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Generalization of intuitionistic fuzzy ideals via $(\tilde{\partial}, \tilde{\wp})$ intuitionistic Q interval-valued fuzzy ideals using regular ordered ternary semigroups

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Abstract

This paper introduces the notion of $\tilde{\partial}$, $\tilde{\wp}$ intuitionistic Q interval-valued fuzzy subsemigroup (IQVFSS), fuzzy left ideal (IQVFLI), fuzzy right ideal (IQVFRI), fuzzy lateral ideal (IQVFLATI), fuzzy ideal (IQVFI), and fuzzy bi-ideal (IQVFBI) of an ordered semigroups. Let $\tilde{\partial}$, $\tilde{\wp}$-IQVFI is a new extension of IQVFI over ternary semigroups $\mathscr{X}$. The subset $\hbar = [\Re\,\Im]$ represents a $(\tilde{\partial}, \tilde{\wp}) - IQVFSS[IQVFLI, IQVFRI, IQVFLATI, IQVFBI]$ of $\mathscr{X}$ if and only if every level subset $\hbar_t$ is an SS $[LI, RI, LAIQVF, TBI]$ of $\mathscr{X}$ for every $t \in (\tilde{\partial}, \tilde{\wp}]$. A few examples can be presented to demonstrate our results.

Keywords: IQVFSS, IQVFLI, IQVFRI, IQVFLATI, IQVFBI.

1 Introduction

D. H. Lehmer initially introduced triplexes, which are ternary algebraic systems, in 1932.[1] Triplexes, ternary algebraic systems that prove to be commutative ternary groups, are the subject of his investigation. The concept of a semiring was initially put out by Vandiver in 1934. In 1962, Hestenes[2] used the idea of ternary algebra to matrices and linear transformation. The fuzzy set (FS) theory, first presented by Zadeh,[3] is the most effective approach to dealing with ambiguity and uncertainty. If an element in an FS has a single value inside the interval, it is regarded as a member degree (MD). However, the degree of non-membership degree (NMD) could not always be equal to one minus the MD, uncertain theories, such as FS,[3] intuitionistic FS (IFS),[4] Pythagorean FS (PFS),[5] and spherical FS (SFS).[6] An FS is made up of sets of various grades, such as MG, which range from 0 to 1. MG is the classification for IFS regardless of the assertion made by Atanassov[4] that NMG can only be worth 1. Using PFS logic, Yager[5] built the generalized MG and NMG, which has a maximum value of 1 and is based on the square of the MGs and NMGs. The neutral condition, which is neither positive nor negative, cannot be adequately described by these concepts. The practical applications of FS extensions were discussed by Al-Husband et al.[7]-.[10] He investigated their characteristics in a manner similar to that of set theory. Rosenfeld[11] created fuzzy subgroups and listed some of their characteristics in 1971. Fuzzy semigroups were first presented by Kuroki[12] as an expansion of classical semigroups. Some fuzzy semigroup characterisation was developed by Mordeson.[13] Sen et al. supplied the $\beth$-semigroups' characteristics.[14, 15] Kehayopula looked at the ordered $\beth$-semigroup.[16] Somsak Lekkoksung used ordered semigroups to investigate Q-fuzzy ideals.[17][18] Kehayopula et al. started the research on fuzzy ordered semigroups.

Initial proposals for the $(\tilde{\partial}, \tilde{\wp})$ fuzzy bi-ideal and FSS were made by Muhammad Khan et al.[19] Numerous scholars have recently examined the idea of IFS, NSS, and its characterisation[20]-.[24] An IFS with normal subbisemiring was introduced by Palanikumar et al.[25] Hila et al.[26] explored bi-ideals on ordered semigroups. Dutta T.K. et al. introduced novel concepts using prime ideals of ternary semirings.[27] A number of prime bi-ideals of the rings have been studied by Palanikumar et al.[28, 29] The several ideals of different algebraic approach were examined by Palanikumar et al.[30–34] The notion of various operators, including averaging, geometric, and its generalized forms, was explored by new researchers Hatamleh et al.[35]-,[41] Bataihah,[42] and Hazaymeh.[43] We study ordered ternary semigroups based on $(\tilde{\partial}, \tilde{\wp})$ ternary IFS and provide examples to show their properties.

2 Basic concepts

Definition 2 .1. Let $\beth$ and $\beth_1$ be subsets of $\mathscr{X}$. Then

1. $(\beth] = \{t \in \mathscr{X} \mid t \leqslant h \, for\, some\, h \in \beth\}$,
2. $\beth\beth_1 = \{ab : a \in \beth,\, b \in \beth_1\}$,
3. $\beth_a = \{(b, c) \in \mathscr{X} \times \mathscr{X} \mid a \leqslant bc\}$.

Definition 2.2. A fuzzy subset $\eth_2$ of an ordered semigroup $\mathscr{X}$ is called a FRI(FLI) of $\mathscr{X}$ if

1. $a \leqslant b \Rightarrow \eth_2(a) \geqslant \eth_2(b)$ for all $a, b \in \mathscr{X}$,
2. $\eth_2(ab) \geqslant \eth_2(a)$ (resp. $\eth_2(ab) \geqslant \eth_2(b)$) for all $a, b \in \mathscr{X}$,

Definition 2.3. If $(\mathscr{X}, +)$ is a commutative semigroup and ternary multiplication meets the following conditions, then

1. $(fgh)ij = f(ghi)j = fg(hij)$,
2. $(f + g)hi = fhi + ghi$,
3. $f(g + h)i = fgi + fhi$,
4. $fg(h + i) = fgh + fgi$ for all $f, g, h, i, j \in \mathscr{X}$

Definition 2.4. The subset K of $\mathscr{X}$ is called a

1. SS if $\upsilon_1\upsilon_2\upsilon_3 \in K$ for all $\upsilon_1, \upsilon_2, \upsilon_3 \in K$.
2. right (lateral, left) ideal if $is_1s_2 \in K(s_1is_2 \in K, s_1s_2i \in K)$ for all $s_1, s_2 \in \mathscr{X}$ and $i \in K$.

Corollary 2.5. *If $\mathscr{X}$ is regular if and only if RI $\beth$, LATIF $\beth_1$ and LI $\beth_2$ of $\mathscr{X}$, then* $(\beth \curlywedge \beth_1 \curlywedge \beth_2] = (\beth * \beth_1 * \beth_2]$.

3 $(\tilde{\partial}, \tilde{\wp})$ ternary intuitionistic Q interval-valued fuzzy ideals

Here, $\mathscr{X}$ represents an ordered ternary semigroup. Assuming $(\tilde{\partial}, \tilde{\wp}) \in [0, 1]$ and $0 \leqslant \tilde{\partial} \prec \tilde{\wp} \leqslant 1$, both $(\tilde{\partial}, \tilde{\wp})$ are arbitrary fixed points.

Definition 3.1. An IVFS N and Q be any set, then the pair $N \times Q$ is called an IQVFS. Let $N = [\tilde{\Re}_N, \tilde{\Im}_N]$ of $\mathscr{X}$ is called a $(\tilde{\partial}, \tilde{\wp})$ IQVFSS of $\mathscr{X}$ if

1. $\eth_1 \leqslant \eth_3 \Rightarrow \tilde{\Re}(\eth_1) \geqslant \tilde{\Re}(\eth_3)$,
2. $\max\{\tilde{\Re}(\eth_1\eth_2\eth_3, \breve{a}), \tilde{\partial}\} \geqslant \min\{\tilde{\Re}(\eth_1, \breve{a}), \tilde{\Re}(\eth_2, \breve{a}), \tilde{\Re}(\eth_3, \breve{a}), \tilde{\wp}\}$,
3. $\min\{\tilde{\Im}(\eth_1\eth_2\eth_3, \breve{a}), \tilde{\partial}\} \leqslant \max\{\tilde{\Im}(\eth_1, \breve{a}), \tilde{\Im}(\eth_2, \breve{a}), \tilde{\Im}(\eth_3, \breve{a}), \tilde{\wp}\}$ for all $\eth_1, \eth_2, \eth_3 \in \mathscr{X}$ and $\breve{a} \in Q$.

Example 3.2. Let $\mathscr{X} = \{a, b, c, d\}$ with the following Cayley table:

$*$	a	b	c	d
a	l	l	l	l
b	l	m	n	o
c	l	n	n	n
d	l	n	n	n

$*$	a	b	c	d
l	a	a	a	a
m	a	b	c	d
n	a	c	c	c
o	a	c	c	c

$\leqslant := \{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(d,c),(d,d)\}$.
Define the mapping $N = [\tilde{\Re}_N, \tilde{\Im}_N] : \mathscr{X} \times \mathscr{X} \times \mathscr{X} \to [0,1]$.

$$\tilde{\Re}(\kappa,\breve{a}) = \begin{cases} [0.6, 0.65] & if\ \kappa = a \\ [0.4, 0.45] & if\ \kappa = b \\ [0.1, 0.15] & if\ \kappa = c \\ [0.2, 0.25] & if\ \kappa = d \end{cases} \quad \tilde{\Im}(\kappa,\breve{a}) = \begin{cases} [0.3, 0.35] & if\ \kappa = a \\ [0.35, 0.4] & if\ \kappa = b \\ [0.45, 0.5] & if\ \kappa = c \\ [0.4, 0.45] & if\ \kappa = d \end{cases}$$

Then N is a $([0.5, 0.55], [0.65, 0.7])$ $IQVFSS$ of $\mathscr{X}$.

Definition 3.3. A IQVFS N of $\mathscr{X}$ is called a $(\tilde{\partial}, \tilde{\wp})$-IQVFBI of $\mathscr{X}$ if

1. If $\eth_1 \leqslant \eth_3$, then $\tilde{\Re}(\eth_1) \geqslant \tilde{\Re}(\eth_3)$ and $\tilde{\Im}(\eth_1) \leqslant \tilde{\Im}(\eth_3)$,
2. $\max\{\tilde{\Re}(\eth_1\eth_2\eth_3,\breve{a}), \tilde{\partial}\} \geqslant \min\{\tilde{\Re}(\eth_1,\breve{a}), \tilde{\Re}(\eth_3,\breve{a}), \tilde{\wp}\}$,
   $\min\{\tilde{\Im}(\eth_1\eth_2\eth_3,\breve{a}), \tilde{\partial}\} \leqslant \max\{\tilde{\Im}(\eth_1,\breve{a}), \tilde{\Im}(\eth_3,\breve{a}), \tilde{\wp}\}$,
3. $\max\{\tilde{\Re}(\eth_1\eth_2\eth_3\eth_4\eth_5,\breve{a}), \tilde{\partial}\} \geqslant \min\{\tilde{\Re}(\eth_1,\breve{a}), \tilde{\Re}(\eth_5,\breve{a}), \tilde{\wp}\}$,
   $\min\{\tilde{\Im}(\eth_1\eth_2\eth_3\eth_4\eth_5,\breve{a}), \tilde{\partial}\} \leqslant \max\{\tilde{\Im}(\eth_1,\breve{a}), \tilde{\Im}(\eth_5,\breve{a}), \tilde{\wp}\}$, for $\eth_1, \eth_2, \eth_3, \eth_4, \eth_5, \in \mathscr{X}$.

4 Level set concepts

Theorem 4.1. *A subset $\hbar_{\tilde{\partial}}$ is a $\tilde{\Re}_\partial$ is a $(\tilde{\partial}, \tilde{\wp})$-IQVFSS (IQVFLI,IQVFLATI, IQVFRI, IQVFBI) of $\mathscr{X}$. Then the lower level set $\tilde{\Re}_\partial$ is an SS (LI,LATIF, RI, TBI) of $\mathscr{X}$, where $\tilde{\Re}_\partial = \{\eth_1 \in \mathscr{X} | \tilde{\Re}(\eth_1,\breve{a}) \succ \tilde{\partial}\}$ and $\tilde{\Im}_\partial = \{\eth_1 \in \mathscr{X} | \tilde{\Re}(\eth_1,\breve{a}) \prec \tilde{\partial}\}$.*

Proof. Suppose that $\hbar_{\tilde{\partial}}$ is a $(\tilde{\partial}, \tilde{\wp})$-IQVFSS of $\mathscr{X}$. Let $\eth_1, \eth_2, \eth_3 \in \mathscr{X}$ such that $\eth_1, \eth_2, \eth_3 \in \tilde{\Re}_\partial$. Then $\tilde{\Re}(\eth_1,\breve{a}) \succ \tilde{\partial}, \tilde{\Re}(\eth_2,\breve{a}) \succ \tilde{\partial}, \tilde{\Re}(\eth_3,\breve{a}) \succ \tilde{\partial}$.
Therefore $\max\{\tilde{\Re}(\eth_1\eth_2\eth_3,\breve{a}), \tilde{\partial}\} \geqslant \min\{\tilde{\Re}(\eth_1,\breve{a}), \tilde{\Re}(\eth_2,\breve{a}), \tilde{\Re}(\eth_3,\breve{a}), \tilde{\wp}\} \succ \min\{\tilde{\partial}, \tilde{\partial}, \tilde{\partial}, \tilde{\wp}\} = \tilde{\partial}$. Hence $\tilde{\Re}(\eth_1\eth_2\eth_3,\breve{a}) \succ \tilde{\partial}$. It shows that $\eth_1\eth_2\eth_3 \in \tilde{\Re}_\partial$. Therefore $\tilde{\Re}_\partial$ is a SS of $\mathscr{X}$. Let $\eth_1, \eth_2, \eth_3 \in \mathscr{X}$ such that $\eth_1, \eth_2, \eth_3 \in \tilde{\Im}_\partial$. Then $\tilde{\Im}(\eth_1,\breve{a}) \prec \tilde{\partial}, \tilde{\Im}(\eth_2,\breve{a}) \prec \tilde{\partial}\tilde{\Im}(\eth_3,\breve{a}) \prec \tilde{\partial}$. Therefore $\min\{\tilde{\Im}(\eth_1\eth_2\eth_3,\breve{a}), \tilde{\partial}\} \leqslant \max\{\tilde{\Im}(\eth_1,\breve{a}), \tilde{\Im}(\eth_2,\breve{a}), \tilde{\Im}(\eth_3,\breve{a}), \tilde{\wp}\} \prec \max\{\tilde{\partial}, \tilde{\partial}, \tilde{\partial}, \tilde{\wp}\} = \tilde{\wp}$. Hence $\tilde{\Im}(\eth_1\eth_2\eth_3,\breve{a}) \prec \tilde{\partial}$. It shows that $\eth_1\eth_2\eth_3 \in \tilde{\Im}_\partial$. Therefore $\tilde{\Im}_\partial$ is a SS of $\mathscr{X}$. Therefore $\hbar_{\tilde{\partial}}$ is a SS of $\mathscr{X}$.

Theorem 4.2. *A subset $\sqsupset$ of $\mathscr{X}$ is a SS $[LI, LATIF, RI, TBI]$ of $\mathscr{X}$ if and only if the IQVFS $\hbar = [\tilde{\Re}, \tilde{\Im}]$ of $\mathscr{X}$ is defined as*

$$\tilde{\Re}(\eth_1,\breve{a}) = \begin{cases} \geqslant \tilde{\wp}\, for\, all\ \ \eth_1 \in (\sqsupset] \\ \tilde{\partial}\, for\, all\ \eth_1 \notin (\sqsupset] \end{cases} \quad \tilde{\Im}(\eth_1,\breve{a}) = \begin{cases} \leqslant \tilde{\wp}\, for\, all\ \ \eth_1 \in (\sqsupset] \\ \tilde{\partial}\, for\, all\ \eth_1 \notin (\sqsupset] \end{cases}$$

is a $(\tilde{\partial}, \tilde{\wp})IQVFSS[IQVFLI, IQVFLATI, IQVFRI, IQVFBI]$ of $\mathscr{X}$.

Proof. Suppose that $\sqsupset$ is an SS of $\mathscr{X}$. Let $\eth_1, \eth_2, \eth_3 \in \mathscr{X}$ be such that $\eth_1, \eth_2, \eth_3 \in (\sqsupset]$ then $\eth_1\eth_2\eth_3 \in (\sqsupset]$. Hence $\tilde{\Re}(\eth_1\eth_2\eth_3,\breve{a}) \geqslant \tilde{\wp}$ and $\tilde{\Im}(\eth_1\eth_2\eth_3,\breve{a}) \leqslant \tilde{\wp}$.
Thus $\max\{\tilde{\Re}(\eth_1\eth_2\eth_3,\breve{a}), \tilde{\partial}\} \geqslant \tilde{\wp} = \min\{\tilde{\Re}(\eth_1,\breve{a}), \tilde{\Re}(\eth_2,\breve{a}), \tilde{\Re}(\eth_3,\breve{a}), \tilde{\wp}\}$ and $\min\{\tilde{\Im}(\eth_1\eth_2\eth_3,\breve{a}), \tilde{\partial}\} \leqslant \tilde{\wp} = \max\{\tilde{\Im}(\eth_1,\breve{a}), \tilde{\Im}(\eth_2,\breve{a}), \tilde{\Im}(\eth_3,\breve{a}), \tilde{\wp}\}$.
If $\eth_1 \notin (\sqsupset]$ or $\eth_2 \notin (\sqsupset]$ or $\eth_3 \notin (\sqsupset]$, then $\min\{\tilde{\Re}(\eth_1,\breve{a}), \tilde{\Re}(\eth_2,\breve{a}), \tilde{\Re}(\eth_3,\breve{a}), \tilde{\wp}\} = \tilde{\partial}$ and
$\max\{\tilde{\Im}(\eth_1,\breve{a}), \tilde{\Im}(\eth_2,\breve{a}), \tilde{\Im}(\eth_3,\breve{a}), \tilde{\wp}\} = \tilde{\wp}$.

That is $\max\{\tilde{\Re}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} \geqslant \min\{\tilde{\Re}(ð_1,\breve{a}),\tilde{\Re}(ð_2,\breve{a}),\tilde{\Re}(ð_3,\breve{a}),\tilde{\wp}\}$ and
$\min\{\tilde{\Im}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} \leqslant \max\{\tilde{\Im}(ð_1,\breve{a}),\tilde{\Im}(ð_2,\breve{a}),\tilde{\Im}(ð_3,\breve{a}),\tilde{\wp}\}$.
Therefore $\hbar$ is a $(\tilde{\partial},\tilde{\wp})$ IQVFSS of $\mathscr{X}$.
Conversely assume that $\hbar = [\tilde{\Re},\tilde{\Im}]$ is a $(\tilde{\partial},\tilde{\wp})$-IQVFSS of $\mathscr{X}$. Let $ð_1ð_2ð_3 \in (\sqsupset]$. Then $\tilde{\Re}(ð_1,\breve{a}) \geqslant \tilde{\wp}, \tilde{\Re}(ð_2,\breve{a}) \geqslant \tilde{\wp}, \tilde{\Re}(ð_3,\breve{a}) \geqslant \tilde{\wp}$ and $\tilde{\Im}(ð_1,\breve{a}) \leqslant \tilde{\wp}, \tilde{\Im}(ð_2,\breve{a}) \leqslant \tilde{\wp}, \tilde{\Im}(ð_3,\breve{a}) \leqslant \tilde{\wp}$. Now $\hbar = [\tilde{\Re},\tilde{\Im}]$ is a $(\tilde{\partial},\tilde{\wp})$-IQVFSS of $\mathscr{X}$. Therefore $\max\{\tilde{\Re}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} \geqslant \min\{\tilde{\Re}(ð_1,\breve{a}),\tilde{\Re}(ð_2,\breve{a}),\tilde{\Re}(ð_3,\breve{a}),\tilde{\wp}\} \geqslant \min\{\tilde{\wp},\tilde{\wp},\tilde{\wp},\tilde{\wp}\} = \tilde{\wp}$ and $\min\{\tilde{\Im}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} \leqslant \max\{\tilde{\Im}(ð_1,\breve{a}),\tilde{\Im}(ð_2,\breve{a}),\tilde{\Im}(ð_3,\breve{a}),\tilde{\wp}\} \leqslant \max\{\tilde{\wp},\tilde{\wp},\tilde{\wp},\tilde{\wp}\} = \tilde{\wp}$. It follows that $ð_1ð_2ð_3 \in (\sqsupset]$. Therefore $\sqsupset$ is a SS of $\mathscr{X}$.

Theorem 4.3. *A subset $\hbar = [\tilde{\Re},\tilde{\Im}]$ is a $(\tilde{\partial},\tilde{\wp}) - IQVFSS[IQVFLI, IQVFLATI, IQVFRI, IQVFBI]$ of $\mathscr{X}$ if and only if each level subset $\hbar_t$ is a SS [LI,LAIQVF,RI,TBI] of $\mathscr{X}$ for all $t \in (\tilde{\partial},\tilde{\wp}]$.*

Proof. Assume that $\hbar_t$ is a SS of $\mathscr{X}$ for each $t \in [0,1]$.
Let $t = \min\{\tilde{\Re}(ð_1,\breve{a}),\tilde{\Re}(ð_2,\breve{a}),\tilde{\Re}(ð_3,\breve{a})\}$. Then $ð_1,ð_2,ð_3 \in \tilde{\Re}_t$ for each $ð_1,ð_2,ð_3 \in \mathscr{X}$.
Thus $\max\{\tilde{\Re}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} \geqslant t = \min\{\tilde{\Re}(ð_1,\breve{a}),\tilde{\Re}(ð_2,\breve{a}),\tilde{\Re}(ð_3,\breve{a}),\tilde{\wp}\}$.
Let $t = \max\{\tilde{\Im}(ð_1,\breve{a}),\tilde{\Im}(ð_2,\breve{a}),\tilde{\Im}(ð_3,\breve{a})\}$.
Then $ð_1,ð_2,ð_3 \in \tilde{\Im}_t$ for each $ð_1,ð_2,ð_3 \in \mathscr{X}$.
Thus $\min\{\tilde{\Im}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} \leqslant t = \max\{\tilde{\Im}(ð_1,\breve{a}),\tilde{\Im}(ð_2,\breve{a}),\tilde{\Im}(ð_3,\breve{a}),\tilde{\wp}\}$.
This shows that $\hbar^t$ is IQVFSS of $\mathscr{X}$.
Conversely, assume that $\hbar^t$ is a IQVFSS of $\mathscr{X}$.
For each $t \in [0,1]$ and $ð_1,ð_2,ð_3 \in \tilde{\Re}_t$.
We have $\tilde{\Re}(ð_1,\breve{a}) \geqslant t, \tilde{\Re}(ð_2,\breve{a}) \geqslant t, \tilde{\Re}(ð_3,\breve{a}) \geqslant t$. Since $\tilde{\Re}$ is a SS of $\mathscr{X}$,
$\max\{\tilde{\Re}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} \geqslant \min\{\tilde{\Re}(ð_1,\breve{a}),\tilde{\Re}(ð_2,\breve{a}),\tilde{\Re}(ð_3,\breve{a}),\tilde{\wp}\} \geqslant t$.
This implies that $ð_1ð_2ð_3 \in \tilde{\Re}_t$.
We have $\tilde{\Im}(ð_1,\breve{a}) \leqslant t, \tilde{\Im}(ð_2,\breve{a}) \leqslant t, \tilde{\Im}(ð_3,\breve{a}) \leqslant t$.
Since $\tilde{\Im}$ is a SS of $\mathscr{X}$,$\min\{\tilde{\Im}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} \leqslant \max\{\tilde{\Im}(ð_1,\breve{a}),\tilde{\Im}(ð_2,\breve{a}),\tilde{\Im}(ð_3,\breve{a}),\tilde{\wp}\} \leqslant t$.
This implies that $ð_1ð_2ð_3 \in \tilde{\Im}_t$.
Therefore $\hbar_t$ is a SS of $\mathscr{X}$ for each $t \in (\tilde{\partial},\tilde{\wp}]$.

Example 4.4. Every IQVFSS $\hbar$ of $\mathscr{X}$ is a $(\tilde{\partial},\tilde{\wp})$-IQVFSS of $\mathscr{X}$, but reverse need not be true.

$$\tilde{\Re}(\kappa,\breve{a}) = \begin{cases} [0.42, 0.47] & if\ \kappa = a \\ [0.35, 0.40] & if\ \kappa = b \\ [0.25, 0.30] & if\ \kappa = c \\ [0.30, 0.35] & if\ \kappa = d \end{cases} \quad \tilde{\Im}(\kappa,\breve{a}) = \begin{cases} [0.27, 0.32] & if\ \kappa = a \\ [0.32, 0.37] & if\ \kappa = b \\ [0.42, 0.47] & if\ \kappa = c \\ [0.37, 0.42] & if\ \kappa = d \end{cases}$$

Here, $\hbar$ is a $([0.33, 0.38], [0.47, 0.52])$-IQVFSS of $\mathscr{X}$, but not a IQVFSS.
Since $\tilde{\Re}(dbd) = 0.25 \ngeqslant \min\{\tilde{\Re}(d,q),\tilde{\Re}(d,q)\} = 0.30$ and $\tilde{\Im}(dbd) = 0.42 \nleqslant \max\{\tilde{\Im}(d,q),\tilde{\Im}(d,q)\} = 0.37$.

Definition 4.5. If $\curlywedge_{\sqsupset}$ is the characteristic function is defined as

$$(\curlywedge_{\sqsupset}^{\top})_{\tilde{\partial}}^{\tilde{\wp}}(ð_1,\breve{a}) = \begin{cases} \tilde{\wp} & if\, ð_1 \in (\sqsupset] \\ \tilde{\partial} & if\, ð_1 \notin (\sqsupset] \end{cases} \quad (\curlywedge_{\sqsupset}^{F})_{\tilde{\partial}}^{\tilde{\wp}}(ð_1,\breve{a}) = \begin{cases} \tilde{\partial} & if\, ð_1 \in (\sqsupset] \\ \tilde{\wp} & if\, ð_1 \notin (\sqsupset] \end{cases}$$

Theorem 4.6. *A non empty subset $\sqsupset$ of $\mathscr{X}$ is a SS $[LI, LAIQVF, RI, TBI]$ of $\mathscr{X}$ if and only if subset $\curlywedge_{(\sqsupset]}$ is a $(\tilde{\partial},\tilde{\wp})$-$IQVFSS[IQVFLI, IQVFLATI, IQVFRI, IQVFBI]$ of $\mathscr{X}$.*

Proof. Assume that $\sqsupset$ is a SS of $\mathscr{X}$. Then $\curlywedge_{(\sqsupset]}$ is a IQVFSS of $\mathscr{X}$ and hence $\curlywedge_{(\sqsupset]}$ is an $(\tilde{\partial},\tilde{\wp})$-IQVFSS of $\mathscr{X}$.
Conversely, Let $\curlywedge_{(\sqsupset]}$ is an $(\tilde{\partial},\tilde{\wp})$-IQVFSS of $\mathscr{X}$. Let $ð_1,ð_2,ð_3 \in \mathscr{X}$ be such that $ð_1,ð_2,ð_3 \in (\sqsupset]$. Then $\curlywedge_{(\sqsupset]}^{\top}(ð_1,\breve{a}) = \tilde{\wp}, \curlywedge_{(\sqsupset]}^{\top}(ð_2,\breve{a}) = \tilde{\wp}, \curlywedge_{(\sqsupset]}^{\top}(ð_3,\breve{a}) = \tilde{\wp}$. Since $\curlywedge_{(\sqsupset]}^{\top}$ is a $(\tilde{\partial},\tilde{\wp})$IQVFSS. Consider

$$\begin{aligned} \max\{\curlywedge_{(\sqsupset]}^{\top}(ð_1ð_2ð_3,\breve{a}),\tilde{\partial}\} &\geqslant \min\{\curlywedge_{(\sqsupset]}^{\top}(ð_1,\breve{a}),\curlywedge_{(\sqsupset]}^{\top}(ð_2,\breve{a}),\curlywedge_{(\sqsupset]}^{\top}(ð_3,\breve{a}),\tilde{\wp}\} \\ &= \min\{\tilde{\wp},\tilde{\wp},\tilde{\wp},\tilde{\wp}\} \\ &= \tilde{\wp} \end{aligned}$$

as $\tilde{\partial} \prec \tilde{\wp}$, this implies that $\curlywedge^{\top}_{(\sqsupset]}(\eth_1\eth_2\eth_3, \breve{a}) \geqslant \tilde{\wp}$. Thus $\eth_1\eth_2\eth_3 \in (\sqsupset]$. Thus $\eth_1\eth_2\eth_3 \in (\sqsupset]$.
Let $\eth_1, \eth_2, \eth_3 \in \mathscr{X}$ be such that $\eth_1, \eth_2, \eth_3 \in (\sqsupset]$. Then $\curlywedge^{F}_{(\sqsupset]}(\eth_1, \breve{a}) = \tilde{\partial}, \curlywedge^{F}_{(\sqsupset]}(\eth_2, \breve{a}) = \tilde{\partial}, \curlywedge^{F}_{(\sqsupset]}(\eth_3, \breve{a}) = \tilde{\partial}$. Since $\curlywedge^{F}_{(\sqsupset]}$ is a $(\tilde{\partial}, \tilde{\wp})$IQVFSS. Consider

$$\begin{aligned}\min\{\curlywedge^{F}_{(\sqsupset]}(\eth_1\eth_2\eth_3, \breve{a}), \tilde{\partial}\} &\leqslant \max\{\curlywedge^{F}_{(\sqsupset]}(\eth_1, \breve{a}), \curlywedge^{F}_{(\sqsupset]}(\eth_2, \breve{a}), \curlywedge^{F}_{(\sqsupset]}(\eth_3, \breve{a}), \tilde{\wp}\}\\ &= \max\{\tilde{\partial}, \tilde{\partial}, \tilde{\partial}, \tilde{\wp}\}\\ &= \tilde{\wp}\end{aligned}$$

as $\tilde{\partial} \prec \tilde{\wp}$, this implies that $\curlywedge^{F}_{(\sqsupset]}(\eth_1\eth_2\eth_3, \breve{a}) \leqslant \tilde{\partial}$. Thus $\eth_1\eth_2\eth_3 \in (\sqsupset]$. Thus $\eth_1\eth_2\eth_3 \in (\sqsupset]$.
Therefore $\sqsupset$ is a SS of $\mathscr{X}$.
Let $\eth_1, \eth_2, \eth_3 \in \mathscr{X}$ be such that $\eth_1, \eth_2, \eth_3 \notin (\sqsupset]$. Then $\curlywedge^{\top}_{(\sqsupset]}(\eth_1, \breve{a}) = \tilde{\partial}, \curlywedge^{\top}_{(\sqsupset]}(\eth_2, \breve{a}) = \tilde{\partial}, \curlywedge^{\top}_{(\sqsupset]}(\eth_3, \breve{a}) = \tilde{\partial}$. Since $\curlywedge^{\top}_{(\sqsupset]}$ is a $(\tilde{\partial}, \tilde{\wp})$IQVFSS.

$$\begin{aligned}\max\{\curlywedge^{\top}_{(\sqsupset]}(\eth_1\eth_2\eth_3, \breve{a}), \tilde{\partial}\} &\geqslant \min\{\curlywedge^{\top}_{(\sqsupset]}(\eth_1, \breve{a}), \curlywedge^{\top}_{(\sqsupset]}(\eth_2, \breve{a}), \curlywedge^{\top}_{(\sqsupset]}(\eth_3, \breve{a}), \tilde{\wp}\}\\ &= \min\{\tilde{\partial}, \tilde{\partial}, \tilde{\partial}, \tilde{\wp}\}\\ &= \tilde{\partial}\end{aligned}$$

as $\tilde{\partial} \prec \tilde{\wp}$, this implies that $\curlywedge^{\top}_{(\sqsupset]}(\eth_1\eth_2\eth_3, \breve{a}) \geqslant \tilde{\partial}$. Thus $\eth_1\eth_2\eth_3 \notin (\sqsupset]$.

Let $\eth_1, \eth_2, \eth_3 \in \mathscr{X}$ be such that $\eth_1, \eth_2, \eth_3 \notin (\sqsupset]$. Then $\curlywedge^{F}_{(\sqsupset]}(\eth_1, \breve{a}) = \tilde{\wp}, \curlywedge^{F}_{(\sqsupset]}(\eth_2, \breve{a}) = \tilde{\wp}, \curlywedge^{F}_{(\sqsupset]}(\eth_3, \breve{a}) = \tilde{\wp}$. Since $\curlywedge^{F}_{(\sqsupset]}$ is a $(\tilde{\partial}, \tilde{\wp})$IQVFSS.

$$\begin{aligned}\min\{\curlywedge^{F}_{(\sqsupset]}(\eth_1\eth_2\eth_3, \breve{a}), \tilde{\partial}\} &\leqslant \max\{\curlywedge^{F}_{(\sqsupset]}(\eth_1, \breve{a}), \curlywedge^{F}_{(\sqsupset]}(\eth_2, \breve{a}), \curlywedge^{F}_{(\sqsupset]}(\eth_3, \breve{a}), \tilde{\wp}\}\\ &= \max\{\tilde{\wp}, \tilde{\wp}, \tilde{\wp}, \tilde{\wp}\}\\ &= \tilde{\wp}\end{aligned}$$

as $\tilde{\partial} \prec \tilde{\wp}$, this implies that $\curlywedge^{F}_{(\sqsupset]}(\eth_1\eth_2\eth_3, \breve{a}) \leqslant \tilde{\wp}$. Thus $\eth_1\eth_2\eth_3 \notin (\sqsupset]$. Therefore $\sqsupset$ is a SS of $\mathscr{X}$.

Definition 4.7. For three IQVFSs $\hbar$, $\eth_2$ and $\varkappa$ of $\mathscr{X}$, their product $\hbar * \eth_2 * \varkappa$ is defined as

$$(\hbar^{\top} * \eth_2^{\top} * \varkappa^{\top})(\eth, \breve{a}) = \begin{cases}\sup\limits_{(r,s,t)\in\sqsupset_{\eth}} \{\hbar^{\top}(r)\triangledown\eth_2^{\top}(s)\triangledown\varkappa^{\top}(t)\}\ if\ \sqsupset_{\eth} \neq 0\\ [0,0]\ otherwise\end{cases}$$

$$(\hbar^{F} * \eth_2^{F} * \varkappa^{F})(\eth, \breve{a}) = \begin{cases}\inf\limits_{(r,s,t)\in\sqsupset_{\eth}} \{\hbar^{F}(r) \vartriangle \eth_2^{F}(s) \vartriangle \varkappa^{F}(t)\}\ if\ \sqsupset_{\eth} \neq 0\\ [1,1]\ otherwise\end{cases}$$

Definition 4.8. Let $\hbar$ be subset of $\mathscr{X}$, we define the subset $(\tilde{\Re})^{\wp}_{\partial}(\eth, \breve{a}) = \{\tilde{\Re}(\eth, \breve{a})\triangledown\tilde{\wp}\} \vartriangle \tilde{\partial}$, $(\tilde{\Im})^{\wp}_{\partial}(\eth, \breve{a}) = \{\tilde{\Im}(\eth, \breve{a}) \vartriangle \tilde{\wp}\}\triangledown\tilde{\partial}$, for all $\eth \in \mathscr{X}$.

Lemma 4.9. *Let $\sqsupset$, $\sqsupset_1$ and $\sqsupset_2$ be subsets of $\mathscr{X}$. Then*

1. $(\curlywedge_{(\sqsupset]}\triangledown\curlywedge_{(\sqsupset_1]}\triangledown\curlywedge_{(\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}} = (\curlywedge_{(\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}$,
2. $(\curlywedge_{(\sqsupset]} \vartriangle \curlywedge_{(\sqsupset_1]} \vartriangle \curlywedge_{(\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}} = (\curlywedge_{(\sqsupset\curlyvee\sqsupset_1\curlyvee\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}$,
3. $(\curlywedge_{(\sqsupset]}*\curlywedge_{(\sqsupset_1]}*\curlywedge_{(\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}} = (\curlywedge_{(\sqsupset\sqsupset_1\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}$.

Proof. (3) Let $\eth_1 \in \mathscr{X}$. If $\eth_1 \in (\sqsupset\sqsupset_1\sqsupset_2]$, then $(\curlywedge_{(\sqsupset\sqsupset_1\sqsupset_2]})(\eth_1, \breve{a}) = \tilde{\wp}$.
Since $\eth_1 \leqslant abc$, $a \in (\sqsupset]$,$b \in (\sqsupset_1]$ and $c \in (\sqsupset_2]$. We have $(a, b, c) \in \sqsupset_{\eth_1}$ and $\sqsupset_{\eth_1} \neq 0$.

$$\begin{aligned}(\leftthreetimes^{\top}_{(\sqsupset]} * \leftthreetimes^{\top}_{(\sqsupset_1]} * \leftthreetimes^{\top}_{(\sqsupset_2]})(\eth_1, \check{a}) &= \sup_{\eth_1 = xyz} \min\{\leftthreetimes^{\top}_{(\sqsupset]}(x, \check{a}), \leftthreetimes^{\top}_{(\sqsupset_1]}(y, \check{a}), \leftthreetimes^{\top}_{(\sqsupset_2]}(z, \check{a})\} \\ &\geqslant \min\{\leftthreetimes^{\top}_{(\sqsupset]}(a, \check{a}), \leftthreetimes^{\top}_{(\sqsupset_1]}(b, \check{a}), \leftthreetimes^{\top}_{(\sqsupset_2]}(c, \check{a})\} \\ &= \tilde{\wp}\end{aligned}$$

$$\begin{aligned}(\leftthreetimes^{F}_{(\sqsupset]} * \leftthreetimes^{F}_{(\sqsupset_1]})(\eth_1, \check{a}) * \leftthreetimes^{F}_{(\sqsupset_2]})(\eth_1, \check{a}) &= \inf_{\eth_1 = xyz} \max\{\leftthreetimes^{F}_{(\sqsupset]}(x, \check{a}), \leftthreetimes^{F}_{(\sqsupset_1]}(y, \check{a}), \leftthreetimes^{F}_{(\sqsupset_2]}(z, \check{a})\} \\ &\leqslant \max\{\leftthreetimes^{F}_{(\sqsupset]}(a, \check{a}), \leftthreetimes^{F}_{(\sqsupset_1]}(b, \check{a}), \leftthreetimes^{F}_{(\sqsupset_2]}(c, \check{a})\} \\ &= \tilde{\partial}\end{aligned}$$

Therefore $(\leftthreetimes_{(\sqsupset]} * \leftthreetimes_{(\sqsupset_1]} * \leftthreetimes_{(\sqsupset_2]})(\eth_1, \check{a}) = (\leftthreetimes_{(\sqsupset\sqsupset_1\sqsupset_2]})(\eth_1, \check{a})$.
If $\eth_1 \notin (\sqsupset\sqsupset_1\sqsupset_2]$ then $(\leftthreetimes^{\top}_{(\sqsupset\sqsupset_1\sqsupset_2]})(\eth_1, \check{a}) = \tilde{\partial}$ and $(\leftthreetimes^{F}_{(\sqsupset\sqsupset_1\sqsupset_2]})(\eth_1, \check{a}) = \tilde{\wp}$. Since $\eth_1 \leqslant abc$, $a \notin (\sqsupset]$, $b \notin (\sqsupset_1]$ and $c \notin (\sqsupset_2]$. We have

$$\begin{aligned}(\leftthreetimes^{\top}_{(\sqsupset]} * \leftthreetimes^{\top}_{(\sqsupset_1]} * \leftthreetimes^{\top}_{(\sqsupset_2]})(\eth_1, \check{a}) &= \sup_{\eth_1 = xyz} \min\{\leftthreetimes^{\top}_{(\sqsupset]}(x, \check{a}), \leftthreetimes^{\top}_{(\sqsupset_1]}(y, \check{a}), \leftthreetimes^{\top}_{(\sqsupset_2]}(z, \check{a})\} \\ &\geqslant \min\{\leftthreetimes^{\top}_{(\sqsupset]}(a, \check{a}), \leftthreetimes^{\top}_{(\sqsupset_1]}(b, \check{a}), \leftthreetimes^{\top}_{(\sqsupset_2]}(c, \check{a})\} \\ &= \tilde{\partial}\end{aligned}$$

$$\begin{aligned}(\leftthreetimes^{F}_{(\sqsupset]} * \leftthreetimes^{F}_{(\sqsupset_1]} * \leftthreetimes^{F}_{(\sqsupset_2]})(\eth_1, \check{a}) &= \inf_{\eth_1 = xyz} \max\{\leftthreetimes^{F}_{(\sqsupset]}(x, \check{a}), \leftthreetimes^{F}_{(\sqsupset_1]}(y, \check{a}), \leftthreetimes^{F}_{(\sqsupset_2]}(z, \check{a})\} \\ &\leqslant \max\{\leftthreetimes^{F}_{(\sqsupset]}(a, \check{a}), \leftthreetimes^{F}_{(\sqsupset_1]}(b, \check{a}), \leftthreetimes^{F}_{(\sqsupset_2]}(c, \check{a})\} \\ &= \tilde{\wp}\end{aligned}$$

Hence $(\leftthreetimes_{(\sqsupset]} * \leftthreetimes_{(\sqsupset_1]} * \leftthreetimes_{(\sqsupset_2]})(\eth_1, \check{a}) = (\leftthreetimes_{(\sqsupset\sqsupset_1\sqsupset_2]})(\eth_1, \check{a})$.

Theorem 4.10. *Let $\{\sqsupset_i | i \in I\}$ be a family of subsets of $\mathscr{X}$ and $\sqsupset, \sqsupset_2 \subseteq \mathscr{X}$. Then*
(1) $(\sqsupset] \subseteq (\sqsupset_2]$ if and only if $(\leftthreetimes_{(\sqsupset]})^{\tilde{\wp}}_{\tilde{\partial}} \leqslant (\leftthreetimes_{(\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}$
(2) $(\curlywedge_{i \in I} \leftthreetimes_{(\sqsupset_i]})^{\tilde{\wp}}_{\tilde{\partial}} = (\leftthreetimes_{\curlywedge_{i \in I}(\sqsupset_i]})^{\tilde{\wp}}_{\tilde{\partial}}$
(3) $(\curlyvee_{i \in I} \leftthreetimes_{(\sqsupset_i]})^{\tilde{\wp}}_{\tilde{\partial}} = (\leftthreetimes_{\curlyvee_{i \in I}(\sqsupset_i]})^{\tilde{\wp}}_{\tilde{\partial}}$.

5 Regular ordered ternary semigroups

Theorem 5.1. *If $\sqsupset$ is a $(\tilde{\partial}, \tilde{\wp})$-IQVFLI[IQVFSS,IQVFLATI,IQVFRI] of $\mathscr{X}$, then $(\sqsupset)^{\tilde{\wp}}_{\tilde{\partial}}$ is a IQVFLI[IQVFSS, IQVFLATI,IQVFRI] of $\mathscr{X}$.*

Theorem 5.2. *Let $\sqsupset$ be an $(\tilde{\partial}, \tilde{\wp})$IQVFRI, $\sqsupset_1$ be an $(\tilde{\partial}, \tilde{\wp})$IQVFLATI and $\sqsupset_2$ be an $(\tilde{\partial}, \tilde{\wp})$IQVFLI of $\mathscr{X}$. Then $((\sqsupset * \sqsupset_1 * \sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}} \subseteq (\sqsupset \curlywedge \sqsupset_1 \curlywedge \sqsupset_2]^{\tilde{\wp}}_{\tilde{\partial}}$.*

Proof. Let $\sqsupset = [\tilde{\Re}_{\sqsupset}, \tilde{\Im}_{\sqsupset}]$ be an $(\tilde{\partial}, \tilde{\wp})$IQVFRI, $\sqsupset_1 = [\tilde{\Re}_{\sqsupset_1}, \tilde{\Im}_{\sqsupset_1}]$ be an $(\tilde{\partial}, \tilde{\wp})$IQVFLATI and $\sqsupset_2 = [\tilde{\Re}_{\sqsupset_2}, \tilde{\Im}_{\sqsupset_2}]$ be an $(\tilde{\partial}, \tilde{\wp})$IQVFLI of $\mathscr{X}$. Let $(\eth_1, \eth_2, \eth_3, \check{a}) \in I_{\hbar}$. If $I_{\hbar} \neq \emptyset$, then $\hbar \leqslant \eth_1\eth_2\eth_3$. Thus $\tilde{\Re}_{\sqsupset}(\hbar, \check{a}) \geqslant \tilde{\Re}_{\sqsupset}(\eth_1\eth_2\eth_3) \geqslant \tilde{\Re}_{\sqsupset}(\eth_1, \check{a})$ and $\tilde{\Im}_{\sqsupset}(\hbar, \check{a}) \leqslant \tilde{\Im}_{\sqsupset}(\eth_1\eth_2\eth_3) \leqslant \tilde{\Im}_{\sqsupset}(\eth_1, \check{a})$.
Similarly $\tilde{\Re}_{\sqsupset_1}(\hbar, \check{a}) \geqslant \tilde{\Re}_{\sqsupset_1}(\eth_1\eth_2\eth_3) \geqslant \tilde{\Re}_{\sqsupset_1}(\eth_2, \check{a})$ and $\tilde{\Im}_{\sqsupset_1}(\hbar, \check{a}) \leqslant \tilde{\Im}_{\sqsupset_1}(\eth_1\eth_2\eth_3) \leqslant \tilde{\Im}_{\sqsupset_1}(\eth_2, \check{a})$.

Similarly, $\tilde{\Re}_{\beth_2}(\hbar,\check{a}) \geqslant \tilde{\Re}_{\beth_2}(\eth_1\eth_2\eth_3) \geqslant \tilde{\Re}_{\beth_2}(\eth_3,\check{a})$ and $\tilde{\Im}_{\beth_2}(\hbar,\check{a}) \leqslant \tilde{\Im}_{\beth_2}(\eth_1\eth_2\eth_3) \leqslant \tilde{\Im}_{\beth_2}(\eth_3,\check{a})$.
We have

$$\begin{aligned}
(\tilde{\Re}_{(\beth*\beth_1*\beth_2]})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\check{a}) &= (\tilde{\Re}_{(\beth*\beth_1*\beth_2]}(\hbar,\check{a})\nabla\tilde{\wp}) \vartriangle \tilde{\partial} \\
&= \Big[[\sup_{\hbar\leqslant\eth_1\eth_2\eth_3}\{\tilde{\Re}_{\beth}(\eth_1,\check{a})\nabla\tilde{\Re}_{\beth_1}(\eth_2,\check{a})\nabla\tilde{\Re}_{\beth_2}(\eth_3,\check{a})\}\nabla\tilde{\wp}]\Big] \vartriangle \tilde{\partial} \\
&= \Big[\sup_{\hbar\leqslant\eth_1\eth_2\eth_3}\{\tilde{\Re}_{\beth}(\eth_1,\check{a})\nabla\tilde{\Re}_{\beth_1}(\eth_2,\check{a})\nabla\tilde{\Re}_{\beth_2}(\eth_3,\check{a})\}\nabla\tilde{\wp}\nabla\tilde{\wp}\nabla\tilde{\wp}\nabla\tilde{\wp}\Big] \vartriangle \tilde{\partial} \\
&= \Big[\sup_{\hbar\leqslant\eth_1\eth_2\eth_3}\{(\tilde{\Re}_{\beth}(\eth_1,\check{a})\nabla\tilde{\wp})\nabla(\tilde{\Re}_{\beth_1}(\eth_2,\check{a})\nabla\tilde{\wp})\nabla(\tilde{\Re}_{\beth_2}(\eth_3,\check{a})\nabla\tilde{\wp})\}\nabla\tilde{\wp}\Big] \vartriangle \tilde{\partial} \\
&\leqslant (\{(\tilde{\Re}_{\beth}(\hbar,\check{a}) \vartriangle \tilde{\partial})\nabla(\tilde{\Re}_{\beth_1}(\hbar,\check{a}) \vartriangle \tilde{\partial})\nabla(\tilde{\Re}_{\beth_2}(\hbar,\check{a}) \vartriangle \tilde{\partial})\}\nabla\tilde{\wp}) \vartriangle \tilde{\partial} \\
&= \{((\tilde{\Re}_{\beth}(\hbar,\check{a})\nabla\tilde{\Re}_{\beth_1}(\hbar,\check{a})\nabla\tilde{\Re}_{\beth_2}(\hbar,\check{a})) \vartriangle \tilde{\partial})\nabla\tilde{\wp}\} \vartriangle \tilde{\partial} \\
&= \{((\tilde{\Re}_{\beth}\nabla\tilde{\Re}_{\beth_1}\nabla\tilde{\Re}_{\beth_2})(\hbar,\check{a})\nabla\tilde{\wp}\} \vartriangle \tilde{\partial} \\
&= (\tilde{\Re}_{\beth\curlywedge\beth_1\curlywedge\beth_2})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\check{a})
\end{aligned}$$

$$\begin{aligned}
(\tilde{\Im}_{(\beth*\beth_1*\beth_2]})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\check{a}) &= (\tilde{\Im}_{(\beth*\beth_1*\beth_2]}(\hbar,\check{a}) \vartriangle \tilde{\wp})\nabla\tilde{\partial} \\
&= \Big[[\inf_{\hbar\leqslant\eth_1\eth_2\eth_3}\{\tilde{\Im}_{\beth}(\eth_1,\check{a}) \vartriangle \tilde{\Im}_{\beth_1}(\eth_2,\check{a}) \vartriangle \tilde{\Im}_{\beth_2}(\eth_3,\check{a})\} \vartriangle \tilde{\wp}]\Big]\nabla\tilde{\partial} \\
&= \Big[\inf_{\hbar\leqslant\eth_1\eth_2\eth_3}\{\tilde{\Im}_{\beth}(\eth_1,\check{a}) \vartriangle \tilde{\Im}_{\beth_1}(\eth_2,\check{a}) \vartriangle \tilde{\Im}_{\beth_2}(\eth_3,\check{a})\} \vartriangle \tilde{\wp} \vartriangle \tilde{\wp} \vartriangle \tilde{\wp} \vartriangle \tilde{\wp}\Big]\nabla\tilde{\partial} \\
&= \Big[\inf_{\hbar\leqslant\eth_1\eth_2\eth_3}\{(\tilde{\Im}_{\beth}(\eth_1,\check{a}) \vartriangle \tilde{\wp}) \vartriangle (\tilde{\Im}_{\beth_1}(\eth_2,\check{a}) \vartriangle \tilde{\wp}) \vartriangle (\tilde{\Im}_{\beth_2}(\eth_3,\check{a}) \vartriangle \tilde{\wp})\} \vartriangle \tilde{\wp}\Big]\nabla\tilde{\partial} \\
&\geqslant (\{(\tilde{\Im}_{\beth}(\hbar,\check{a})\nabla\tilde{\partial}) \vartriangle (\tilde{\Im}_{\beth_1}(\hbar,\check{a})\nabla\tilde{\partial}) \vartriangle (\tilde{\Im}_{\beth_2}(\hbar,\check{a})\nabla\tilde{\partial})\} \vartriangle \tilde{\wp})\nabla\tilde{\partial} \\
&= \{((\tilde{\Im}_{\beth}(\hbar,\check{a}) \vartriangle \tilde{\Im}_{\beth_1}(\hbar,\check{a}) \vartriangle \tilde{\Im}_{\beth_2}(\hbar,\check{a}))\nabla\tilde{\partial}) \vartriangle \tilde{\wp}\}\nabla\tilde{\partial} \\
&= \{((\tilde{\Im}_{\beth} \vartriangle \tilde{\Im}_{\beth_1} \vartriangle \tilde{\Im}_{\beth_2})(\hbar,\check{a}) \vartriangle \tilde{\wp}\}\nabla\tilde{\partial} \\
&= (\tilde{\Im}_{\beth\curlyvee\beth_1\curlyvee\beth_2})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\check{a})
\end{aligned}$$

Let $\eth_1,\eth_2,\eth_3 \notin I_\hbar$. If $I_\hbar = \emptyset$, then $(\tilde{\Re}_{\beth} * \beth_1 * \tilde{\Re}_{\beth_2})(\hbar,\check{a}) = 0$ and $(\tilde{\Im}_{\beth} * \beth_1 * \tilde{\Im}_{\beth_2})(\hbar,\check{a}) = 1$ such that $\hbar \leqslant \eth_1\eth_2\eth_3$.

$$\begin{aligned}
(\tilde{\Re}_{(\beth*\beth_1*\beth_2]})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\check{a}) &= (\tilde{\Re}_{(\beth*\beth_1*\beth_2]}(\hbar,\check{a})\nabla\tilde{\wp}) \vartriangle \tilde{\partial} \\
&= 0 \vartriangle \tilde{\partial} \\
&\leqslant (\tilde{\Re}_{\beth\curlywedge\beth_1\curlywedge\beth_2}(\hbar,\check{a})\nabla\tilde{\wp}) \vartriangle \tilde{\partial} \\
&= (\tilde{\Re}_{\beth\curlywedge\beth_1\curlywedge\beth_2}(\hbar,\check{a})\nabla\tilde{\wp})
\end{aligned}$$

$$\begin{aligned}
(\tilde{\Im}_{(\beth*\beth_1*\beth_2]})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\check{a}) &= (\tilde{\Im}_{(\beth*\beth_1*\beth_2]}(\hbar,\check{a}) \vartriangle \tilde{\wp})\nabla\tilde{\partial} \\
&= 1\nabla\tilde{\partial} \\
&= \tilde{\partial} \\
&\geqslant (\tilde{\Im}_{\beth\curlyvee\beth_1\curlyvee\beth_2}(\hbar,\check{a}) \vartriangle \tilde{\wp})\nabla\tilde{\partial} \\
&= (\tilde{\Im}_{\beth\curlyvee\beth_1\curlyvee\beth_2}(\hbar,\check{a}) \vartriangle \tilde{\wp})
\end{aligned}$$

Therefore $((\beth * \beth_1 * \beth_2])^{\tilde{\wp}}_{\tilde{\partial}} \subseteq ((\beth \curlywedge \beth_1 \curlywedge \beth_2])^{\tilde{\wp}}_{\tilde{\partial}}$.

Theorem 5.3. *Let $\mathscr{L}$ is regular, $\beth$ be an $(\tilde{\partial},\tilde{\wp})$IQVFRI, $\beth_1$ be an $(\tilde{\partial},\tilde{\wp})$ IQVFLATI and $\beth_2$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLI of $\mathscr{L}$ if and only if $((\beth * \beth_1 * \beth_2])^{\tilde{\wp}}_{\tilde{\partial}} = ((\beth \curlywedge \beth_1 \curlywedge \beth_2])^{\tilde{\wp}}_{\tilde{\partial}}$.*

Proof. Let $\mathscr{X}$ regular and $\sqsupset$ be an $(\tilde{\partial},\tilde{\wp})$IQVFRI, $\sqsupset_1$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLATI and $\sqsupset_2$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLI of $\mathscr{X}$. Let $(\eth_1,\eth_3)\in I_\hbar$. If $I_\hbar\neq\emptyset$, then $\hbar\leqslant\eth_1\eth_2\eth_3$. Thus, $\tilde{\Re}_{\sqsupset}(\hbar,\breve{a})\geqslant\tilde{\Re}_{\sqsupset}(\eth_1\eth_2\eth_3)\geqslant\tilde{\Re}_{\sqsupset}(\eth_1,\breve{a})$ and $\tilde{\Im}_{\sqsupset}(\hbar,\breve{a})\leqslant\tilde{\Im}_{\sqsupset}(\eth_1\eth_2\eth_3)\leqslant\tilde{\Im}_{\sqsupset}(\eth_1,\breve{a})$.
Similarly $\tilde{\Re}_{\sqsupset_1}(\hbar,\breve{a})\geqslant\tilde{\Re}_{\sqsupset_1}(\eth_1\eth_2\eth_3)\geqslant\tilde{\Re}_{\sqsupset_1}(\eth_2,\breve{a})$ and $\tilde{\Im}_{\sqsupset_1}(\hbar,\breve{a})\leqslant\tilde{\Im}_{\sqsupset_1}(\eth_1\eth_2\eth_3)\leqslant\tilde{\Im}_{\sqsupset_1}(\eth_2,\breve{a})$.
Similarly, $\tilde{\Re}_{\sqsupset_2}(\hbar,\breve{a})\geqslant\tilde{\Re}_{\sqsupset_2}(\eth_1\eth_2\eth_3)\geqslant\tilde{\Re}_{\sqsupset_2}(\eth_3,\breve{a})$ and $\tilde{\Im}_{\sqsupset_2}(\hbar,\breve{a})\leqslant\tilde{\Im}_{\sqsupset_2}(\eth_1\eth_2\eth_3)\leqslant\tilde{\Im}_{\sqsupset_2}(\eth_3,\breve{a})$.
For $\hbar\in\mathscr{X}$, there exists $x\in\mathscr{X}$ such that $\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar$. Then $\hbar,(z_1\hbar z_2\hbar z_3),\hbar\in I_\hbar$. We have

$$\begin{aligned}
&(\tilde{\Re}_{(\sqsupset*\sqsupset_1*\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\breve{a})\\
&=(\tilde{\Re}_{(\sqsupset*\sqsupset_1*\sqsupset_2]}(\hbar,\breve{a})\triangledown\tilde{\wp})\vartriangle\tilde{\partial}\\
&=\Big[[\sup_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar}\{\tilde{\Re}_{\sqsupset}(\eth_1,\breve{a})\triangledown\tilde{\Re}_{\sqsupset_1}(\eth_2,\breve{a})\triangledown\tilde{\Re}_{\sqsupset_2}(\eth_3,\breve{a})\}\triangledown\tilde{\wp}]\Big]\vartriangle\tilde{\partial}\\
&=\Big[\sup_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar}\{\tilde{\Re}_{\sqsupset}(\eth_1,\breve{a})\triangledown\tilde{\Re}_{\sqsupset_1}(\eth_2,\breve{a})\triangledown\tilde{\Re}_{\sqsupset_2}(\eth_3,\breve{a})\}\triangledown\tilde{\wp}\triangledown\tilde{\wp}\triangledown\tilde{\wp}\triangledown\tilde{\wp}\Big]\vartriangle\tilde{\partial}\\
&=\Big[\sup_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar}\{(\tilde{\Re}_{\sqsupset}(\eth_1,\breve{a})\triangledown\tilde{\wp})\triangledown(\tilde{\Re}_{\sqsupset_1}(\eth_2,\breve{a})\triangledown\tilde{\wp})\triangledown(\tilde{\Re}_{\sqsupset_2}(\eth_3,\breve{a})\triangledown\tilde{\wp})\}\triangledown\tilde{\wp}\Big]\vartriangle\tilde{\partial}\\
&\geqslant(\{(\tilde{\Re}_{\sqsupset}(\hbar,\breve{a})\vartriangle\tilde{\partial})\triangledown(\tilde{\Re}_{\sqsupset_1}(z_1\hbar z_2\hbar z_3)\vartriangle\tilde{\partial})\triangledown(\tilde{\Re}_{\sqsupset_2}(\hbar,\breve{a})\vartriangle\tilde{\partial})\}\triangledown\tilde{\wp})\vartriangle\tilde{\partial}\\
&\geqslant(\{(\tilde{\Re}_{\sqsupset}(\hbar,\breve{a})\vartriangle\tilde{\partial})\triangledown(\tilde{\Re}_{\sqsupset_1}(\hbar,\breve{a})\vartriangle\tilde{\partial})\triangledown(\tilde{\Re}_{\sqsupset_2}(\hbar,\breve{a})\vartriangle\tilde{\partial})\}\triangledown\tilde{\wp})\vartriangle\tilde{\partial}\\
&=\{((\tilde{\Re}_{\sqsupset}(\hbar,\breve{a})\triangledown\tilde{\Re}_{\sqsupset_1}(\hbar,\breve{a})\triangledown\tilde{\Re}_{\sqsupset_2}(\hbar,\breve{a}))\vartriangle\tilde{\partial})\triangledown\tilde{\wp}\}\vartriangle\tilde{\partial}\\
&=\{((\tilde{\Re}_{\sqsupset}\triangledown\tilde{\Re}_{\sqsupset_1}\triangledown\tilde{\Re}_{\sqsupset_2})(\hbar,\breve{a})\triangledown\tilde{\wp}\}\vartriangle\tilde{\partial}\\
&=(\tilde{\Re}_{\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\breve{a})
\end{aligned}$$

$$\begin{aligned}
&(\tilde{\Im}_{(\sqsupset*\sqsupset_1*\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\breve{a})\\
&=(\tilde{\Im}_{(\sqsupset*\sqsupset_1*\sqsupset_2]}(\hbar,\breve{a})\vartriangle\tilde{\wp})\triangledown\tilde{\partial}\\
&=\Big[[\inf_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar}\{\tilde{\Im}_{\sqsupset}(\eth_1,\breve{a})\vartriangle\tilde{\Im}_{\sqsupset_1}(\eth_2,\breve{a})\vartriangle\tilde{\Im}_{\sqsupset_2}(\eth_3,\breve{a})\}\vartriangle\tilde{\wp}]\Big]\triangledown\tilde{\partial}\\
&=\Big[\inf_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar}\{\tilde{\Im}_{\sqsupset}(\eth_1,\breve{a})\vartriangle\tilde{\Im}_{\sqsupset_1}(\eth_2,\breve{a})\vartriangle\tilde{\Im}_{\sqsupset_2}(\eth_3,\breve{a})\}\vartriangle\tilde{\wp}\vartriangle\tilde{\wp}\vartriangle\tilde{\wp}\vartriangle\tilde{\wp}\Big]\triangledown\tilde{\partial}\\
&=\Big[\inf_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar}\{(\tilde{\Im}_{\sqsupset}(\eth_1,\breve{a})\vartriangle\tilde{\wp})\vartriangle(\tilde{\Im}_{\sqsupset_1}(\eth_2,\breve{a})\vartriangle\tilde{\wp})\vartriangle(\tilde{\Im}_{\sqsupset_2}(\eth_3,\breve{a})\vartriangle\tilde{\wp})\}\vartriangle\tilde{\wp}\Big]\triangledown\tilde{\partial}\\
&\leqslant(\{(\tilde{\Im}_{\sqsupset}(\hbar,\breve{a})\triangledown\tilde{\partial})\vartriangle(\tilde{\Im}_{\sqsupset_1}(z_1\hbar z_2\hbar z_3)\triangledown\tilde{\partial})\vartriangle(\tilde{\Im}_{\sqsupset_2}(\hbar,\breve{a})\triangledown\tilde{\partial})\}\vartriangle\tilde{\wp})\triangledown\tilde{\partial}\\
&\leqslant(\{(\tilde{\Im}_{\sqsupset}(\hbar,\breve{a})\triangledown\tilde{\partial})\vartriangle(\tilde{\Im}_{\sqsupset_1}(\hbar,\breve{a})\triangledown\tilde{\partial})\vartriangle(\tilde{\Im}_{\sqsupset_2}(\hbar,\breve{a})\triangledown\tilde{\partial})\}\vartriangle\tilde{\wp})\triangledown\tilde{\partial}\\
&=\{((\tilde{\Im}_{\sqsupset}(\hbar,\breve{a})\vartriangle\tilde{\Im}_{\sqsupset_1}(\hbar,\breve{a})\vartriangle\tilde{\Im}_{\sqsupset_2}(\hbar,\breve{a}))\triangledown\tilde{\partial})\vartriangle\tilde{\wp}\}\triangledown\tilde{\partial}\\
&=\{((\tilde{\Im}_{\sqsupset}\vartriangle\tilde{\Im}_{\sqsupset_1}\vartriangle\tilde{\Im}_{\sqsupset_2})(\hbar,\breve{a})\vartriangle\tilde{\wp}\}\triangledown\tilde{\partial}\\
&=(\tilde{\Im}_{\sqsupset\curlyvee\sqsupset_1\curlyvee\sqsupset_2})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\breve{a})
\end{aligned}$$

Thus, $((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}\supseteq((\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$ and by Theorem 5.2.
Hence, $((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}=((\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$.
Conversely assume that $((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}=((\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$. Let $\sqsupset=(\tilde{\Re}_{\sqsupset},\tilde{\Im}_{\sqsupset})$ be an $(\tilde{\partial},\tilde{\wp})$IQVFRI, $\sqsupset_1=(\tilde{\Re}_{\sqsupset_1},\tilde{\Im}_{\sqsupset_1})$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLATI and $\sqsupset_2=(\tilde{\Re}_{\sqsupset_2},\tilde{\Im}_{\sqsupset_2})$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLI of $\mathscr{X}$. Then by Theorem 4.6, $\curlywedge_{\sqsupset}$ is a $(\tilde{\partial},\tilde{\wp})$IQVFRI, $\curlywedge_{\sqsupset_1}$ is a $(\tilde{\partial},\tilde{\wp})$IQVFLATI and $\curlywedge_{\sqsupset_2}$ be a $(\tilde{\partial},\tilde{\wp})$IQVFLI of $\mathscr{X}$. By Lemma 4.9 and Theorem 4.10, $(\curlywedge_{(\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}=(\curlywedge_{\sqsupset}\curlywedge\curlywedge_{\sqsupset_1}\curlywedge\curlywedge_{\sqsupset_2})^{\tilde{\wp}}_{\tilde{\partial}}=(\curlywedge_{\sqsupset}*\curlywedge_{\sqsupset_1}*\curlywedge_{\sqsupset_2})^{\tilde{\wp}}_{\tilde{\partial}}=(\curlywedge_{(\sqsupset*\sqsupset_1*\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}$. This implies $(\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2]^{\tilde{\wp}}_{\tilde{\partial}}=((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$. Hence by Corollary 2.5, $\mathscr{X}$ is regular.

Theorem 5.4. *Let $\mathscr{X}$ is regular, $\sqsupset$ be an $(\tilde{\partial},\tilde{\wp})$IQVFBI, $\sqsupset_1$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLATI and $\sqsupset_2$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLI of $\mathscr{X}$ if and only if $((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}=((\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$.*

Proof. Let $\mathscr{X}$ be regular semigroup and $\sqsupset$ be an $(\tilde{\partial},\tilde{\wp})$IQVFBI and $\sqsupset_2$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLI of $\mathscr{X}$. Let $(\eth_1,\eth_3)\in I_\hbar$. If $I_\hbar\neq\emptyset$, then $\hbar\leqslant\eth_1\eth_2\eth_3$. Thus $\tilde{\Re}_{\sqsupset}(\hbar,\breve{a})\geqslant\tilde{\Re}_{\sqsupset}(\eth_1\eth_2\eth_3)\geqslant\tilde{\Re}_{\sqsupset}(\eth_1,\breve{a})$ and $\tilde{\Im}_{\sqsupset}(\hbar,\breve{a})\leqslant\tilde{\Im}_{\sqsupset}(\eth_1\eth_2\eth_3)\leqslant\tilde{\Im}_{\sqsupset}(\eth_1,\breve{a})$.

Similarly $\tilde{\Re}_{\sqsupset_1}(\hbar,\breve{a}) \geqslant \tilde{\Re}_{\sqsupset_1}(\eth_1\eth_2\eth_3) \geqslant \tilde{\Re}_{\sqsupset_1}(\eth_2,\breve{a})$ and $\tilde{\Im}_{\sqsupset_1}(\hbar,\breve{a}) \leqslant \tilde{\Im}_{\sqsupset_1}(\eth_1\eth_2\eth_3) \leqslant \tilde{\Im}_{\sqsupset_1}(\eth_2,\breve{a})$.
Similarly, $\tilde{\Re}_{\sqsupset_2}(\hbar,\breve{a}) \geqslant \tilde{\Re}_{\sqsupset_2}(\eth_1\eth_2\eth_3) \geqslant \tilde{\Re}_{\sqsupset_2}(\eth_3,\breve{a})$ and $\tilde{\Im}_{\sqsupset_2}(\hbar,\breve{a}) \leqslant \tilde{\Im}_{\sqsupset_2}(\eth_1\eth_2\eth_3) \leqslant \tilde{\Im}_{\sqsupset_2}(\eth_3,\breve{a})$.
For $\hbar \in \mathscr{X}$, there exists $x \in \mathscr{X}$ such that $\hbar \leqslant \hbar z_1\hbar z_2\hbar z_3\hbar z_4\hbar z_5\hbar$.
Then $\hbar \leqslant (\hbar z_1\hbar z_2\hbar,\breve{a}),(z_3\hbar z_4\hbar z_5),\hbar \in I_\hbar$. We have

$$\begin{aligned}
&(\tilde{\Re}_{(\sqsupset*\sqsupset_1*\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\breve{a})\\
&= (\tilde{\Re}_{(\sqsupset*\sqsupset_1*\sqsupset_2]}(\hbar,\breve{a})\nabla\tilde{\wp}) \vartriangle \tilde{\partial}\\
&= \Big[\big[\sup_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar z_4\hbar z_5\hbar}\{\tilde{\Re}_{\sqsupset}(\eth_1,\breve{a})\nabla\tilde{\Re}_{\sqsupset_1}(\eth_2,\breve{a})\nabla\tilde{\Re}_{\sqsupset_2}(\eth_3,\breve{a})\}\nabla\tilde{\wp}\big]\Big] \vartriangle \tilde{\partial}\\
&= \Big[\sup_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar z_4\hbar z_5\hbar}\{\tilde{\Re}_{\sqsupset}(\eth_1,\breve{a})\nabla\tilde{\Re}_{\sqsupset_1}(\eth_2,\breve{a})\nabla\tilde{\Re}_{\sqsupset_2}(\eth_3,\breve{a})\}\nabla\tilde{\wp}\nabla\tilde{\wp}\nabla\tilde{\wp}\nabla\tilde{\wp}\Big] \vartriangle \tilde{\partial}\\
&= \Big[\sup_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar z_4\hbar z_5\hbar}\{(\tilde{\Re}_{\sqsupset}(\eth_1,\breve{a})\nabla\tilde{\wp})\nabla(\tilde{\Re}_{\sqsupset_1}(\eth_2,\breve{a})\nabla\tilde{\wp})\nabla(\tilde{\Re}_{\sqsupset_2}(\eth_3,\breve{a})\nabla\tilde{\wp})\}\nabla\tilde{\wp}\Big] \vartriangle \tilde{\partial}\\
&\geqslant (\{(\tilde{\Re}_{\sqsupset}(\hbar z_1\hbar z_2\hbar,\breve{a}) \vartriangle \tilde{\partial})\nabla(\tilde{\Re}_{\sqsupset_1}(z_3\hbar z_4\hbar z_5) \vartriangle \tilde{\partial})\nabla(\tilde{\Re}_{\sqsupset_2}(\hbar,\breve{a}) \vartriangle \tilde{\partial})\}\nabla\tilde{\wp}) \vartriangle \tilde{\partial}\\
&\geqslant (\{(\tilde{\Re}_{\sqsupset}(\hbar,\breve{a}) \vartriangle \tilde{\partial})\nabla(\tilde{\Re}_{\sqsupset_1}(\hbar,\breve{a}) \vartriangle \tilde{\partial})\nabla(\tilde{\Re}_{\sqsupset_2}(\hbar,\breve{a}) \vartriangle \tilde{\partial})\}\nabla\tilde{\wp}) \vartriangle \tilde{\partial}\\
&= \{((\tilde{\Re}_{\sqsupset}(\hbar,\breve{a})\nabla\tilde{\Re}_{\sqsupset_1}(\hbar,\breve{a})\nabla\tilde{\Re}_{\sqsupset_2}(\hbar,\breve{a})) \vartriangle \tilde{\partial})\nabla\tilde{\wp}\} \vartriangle \tilde{\partial}\\
&= \{((\tilde{\Re}_{\sqsupset}\nabla\tilde{\Re}_{\sqsupset_1}\nabla\tilde{\Re}_{\sqsupset_2})(\hbar,\breve{a})\nabla\tilde{\wp}\} \vartriangle \tilde{\partial}\\
&= (\tilde{\Re}_{\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\breve{a})
\end{aligned}$$

$$\begin{aligned}
&(\tilde{\Im}_{(\sqsupset*\sqsupset_1*\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\breve{a})\\
&= (\tilde{\Im}_{(\sqsupset*\sqsupset_1*\sqsupset_2]}(\hbar,\breve{a}) \vartriangle \tilde{\wp})\nabla\tilde{\partial}\\
&= \Big[\big[\inf_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar z_4\hbar z_5\hbar}\{\tilde{\Im}_{\sqsupset}(\eth_1,\breve{a}) \vartriangle \tilde{\Im}_{\sqsupset_1}(\eth_2,\breve{a}) \vartriangle \tilde{\Im}_{\sqsupset_2}(\eth_3,\breve{a})\} \vartriangle \tilde{\wp}\big]\Big]\nabla\tilde{\partial}\\
&= \Big[\inf_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar z_4\hbar z_5\hbar}\{\tilde{\Im}_{\sqsupset}(\eth_1,\breve{a}) \vartriangle \tilde{\Im}_{\sqsupset_1}(\eth_2,\breve{a}) \vartriangle \tilde{\Im}_{\sqsupset_2}(\eth_3,\breve{a})\} \vartriangle \tilde{\wp} \vartriangle \tilde{\wp} \vartriangle \tilde{\wp} \vartriangle \tilde{\wp}\Big]\nabla\tilde{\partial}\\
&= \Big[\inf_{\hbar\leqslant\hbar z_1\hbar z_2\hbar z_3\hbar z_4\hbar z_5\hbar}\{(\tilde{\Im}_{\sqsupset}(\eth_1,\breve{a}) \vartriangle \tilde{\wp}) \vartriangle (\tilde{\Im}_{\sqsupset_1}(\eth_2,\breve{a}) \vartriangle \tilde{\wp}) \vartriangle (\tilde{\Im}_{\sqsupset_2}(\eth_3,\breve{a}) \vartriangle \tilde{\wp})\} \vartriangle \tilde{\wp}\Big]\nabla\tilde{\partial}\\
&\leqslant (\{(\tilde{\Im}_{\sqsupset}(\hbar z_1\hbar z_2\hbar,\breve{a})\nabla\tilde{\partial}) \vartriangle (\tilde{\Im}_{\sqsupset_1}(z_3\hbar z_4\hbar z_5)\nabla\tilde{\partial}) \vartriangle (\tilde{\Im}_{\sqsupset_2}(\hbar,\breve{a})\nabla\tilde{\partial})\} \vartriangle \tilde{\wp})\nabla\tilde{\partial}\\
&\leqslant (\{(\tilde{\Im}_{\sqsupset}(\hbar,\breve{a})\nabla\tilde{\partial}) \vartriangle (\tilde{\Im}_{\sqsupset_1}(\hbar,\breve{a})\nabla\tilde{\partial}) \vartriangle (\tilde{\Im}_{\sqsupset_2}(\hbar,\breve{a})\nabla\tilde{\partial})\} \vartriangle \tilde{\wp})\nabla\tilde{\partial}\\
&= \{((\tilde{\Im}_{\sqsupset}(\hbar,\breve{a}) \vartriangle \tilde{\Im}_{\sqsupset_1}(\hbar,\breve{a}) \vartriangle \tilde{\Im}_{\sqsupset_2}(\hbar,\breve{a}))\nabla\tilde{\partial}) \vartriangle \tilde{\wp}\}\nabla\tilde{\partial}\\
&= \{((\tilde{\Im}_{\sqsupset} \vartriangle \tilde{\Im}_{\sqsupset_1} \vartriangle \tilde{\Im}_{\sqsupset_2})(\hbar,\breve{a}) \vartriangle \tilde{\wp}\}\nabla\tilde{\partial}\\
&= (\tilde{\Im}_{\sqsupset\curlyvee\sqsupset_1\curlyvee\sqsupset_2})^{\tilde{\wp}}_{\tilde{\partial}}(\hbar,\breve{a})
\end{aligned}$$

Thus $((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}} \supseteq ((\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$ and by Theorem 5.2 and hence $((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}} = ((\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$. Conversely assume that $((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}} = ((\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$. Let $\sqsupset = (\tilde{\Re}_{\sqsupset},\tilde{\Im}_{\sqsupset})$ be an $(\tilde{\partial},\tilde{\wp})$IQVFBI, $\sqsupset_1 = (\tilde{\Re}_{\sqsupset_1},\Xi_{\sqsupset_1},\tilde{\Im}_{\sqsupset_1})$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLATI and $\sqsupset_2 = (\tilde{\Re}_{\sqsupset_2},\tilde{\Im}_{\sqsupset_2})$ be an $(\tilde{\partial},\tilde{\wp})$IQVFLI of $\mathscr{X}$. Then by Theorem 4.6, $\nwarrow_{\sqsupset}$ is a $(\tilde{\partial},\tilde{\wp})$IQVFBI, $\nwarrow_{\sqsupset_1}$ is a $(\tilde{\partial},\tilde{\wp})$IQVFLATI and $\nwarrow_{\sqsupset_2}$ be a $(\tilde{\partial},\tilde{\wp})$IQVFLI of $\mathscr{X}$. By Lemma 4.9 and Theorem 4.10, $(\nwarrow_{(\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}} = (\nwarrow_{\sqsupset}\curlywedge\nwarrow_{\sqsupset_1}\curlywedge\nwarrow_{\sqsupset_2})^{\tilde{\wp}}_{\tilde{\partial}} = (\nwarrow_{\sqsupset}*\nwarrow_{\sqsupset_1}*\nwarrow_{\sqsupset_2})^{\tilde{\wp}}_{\tilde{\partial}} = (\nwarrow_{(\sqsupset*\sqsupset_1*\sqsupset_2]})^{\tilde{\wp}}_{\tilde{\partial}}$. This implies $(\sqsupset\curlywedge\sqsupset_1\curlywedge\sqsupset_2]^{\tilde{\wp}}_{\tilde{\partial}} = ((\sqsupset*\sqsupset_1*\sqsupset_2])^{\tilde{\wp}}_{\tilde{\partial}}$. Hence by Corollary 2.5, $\mathscr{X}$ is regular.

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