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Power Dominator Equitable Coloring for some Standard Graphs

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Abstract:

A power dominator coloring of a graph G is a proper coloring where each vertex in $V(G)$ power dominates at least one complete color class. The power dominator chromatic number of G is represented by $\chi_{pd}(G)$. A graph G is said to be equitably k-colorable if it can be properly colored with k colors such that the size of any two color classes $C^1, C^2, \dots, C^k$ of G is differ by at most one, i.e, $\left||C^i| - |C^j|\right| \leq 1, 1 \leq i,j \leq k$ and $\chi_e(G)$ represents an equitable chromatic number of G. The power dominator equitable coloring of a graph G is a proper k - colorable if each vertex of G power dominates each and every vertex of some color class $C^1, C^2, \dots C^k$ for which $\left||C^i| - |C^j|\right| \leq 1, 1 \leq i,j \leq k$. In this paper, we obtain the power dominator equitable chromatic number χ_{pde} for some standard graphs.

Keywords: Proper coloring, color class, equitable coloring, power dominator coloring, standard graphs.

AMS Subject Classification: 05C15, 05C69

1. Introduction

In graph theory, Haynes introduced the important and extensively studied idea of domination in graphs [8]. In a graph G, a *dominating set* is a subset S of its vertices $V(G)$, where each vertex outside of S shares at least one adjacency with a vertex inside S. The smallest size among all such dominating sets is termed as the *"domination number",* represented by $\gamma(G)$. S is designated as a γ-set of G if it attains this minimal cardinality [5]. Haynes et al [7] introduced a groundbreaking concept in domination known as *power domination*. This concept finds application within the framework of electric power systems, where a graph G represents the system, with vertices symbolizing electrical nodes and edges representing transmission lines between these nodes. Finding the smallest possible collection of "Phasor Measurement Units" (PMUs) required for efficient system monitoring is the main goal. A connected graph G and a subset X of its vertices are considered, where the set monitored by X denoted by $M(X)$ is defined as follows:

1. Initialize $M(X)$ by adding the vertices in X along with their neighbors.
2. Iterate: While $\exists\, y \in M(X)$ such that all its neighbors except one, denoted by x, are already in $M(X)$ then add x to $M(X)$.

After this process, $M(X)$ represents the set monitored by X. A power dominating set X of G is such that $M(X)$ covers all vertices in G. The smallest size of such a power dominating set is termed as the "*power domination number*" $\gamma_p(G)$. Vast research is going in [3, 7, 9, 13].

The vertex d power dominates the vertices within the set $M(d)$ if they satisfy the following conditions:

1. Each vertex in $N(d) \cup \{d\}$ is included in $M(d)$.
2. For any vertex $c \in V(G)$, c is added to $M(d)$ if there exists a neighbor b in $M(d)$ such that all neighbors of b except c are already in $M(d)$
3. Step 2 is repeated for all vertices in the graph.

The process of giving colors to vertices in a graph G so that no two adjacent vertices have the same color in a suitable coloring of the graph is known as graph coloring, and it is useful in many graph theory applications. Let C^i be the color class i, signifying the collection of all vertices that possess the color i [12]. Each vertex in $V(G)$ dominates every vertex of some color class is said to be a dominator coloring of graph G and $\chi_d(G)$ represents the dominator chromatic number of G [10, 2, 12]. A "*power dominator coloring*" (PDC) of a graph G entails a proper coloring where each vertex $v \in V(G)$ power dominates all the vertices of at least one-color class. The term $\chi_{pd}(G)$ signifies the "power dominator chromatic number of G", as elucidated in reference [1, 10]. A proper coloring of a graph G is said to be *equitably k-colorable* if the number of vertices of any two-color classes $C^1, C^2, \ldots C^k$ of G is differ by at most one. That is $\left||C^i| - |C^j|\right| \leq 1, 1 \leq i, j \leq k$. The term $\chi_e(G)$ represents an "equitable chromatic number of G" [4, 11]. The *power dominator equitable coloring* (PDEC) of a graph G defines each vertex $v \in V(G)$ power dominates all the vertices of at least one color class and also satisfies the inequality $\left||C^i| - |C^j|\right| \leq 1, 1 \leq i, j \leq k$ The notation $\chi_{pde}(G)$ represents the "power dominator equitable chromatic number of G". The power dominator equitable chromatic number χ_{pde} for a few common graphs is obtained in this study.

2. Motivation

The motivation behind this paper lies in addressing a significant problem in graph theory, finding efficient and balanced colorings for graphs that satisfy specific domination properties. However, in certain applications, such as network design or resource allocation, additional constraints need to be considered. The concept of power domination introduces the idea that each vertex in a graph should have influence over all other vertices within its neighborhood. This leads to the notion of PDC, where every vertex power dominates at least one-color class. This ensures a level of connectivity and influence within the graph. Equitable coloring, on the other hand, seeks to balance the sizes of color classes. This is particularly important in scenarios where fairness or resource allocation is a concern. By combining the principles of power dominator coloring and equitable coloring, the paper presented the concept of PDEC. This approach aims to find coloring where every vertex not only dominates at least one-color class but also ensures that the sizes of the color classes are balanced. The research presented in this paper commences a study on this parameter by exploring its properties and determining the "power dominator equitable chromatic number" for some standard graphs. Understanding this parameter can have implications in various real-world applications, such as network communication, social network analysis, and resource allocation in distributed systems. Ultimately, this research contributes to advancing our understanding of graph coloring with additional domination constraints, paving the way for more efficient and equitable solutions in practical scenarios.

3. Preliminaries

Consider an undirected, connected, and simple graph $G = (V, E)$ comprising two non-empty sets V and E, where edges of G are the elements of E, whereas vertices are the components of V. We use Harary's [6] graph theoretic notations. If a path exists between any two vertices in the graph, denoted by u and v, then the graph is said to be connected. The open neighborhood of vertex c comprises all vertices adjacent to c, denoted as $N(c)$. The union of $N(c)$ with the vertex c itself is known as the closed neighborhood of c, or $N[c]$. A path P_n is a sequence of n vertices, denoted as $a_1, a_2, \dots a_n$ that are connected by $n-1$ edges, ensuring that no vertex or edge is repeated within the sequence. A cycle C_n is a closed path having n edges and n vertices with first and last vertex being the same. All pairs of vertices in a complete graph of order n, represented as K_n, are adjacent. In a bipartite graph $K_{m,n}$, there are two separate set of vertices M_1 and M_2. Set M_1 contains m vertices, while M_2 contains n number of vertices. Each vertex in M_1 is exclusively connected to every other vertex in M_2, and the same holds true in reverse. There are no connections between vertices within the same set. A wheel graph W_n with an order of $n+1$, is formed by joining every vertex in cycle C_n to one universal vertex (is a vertex that shares an edge with every other vertex present in the graph). A helm graph H_n is constructed by adding a leaf edge to each vertex of an n-wheel graph, which consists of a cycle of n vertices connected to a single central vertex.

4. Main Results

Theorem 4.1. For path $P_n, n \geq 2,\ \chi_{pde}(P_n) = 2$.

Proof. Let $a_i, 1 \leq i \leq n$ be the n vertices of path P_n and $a_1a_2, a_2a_3, \dots, a_{n-1}a_n$ be the edges of P_n. The odd indices receive color *1* i.e, the vertices $a_{2j+1}, 0 \leq j \leq \lfloor (n-1)/2 \rfloor$ receives color *1* and color *2* to the vertices with the even indices, i.e., to the vertices $a_{2j}, 1 \leq j \leq \lfloor n/2 \rfloor$. Evidently, each vertex $a_i, 1 \leq i \leq n$, power dominates each and every vertex of P_n. Thus, each vertex of P_n power dominates both the color classes C^1 and C^2. Furthermore, the discrepancy occurs whenever n is odd, $\left||C^i| - |C^j|\right| = 1$ and whenever n is even, $\left||C^i| - |C^j|\right| = 0$. Therefore, a minimum of *2* colors is necessary for achieving power dominator equitable coloring. Thus, we get the desired result $\chi_{pde}(P_n) = 2$.

Theorem 4.2. For cycle $C_n, n \geq 3$,

$$\chi_{pde}(C_n) = \begin{cases} 2, & if\ n\ is\ even \\ 3, & if\ n\ is\ odd \end{cases}$$

Proof. We analyze two separate cases for C_n, based on n is even or odd.

Case 1: n is even

Let $C_{2k}, k \geq 2$ be an even cycle with the vertices $a_1, a_2, \dots, a_{2k}, k \geq 2$ and let $a_1a_2, a_2a_3, \dots a_{2k-1}a_{2k}, a_{2k}a_1$ be the edges of C_{2k}. We assign color *1* to the vertices with the odd indices, i.e., to the vertices $a_{2j+1}, 0 \leq j \leq \lfloor (2k-1)/2 \rfloor$ and color *2* to the vertices with the even indices, i.e., to the vertices $a_{2j}, 1 \leq j \leq \lfloor (2k)/2 \rfloor$. Each vertex $a_i, 1 \leq i \leq n$ power dominates all the vertices of C_{2k} and hence each vertex of C_{2k} power dominates both the color classes C^1 and C^2. Here $\left||C^1| - |C^2|\right| = 0$. Therefore $\left||C^i| - |C^j|\right| \leq 1, 1 \leq i, j \leq 2$. Hence $\chi_{pde}(C_{2k}) = 2$.

Case 2: n is odd

Consider the odd cycle $C_{2k+1}, k \geq 1$ with the vertices $a_{2i+1}, 0 \leq i \leq k$. Assigning color *1* and *2* alternatively results violation in assigning color *1* and *2* to a_n, since neighbors of vertex a_n receives color *1* and *2*. For odd values of n at least *3* colors are required. Assign the colors *1, 2* and *3* repeatedly for the vertices $a_{i+1}, 0 \leq i \leq 2k$ in the following manner,

Subcase (i): $n \equiv 0,2 \ (mod\ 3)$
The vertex set $\left\{a_{3k-2}: 1 \leq k \leq \frac{n}{3}\right\}$ receives color *1*. The vertex set $\{a_{3k-1}: 1 \leq k \leq \frac{n}{3}\}$ receives color *2*. The vertices $\left\{a_{3k}: 1 \leq k \leq \frac{n}{3}\right\}$ receives color *3*.
Subcase (ii): $n \equiv 1 \ (mod\ 3)$
The vertex set $\left\{a_{3k-2}: 1 \leq k \leq \frac{n-1}{3}\right\}$ receives color *1*. The vertex set $\{a_{3k-1}: 1 \leq k \leq \frac{n-1}{3}\} \cup \{a_n\}$ receives color *2*. The vertex set $\left\{a_{3k}: 1 \leq k \leq \frac{n-1}{3}\right\}$ receives color *3*. Specifically, vertex a_n receives color *2*, since neighbors of a_n receives color *1* and *3*.

From the above subcases, we clearly see that the graph C_{2k+1} receives proper coloring and each vertex of C_{2k+1} power dominates all the vertices of C_{2k+1} and hence each vertex power dominates all the color classes C^1, C^2 and C^3 and also it satisfies the equitable condition that $\left||C^i| - |C^j|\right| \leq 1, 1 \leq i, j \leq 3$. Hence $\chi_{pde}(C_{2k+1}) = 3$.

Theorem 4.3. For complete graph $K_n, n \geq 1, \chi_{pde}(K_n) = n$.

Proof. Examine the entire graph $K_n, n \geq 1$, consisting of the vertices $a_i, 1 \leq i \leq n$. Since all the vertices in $V(G)$ are next to each other, assign color $i, 1 \leq i \leq n$ to the corresponding vertex $a_i, 1 \leq i \leq n$ respectively. Each vertex in the complete graph K_n power dominates every other vertex in K_n, as well as all color classes. Furthermore, the inequality $\left||C^i| - |C^j|\right| = 0$ is satisfied for all pairs of color classes C^i and C^j, where $1 \leq i, j \leq n$. Hence $\chi_{pde}(K_n) = n$.

Theorem 4.4. For complete bipartite graph $K_{m,n}, m, n \geq 1$

$$\chi_{pde}(K_{m,n}) = \begin{cases} 2, & if\ (m-n) < 2 \\ \left\lceil \frac{m}{n+1} \right\rceil + 1, & if\ (m-n) \geq 2 \end{cases}$$

Proof. Let (M_1, M_2) be the partitions of $K_{m,n}$. Let m be the cardinality of M_1 and n be the cardinality of M_2.

Case 1: $(m-n) < 2$

Each vertex $a_i, 1 \leq i \leq m$ of M_1 power dominates $N[a_i], 1 \leq i \leq m$. Each vertex $a_j, 1 \leq j \leq n$ of M_2 power dominates $N[a_j], 1 \leq j \leq n$. By assigning color *1* to vertices in M_1 and color *2* to vertices in

M_2, we ensure that each vertex in M_1 power dominates the color class C^2, while each vertex in M_2 power dominates the color class C^1. Therefore $\left||C^1| - |C^2|\right| \leq 1$. Hence $\chi_{pde}(K_{m,n}) = 2$.

Case 2: $(m - n) \geq 2$

Without loss of generality, let $m > n$. Each vertex $a_i, 1 \leq i \leq m$, of partition M_1 power dominates $N[a_i], 1 \leq i \leq m$ and each vertex a_j, $1 \leq j \leq n, n > 2$ of partition M_2 power dominates $N[a_j], 1 \leq j \leq n$. Divide the vertices of M_1 into $\left\lceil \frac{m}{n+1} \right\rceil$ number of sets and utilize $\left\lceil \frac{m}{n+1} \right\rceil$ colors to color the vertices in M_1 and assign color *1* to all vertices of M_2. This ensures that every vertex power dominates at least one-color class and maintains the equitable condition. Hence $\chi_{pde}(K_{m,n}) = \left\lceil \frac{m}{n+1} \right\rceil + 1$.

Theorem 4.5. For wheel graph $W_n, n \geq 4$, $\chi_{pde}(W_n) = \left\lceil \frac{n}{2} \right\rceil + 1$

Proof. Consider a_1 as the central vertex and a_i, where $2 \leq i \leq n$ as the vertices located on the cycle of W_n. Assign color *1* to a_1. Each vertex of n wheel graph power dominates the color class C^1. Assign $\left\lceil \frac{n}{2} \right\rceil$ colors to the remaining vertices equitably, so that $\left||C^i| - |C^j|\right| \leq 1 \; \forall \; i$ and j. Therefore $\chi_{pde}(W_n) = \left\lceil \frac{n}{2} \right\rceil + 1$.

Theorem 4.6. For helm graph $H_n, n \geq 3$, $\chi_{pde}(H_n) = n + \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. Let $a_i, 1 \leq i \leq n$ be the vertices on the cycle of $H_n, u_i, 1 \leq i \leq n$ representing the pendent vertices and a_{n+1} denotes the central vertex within the graph H_n. The vertex a_{n+1} power dominates over all vertices of H_n. The pendent vertices u_i for $1 \leq i \leq n$ where $n \geq 4$ power dominates $N[u_i]$ for $1 \leq i \leq n$ where $n \geq 4$. Moreover, the vertices on the cycle $a_i, 1 \leq i \leq n$ power dominates $N[a_i], 1 \leq i \leq n$ where $n \geq 4$. So, assign color i to $\{a_i\}, 1 \leq i \leq n$ respectively. Assign $\left\lceil \frac{n+1}{2} \right\rceil$ colors to the leftover $n + 1$ vertices equitably. The central vertex a_{n+1} power dominates all the color classes. Vertex $u_i, 1 \leq i \leq n$ and $a_i, 1 \leq i \leq n$ power dominates the color class $C^i, 1 \leq i \leq n$ respectively and also it holds the inequality $\left||C^i| - |C^j|\right| \leq 1$, $\forall \; i$ and j. Hence $\chi_{pde}(H_n) = n + \left\lceil \frac{n+1}{2} \right\rceil$

5. Conclusion

The inequality $\chi(G) \leq \chi_{pd}(G) \leq \chi_e(G) \leq \chi_{pde}(G)$ provides a concise framework for understanding the precise values of power dominator equitable chromatic numbers across various standard graphs, such as $P_n, C_n, K_n, K_{m,n}$, W_n and H_n. The graph G, exhibiting $\chi(G) = \chi_{pd}(G) = \chi_e(G) = \chi_{pde}(G)$ encompasses P_n, C_n and K_n whereas for $K_{m,n}$, all these parameters are equal when $(m - n) < 2$. For wheel graph W_n, $\chi_e(W_n) = \chi_{pde}(W_n)$. For helm graph H_n, $\chi(H_n) = \chi_e(H_n)$ and $\chi_{pde}(H_n)$ does not coincide with other three parameters. Determining the power dominator equitable chromatic numbers for diverse graphs remains a task reserved for future investigation.

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