lambda_3\Phi_p([d(T\varpi_{n-1},\varpi_n)+d(T\varpi_n,\varpi_{n-1})])$$

$$+\lambda_4\Phi_p\left(\left[\frac{d(\varpi_n,T\varpi_n)d(\varpi_{n-1},T\varpi_{n-1})}{d(\varpi_n,\varpi_{n-1})}\right]\right)$$

$$+\lambda_5\Phi_p\left(\left[\frac{d(\varpi_n,T\varpi_{n-1})d(\varpi_{n-1},T\varpi_n)}{d(\varpi_n,\varpi_{n-1})}\right]\right)$$

$$+\lambda_6\Phi_p\left(\left[\frac{d(\varpi_n,T\varpi_n)d(\varpi_{n-1},T\varpi_{n-1})+d(\varpi_n,T\varpi_{n-1})d(\varpi_{n-1},T\varpi_n)}{d(\varpi_n,\varpi_{n-1})}\right]\right)$$

$$+\lambda_7\Phi_p\left(\left[\frac{d(\varpi_n,T\varpi_n)d(\varpi_{n-1},T\varpi_{n-1})+d(\varpi_n,T\varpi_{n-1})d(\varpi_{n-1},T\varpi_n)}{d(\varpi_n,\varpi_{n-1})}\right]\right)$$

$$+\lambda_8\Phi_p\left(\left[\frac{d(\varpi_n,T\varpi_{n-1})[d(\varpi_n,T\varpi_n)+d(\varpi_{n-1},T\varpi_{n-1})+d(\varpi_n,T\varpi_{n-1})+d(\varpi_{n-1},T\varpi_n)]}{d(\varpi_n,T\varpi_n)+d(\varpi_{n-1},T\varpi_{n-1})+d(\varpi_n,T\varpi_{n-1})+d(\varpi_{n-1},T\varpi_n)}\right]\right)$$

$$\leq \lambda_1\Phi_p(d(\varpi_n,\varpi_{n-1})+\lambda_2[d(\varpi_{n+1},\varpi_n)+d(\varpi_n,\varpi_{n-1})]$$

$$+\lambda_3\Phi_p[d(\varpi_n,\varpi_n)+d(\varpi_{n+1},\varpi_{n-1})]+\lambda_4\Phi_p\left[\frac{d(\varpi_n,\varpi_{n+1})d(\varpi_{n-1},\varpi_n)}{d(\varpi_n,\varpi_{n-1})}\right]$$

$$+\lambda_5\Phi_p\left[\frac{d(\varpi_n,\varpi_n)d(\varpi_{n-1},\varpi_{n+1})}{d(\varpi_n,\varpi_{n-1})}\right]$$

$$+\lambda_6\Phi_p\left[\frac{d(\varpi_n,\varpi_{n+1})d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\varpi_n)d(\varpi_{n-1},\varpi_{n+1})}{d(\varpi_n,\varpi_{n-1})}\right]$$

$$+\lambda_7\Phi_p\left[\frac{d(\varpi_n,\varpi_{n+1})d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\varpi_n)d(\varpi_{n-1},\varpi_{n+1})}{d(\varpi_n,\varpi_{n-1})}\right]$$

$$+\lambda_8\Phi_p\left[\frac{d(\varpi_n,\varpi_n)[d(\varpi_n,\varpi_{n+1})+d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\varpi_n)+d(\varpi_{n-1},\varpi_{n+1})]}{d(\varpi_n,\varpi_{n+1})+d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\varpi_n)+d(\varpi_{n-1},\varpi_{n+1})}\right]$$

From the property of Φ_p Operator,

$$\Phi_p(d(\varpi_{n+1},\varpi_n))\leq \lambda_1\Phi_p\big(d(\varpi_n,\varpi_{n-1})\big)+\lambda_2\Phi_p(d(\varpi_{n+1},\varpi_n))+\lambda_2\Phi_p(d(\varpi_n,\varpi_{n-1}))$$

$$+\lambda_3\Phi_p[d(\varpi_{n+1},\varpi_n)+d(\varpi_n,\varpi_{n-1})]+\lambda_4\Phi_p\big(d(\varpi_n,\varpi_{n+1})\big)$$

$$+\lambda_6\Phi_p(d(\varpi_n,\varpi_{n+1}))+\lambda_7\Phi_p\big(d(\varpi_n,\varpi_{n+1})\big)$$

$$\Phi_p\big(d(\varpi_{n+1},\varpi_n)\big)-\lambda_2\Phi_p\big(d(\varpi_{n+1},\varpi_n)\big)-\lambda_3\Phi_p\big(d(\varpi_{n+1},\varpi_n)\big)-\lambda_4\Phi_p\big(d(\varpi_n,\varpi_{n+1})\big)$$

$$-\lambda_6\Phi_p(d(\varpi_n,\varpi_{n+1}))-\lambda_7\Phi_p(d(\varpi_n,\varpi_{n+1}))$$

$$\leq \lambda_1\Phi_p(d(\varpi_n,\varpi_{n-1}))+\lambda_2\Phi_p(d(\varpi_n,\varpi_{n-1}))+\lambda_3\Phi_p(d(\varpi_n,\varpi_{n-1}))$$

$$d(\varpi_{n+1},\varpi_n)\leq\left(\frac{\Phi_p(\lambda_1)+\Phi_p(\lambda_2)+\Phi_p(\lambda_3)}{1-\Phi_p(\lambda_2)-\Phi_p(\lambda_3)-\Phi_p(\lambda_4)-\Phi_p(\lambda_6)-\Phi_p(\lambda_7)}\right)d(\varpi_n,\varpi_{n-1})$$

On applying the same process, we get

$$d(\varpi_{n+1},\varpi_n)\leq\left(\frac{\Phi_p(\lambda_1)+\Phi_p(\lambda_2)+\Phi_p(\lambda_3)}{1-\Phi_p(\lambda_2)-\Phi_p(\lambda_3)-\Phi_p(\lambda_4)-\Phi_p(\lambda_6)-\Phi_p(\lambda_7)}\right)^n d(\varpi_1,\varpi_0)$$

$d(\varpi_{n+1},\varpi_n)\leq \delta^n\, d(\varpi_1,\varpi_0)$ where $\delta=\left(\frac{\Phi_p(\lambda_1)+\Phi_p(\lambda_2)+\Phi_p(\lambda_3)}{1-\Phi_p(\lambda_2)-\Phi_p(\lambda_3)-\Phi_p(\lambda_4)-\Phi_p(\lambda_6)-\Phi_p(\lambda_7)}\right)<1$

By triangular inequality, we have for $m > n$

$$d(\varpi_n, \varpi_m) \leq d(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \varpi_{n+2}) + d(\varpi_{n+2}, \varpi_{n+3}) + \cdots + d(\varpi_{m-1}, \varpi_n)$$
$$\leq (\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}) d(\varpi_1, \varpi_0)$$

Therefore, $d(\varpi_n, \varpi_m) \leq \frac{\delta^n}{1-\delta} d(\varpi_0, T\varpi_0) \to 0$ as $m, n \to \infty$

So, $\{\varpi_n\}$ is Cauchy sequence in ϖ, so by completeness of X, there is a point $u \in X$, such that $\varpi_n \to u$ as $n \to \infty$. Further, the continuity of T in X implies

$$T(u) = T\left(\lim_{n\to\infty} \varpi_n\right) = \lim_{n\to\infty} T\varpi_n = \lim_{n\to\infty} \varpi_{n+1} = u$$

Therefore, u is a fixed point of T in ϖ.

Suppose if there is any other $\varpi_1 \neq \varpi_2$ in X such that $T(\varpi_2) = \varpi_2$, then $d(\varpi_1, \varpi_2) = d(T\varpi_1, T\varpi_2)$

$$\Phi_p\big(d(\varpi_1, \varpi_2)\big) \leq \lambda_1 \Phi_p\big(d(\varpi_1, \varpi_2)\big) + \lambda_2 \Phi_p([d(T\varpi_1, \varpi_2) + d(T\varpi_2, \varpi_2)])$$
$$+\lambda_3 \Phi_p([d(T\varpi_2, \varpi_1) + d(T\varpi_1, \varpi_2)]) + \lambda_4 \Phi_p\left(\left[\frac{d(\varpi_1, T\varpi_2) d(\varpi_2, T\varpi_2)}{d(\varpi_1, \varpi_2)}\right]\right)$$
$$+\lambda_5 \Phi_p\left(\left[\frac{d(\varpi_1, T\varpi_2) d(\varpi_2, T\varpi_1)}{d(\varpi_1, \varpi_2)}\right]\right)$$
$$+\lambda_6 \Phi_p\left(\left[\frac{d(\varpi_1, T\varpi_1) d(\varpi_2, T\varpi_2) + d(\varpi_1, T\varpi_2) d(\varpi_2, T\varpi_1)}{d(\varpi_1, \varpi_2)}\right]\right)$$
$$+\lambda_7 \Phi_p\left(\left[\frac{d(\varpi_1, T\varpi_1) d(\varpi_2, T\varpi_2) + d(\varpi_1, T\varpi_2) d(\varpi_2, T\varpi_1)}{d(\varpi_1, v)}\right]\right)$$
$$+\lambda_8 \Phi_p\left(\left[\frac{d(\varpi_1, T\varpi_2)[d(\varpi_1, T\varpi_1) + d(\varpi_2, T\varpi_2) + d(\varpi_1, T\varpi_2) + d(\varpi_2, T\varpi_1)]}{d(\varpi_1, T\varpi_2) + d(\varpi_2, T\varpi_2) + d(\varpi_1, T\varpi_2) + d(\varpi_2, T\varpi_1)}\right]\right)$$

$$\Phi_p\big(d(\varpi_1, \varpi_2)\big) \leq \lambda_1 \Phi_p\big(d(\varpi_1, \varpi_2)\big) + \lambda_2 \Phi_p([d(\varpi_1, \varpi_1) + d(\varpi_2, \varpi_2)])$$
$$+\lambda_3 \Phi_p([d(\varpi_2, \varpi_1) + d(\varpi_1, \varpi_2)]) + \lambda_4 \Phi_p\left(\left[\frac{d(\varpi_1, \varpi_1) d(\varpi_2, \varpi_2)}{d(\varpi_1, \varpi_2)}\right]\right)$$
$$+\lambda_5 \Phi_p\left(\left[\frac{d(\varpi_1, \varpi_2) d(\varpi_2, \varpi_1)}{d(\varpi_1, \varpi_2)}\right]\right) + \lambda_6 \Phi_p\left(\left[\frac{d(\varpi_1, \varpi_1) d(\varpi_2, \varpi_2) + d(\varpi_1, \varpi_2) d(\varpi_2, \varpi_1)}{d(\varpi_1, \varpi_2)}\right]\right)$$
$$+\lambda_7 \Phi_p\left(\left[\frac{d(\varpi_1, \varpi_1) d(\varpi_2, \varpi_2) + d(\varpi_1, \varpi_2) d(\varpi_2, \varpi_1)}{d(\varpi_1, \varpi_2)}\right]\right)$$
$$+\lambda_8 \Phi_p\left(\left[\frac{d(\varpi_1, \varpi_2)[d(\varpi_1, \varpi_1) + d(\varpi_2, \varpi_2) + d(\varpi_1, \varpi_2) + d(\varpi_2, \varpi_1)]}{d(\varpi_1, \varpi_2) + d(\varpi_2, \varpi_2) + d(\varpi_1, \varpi_2) + d(\varpi_2, \varpi_1)}\right]\right)$$

$$\Phi_p\big(d(\varpi_1, \varpi_2)\big) \leq \lambda_1 \Phi_p\big(d(\varpi_1, \varpi_2)\big) + 2\lambda_3 \Phi_p\big(d(\varpi_1, \varpi_2)\big) + \lambda_5 \Phi_p\big(d(\varpi_1, \varpi_2)\big)$$
$$+\lambda_6\big(d(\varpi_1, \varpi_2)\big) + \lambda_7 \Phi_p\big(d(\varpi_1, \varpi_2)\big) + \lambda_8 \Phi_p(d(\varpi_1, \varpi_2))$$

$$d(\varpi_1, \varpi_2) \leq \big(\Phi_p(\lambda_1) + 2\Phi_p(\lambda_3) + \Phi_p(\lambda_5) + \Phi_p(\lambda_6) + \Phi_p(\lambda_7) + \Phi_p(\lambda_8)\big) d(\varpi_1, \varpi_2)$$

Which is contradiction. Hence ϖ_1 is a fixed point

Corollary 2.2: Let T be continues self map, defined on a complete metric space X. Further T satisfies the following conditions

$$\Phi_p\big(d(T\varpi, T\zeta)\big) \leq \lambda_1 \Phi_p\big(d(\varpi, \zeta)\big) + \lambda_2([d(T\varpi, \varpi) + d(T\zeta, \zeta)])$$
$$+\lambda_3 \Phi_p([d(T\zeta, \varpi) + d(T\varpi, \zeta)])$$
$$+\lambda_4 \Phi_p\left(\left[\frac{d(\varpi, T\varpi)d(y, T\zeta)}{d(\varpi, \zeta)}\right]\right) + \lambda_5 \Phi_p\left(\left[\frac{d(\varpi, T\zeta)d(\zeta, T\varpi)}{d(\varpi, \zeta)}\right]\right)$$

For all $\varpi, \zeta \in X, \varpi \neq \zeta$ and $\Phi_p(\lambda_1) + 2\Phi_p(\lambda_2) + 2\Phi_p(\lambda_3) + \Phi_p(\lambda_4) < 1$ then T has unique fixed point in T.

Proof: The proof is comes instead of $\lambda_6 = \lambda_7 = \lambda_8 = 0$ instead of above theorem.

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