Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 7s (2025)

Diophantine Kites: Rational Diagonals and Integer Area Constructions

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Article History:

Received: *25-10-2024*

Revised: *02-12-2024*

Accepted: *12-12-2024*

Abstract:

In this paper, we aim to collect all kites with certain fixed sides in integers, rational diagonals and integer areas. We develop mathematical procedure and python programming to collect all those kites. Also, experimental analyses are done by means of examples.

Keywords: Diophantine equations, Geometrical shape, Integer area, Kite, Rational area, Rational diagonals.

2010 MSC Subject Classification: 57N25, 97G30, 97F40, 11D09, 11D99.

1. Introduction

Geometry is one of the most interesting areas of mathematics. The basic geometry covers a range of shapes and deals with their properties along with the applications. In earlier days itself, Mathematicians involved themselves in finding some parameters regarding such shapes. Some basic shapes are square, triangle, rectangle, and circle. Area and perimeter are the most used parameters by common people.

To make the process of finding the area and perimeter much easier, early Mathematicians launched formulas using the terms which are needed to construct that particular shape. For example, if the side of a square is a, then its area is a^2 and perimeter is $4a$. So many regular shapes are seen along with those formulas.

As an extension, researchers constructed some new shapes by adjoining existing known shapes. Also, they paved an unchallenging way to calculate its area. One such shape is Kite. It looks like the joining of two triangles under some considerations. Using diagonals of kites, we can easily calculate its area. But here the task is to find the area of the kite by splitting it into two isosceles triangles, then find those triangle's area, and adding them up. Heron's formula is used to find the area of those triangles, which is stated as ``If a, b and c are sides of a triangle, then its area is given as $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ ".

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Vol 32 No. 7s (2025)

In [14], (Rusin, 1998) collected rational triangles with equal area. Inspired by his work, this paper was developed. Also in [3], (Dickson, 2013) provided some key ideas for the creation of this problem. Also [(Aassila, 2001), (Hendrik W. Lenstra, 2002), (Nelsen, 2020), (Robin McLean, 1988), (Robin McLean, 1988), (Rusin, 1998), (Gopalan, Kannan, Manju, & Raja, 2016), (Manju, Kannan, & Raja, 2017), (Mahalakshmi, Kannan, & Narasimman, 2022), (Kannan & Mahalakshmi, Some Annotations on Almost And Pseudo Almost Equilateral Rational Rectangles, 2022)] are considered as roots for this paper.

To do our work, we demand the need of classical branch of Mathematics, the Number Theory. Throughout this paper, the vast research area Diophantine Analysis, the study of Diophantine equations, its solutions and its applications, is employed (Dickson, 2013). Especially, the well-known Pythagorean equations and its solutions are used to collect the required kites.

In this paper, we work on four types of kites (based on sides). In section 2, we display the preliminary concepts needed for this work. Section 3 collects all kites with sides n and $n+1$ and having rational diagonal and integer area whereas in section 4, kites with sides n and $n-1$ and with integer area are gathered. In sections 5 and 6, we focus on the same but with sides $n, n \pm r$ respectively. In each section, we include python coding to collect all required kites along with the output and experimental analysis by an example.

2. Preliminaries

Pythagorean equations are among the most pre-eminent Diophantine equations. These equations are of the form $x^2 + y^2 = z^2$. (Titu, Dorin, & Ion, 2010) Depicts that the general solution of the Pythagorean equation $x^2 + y^2 = z^2$ over integers is of the form $x = k(m^2 - l^2)$, $y = 2kml$ and $z = k(m^2 + l^2)$ where $k, m, l \in \mathbb{Z}$ and $m > l$.

A quadrilateral known as a kite has four sides that are grouped into two adjacent pairs of equal-length sides, and the diagonals cross each other at right angles.

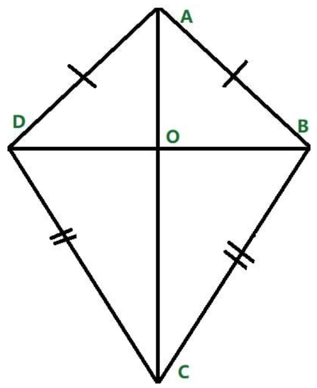

Figure 1: Example of a Kite

Note 2.1.: Throughout this paper, $n \in \mathbb{N}$ and $r \in \mathbb{N} - \{1\}$.

3. Kite with Sides $n, n+1$

We address the kites with sides n and $n+1$ in this part and gather all kites with rational diagonal and integer area by employing the Pythagorean equation and its solutions.

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Let us consider a kite ABCD with sides n and $n + 1$ (Figure 2).

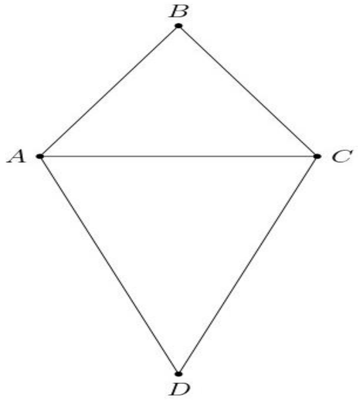

Figure 2: Kite Used in Sections 3 and 5

Take $AB = n = BC$ and $AD = n + 1 = DC$. Let $AC = d$. Then the kite $ABCD$ can be split into two isosceles triangles ABC and ADC. Thus

$$\text{Area of } ABCD = \text{Area of } \Delta\, ABC + \text{Area of } \Delta\, ADC$$

The semi perimeter for $\Delta\, ABC$ is $s = \frac{n+n+d}{2} = \frac{2n+d}{2}$. Thus by applying Heron's formula, we obtain

$$\text{Area of } \Delta\, ABC = \frac{d}{4}\sqrt{4n^2 - d^2} \quad (1)$$

The semi perimeter for $\Delta\, ADC$ is $s = \frac{n+1+n+1+d}{2} = \frac{2n+2+d}{2}$. Thus by applying Heron's formula, we obtain

$$\text{Area of } \Delta\, ADC = \frac{d}{4}\sqrt{(2n+2)^2 - d^2} \quad (2)$$

This implies the fact that the Area of $ABCD \in \mathbb{Z}$, if and only if the Area of $\Delta\, ABD$ and Area of $\Delta\, ADC$ both must be rational numbers.

Area of $\Delta\, ABC$ over $\mathbb{Q}$

``Equation (1)" implies that Area of $\Delta\, ABC \in \mathbb{Q}$ if $4n^2 - d^2 = t_1^2$ for some $t_1 \in \mathbb{Q}$. Now we reduced the problem of finding rational area for $\Delta\, ABC$ to solve the equation $t_1^2 + d^2 = (2n)^2$ over $\mathbb{Q}$. Let's first solve the equation $t_1^2 + d^2 = 1$ over $\mathbb{Q}$ to get started.

Assume that $t_1 = \frac{p}{q}$ and $d = \frac{r}{s}$ for some $p, q, r, s \in \mathbb{Z}$. Then we acquire the Pythagorean type equation $(ps)^2 + (qr)^2 = (qs)^2$. This equation has to be solved over integers.

From section 2, we have $t_1 = \frac{p}{q} = \frac{m^2 - l^2}{m^2 + l^2}$ and $d = \frac{r}{s} = \frac{2ml}{m^2 + l^2}$ for some $m, l \in \mathbb{Z}$. Hence the rational solutions for the equation $t_1^2 + d^2 = (2n)^2$ are $t_1 = 2n\left(\frac{m^2 - l^2}{m^2 + l^2}\right)$ and $d = 4n\left(\frac{ml}{m^2 + l^2}\right)$ for some $m, l \in \mathbb{Z}$ and $m > l$.

Area of $\Delta\, ADC$ over $\mathbb{Q}$

``Equation (2)'' implies that Area of $\Delta\, ADC \in \mathbb{Q}$ if $(2n+2)^2 - d^2 = t_2^2$ for some $t_2 \in \mathbb{Q}$. As in the above case, we have $t_2 = (2n+2)\left(\frac{a^2-b^2}{a^2+b^2}\right)$ and $d = (2n+2)\left(\frac{2ab}{a^2+b^2}\right)$ for some $a, b \in \mathbb{Z}$ and $a > b$.

To find required n

Since the value d is equal in both the areas, we can equate them and find n as

$$n = \frac{ab(m^2+l^2)}{ml(a^2+b^2) - ab(m^2+l^2)}$$

3.1. **Python coding for generation of required kites with sides n and $n+1$**

```
#kite with sides n,n+1
import math
from fractions import Fraction as frac
def kite():
 print('\tm\tl\ta\tb\tn\tn+1\td\tArea')
 for l in range(1,k+1):
  for m in range(l+1,k+1):
    for b in range(1,k+1):
      for a in range(b+1,k+1):
         x = (a*b)*(m**2 + l**2)
         y = (m*l)*(a**2+b**2)-(a*b)*(m**2+l**2)
         if y > 0:
           if x % y ==0:
             n = x//y
             d = frac(4*n*m*l,m**2+l**2)
             p = math.sqrt((4*(n**2)*(d**2))-d**4)
             q = math.sqrt((d**2)*(2*n +2)**2- d**4)
             if (p+q)% 4 ==0:
              area = (p+q)//4
              print('\t',m,'\t',l,'\t',a,'\t',b,'\t',n,'\t',n+1,'\t',d,'\t',area)
k=int(input("Enter maximum range:"))
#k is the maximum range for m,l,a,b
kite()
```

In this section we display python coding to collect kites with sides $n, n+1$, rational diagonals and integer area followed by its output.

Figure 3: Coding 1: Kite with sides $n, n+1$

```
Enter maximum range:10
     m      l      a      b      n      n+1      d       Area
     2      1      3      1      3      4      24/5     12.0
     2      1      6      2      3      4      24/5     12.0
     2      1      9      3      3      4      24/5     12.0
     4      1      9      2      9      10     144/17            72.0
     3      2      7      4      14     15     336/13            168.0
     4      2      6      2      3      4      24/5     12.0
     4      2      9      3      3      4      24/5     12.0
     8      2      9      2      9      10     144/17            72.0
     4      3      6      3      5      6      48/5     24.0
     4      3      6      4      25     26     48       408.0
     4      3      8      4      5      6      48/5     24.0
     4      3      10     5      5      6      48/5     24.0
     4      3      9      6      25     26     48       408.0
     6      3      9      3      3      4      24/5     12.0
     7      3      10     4      20     21     840/29            420.0
     6      4      7      4      14     15     336/13            168.0
     7      4      8      4      13     14     112/5    168.0
     7      4      10     5      13     14     112/5    168.0
     6      5      10     8      122    123    240      5880.0
     7      5      10     7      148    149    280      13860.0
     7      6      8      6      34     35     336/5    504.0
     8      6      9      6      25     26     48       408.0
>>
```

Figure 4: Output: Coding 1

3.2. **Experimental Analysis by an Example**

Usual formula for area of kite is $\frac{1}{2}dd_1$, where d and d_1 are diagonals. In figure 1, take the diagonals as $BD(= d)$, $AC(= d_1)$ and the sides as $AD = n = AB$ and $BC = n + 1 = CD$. Applying Pythagoras theorem for $\Delta\, AOB$ and $\Delta\, BOC$, we get $OA = \frac{1}{2}\sqrt{4n^2 - d^2}$ and $OC = \frac{1}{2}\sqrt{4(n+1)^2 - d^2}$ respectively.

This gives $d_1 = OA + OC = \frac{1}{2}\left[\sqrt{4n^2 - d^2} + \sqrt{4(n+1)^2 - d^2}\right]$.

So the area is $\frac{1}{4}d\left[\sqrt{4n^2 - d^2} + \sqrt{4(n+1)^2 - d^2}\right]$.

Now, let us verify this by an example. Take $n = 20$ and $d = \frac{840}{29}$. Then we obtain $d_1 = 29$. This gives the area 420, is same as the one found from the python coding.

4. Kite with sides $n, n-1$

Utilizing the Pythagorean equation and its solutions, we examine the kites with sides n and $n - 1$ in this section and generate all kites with integer area.

Let us consider a kite $ABCD$ with sides n and $n - 1$ (Figure 5).

Take $AB = n = BC$ and $AD = n - 1 = DC$. Let $AC = d$. Then the kite $ABCD$ can be split into two isosceles triangles ABC and ADC.

Thus, Area of $ABCD$ = Area of $\Delta\, ABC$ + Area of $\Delta\, ADC$

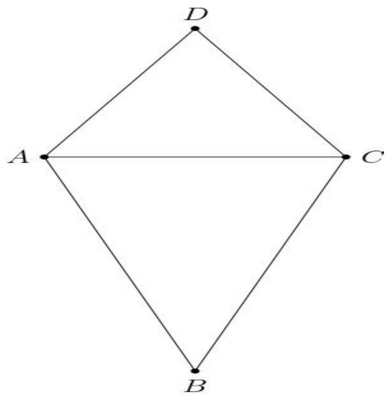

Figure 5: Kite Used in Sections 4 and 6

The semi perimeter for $\Delta\, ABC$ is $s = \frac{n+n+d}{2} = \frac{2n+d}{d}$. Thus by applying Heron's formula, we obtain

Area of $\Delta\, ABC \; = \frac{d}{4}\sqrt{4n^2 - d^2}$ (3)

The semi perimeter for $\Delta\, ADC$ is $s = \frac{n-1+n-1+d}{2} = \frac{2n-2+d}{2}$. Thus by applying Heron's formula, we obtain

Area of $\Delta\, ADC \; = \; \frac{d}{4}\sqrt{(2n-2)^2 - d^2}$ (4)

This implies the fact that the Area of $ABCD \in \mathbb{Z}$, if and only if the Area of $\Delta\, ABD$ and Area of $\Delta\, ADC$ both must be rational numbers.

Area of $\boldsymbol{\Delta\, ABC}$ over $\mathbb{Q}$

``Equation (3)'' implies that Area of $\Delta\, ABC \in \mathbb{Q}$ if $t_1 = 2n\left(\frac{m^2-l^2}{m^2+l^2}\right)$ and $d = 4n\left(\frac{ml}{m^2+l^2}\right)$ for some $m, l \in \mathbb{Z}$ and $m > l$.

Area of $\boldsymbol{\Delta\, ADC}$ over $\mathbb{Q}$

``Equation (4)'' implies that Area of $\Delta\, ADC \in \mathbb{Q}$ if $(2n-2)^2 - d^2 = t_2^2$ for some $t_2 \in \mathbb{Q}$. As in the above case, we have $t_2 = (2n-2)\frac{(a^2-b^2)}{(a^2+b^2)}$ and $d = (2n-2)\left(\frac{2ab}{a^2+b^2}\right)$ for some $a, b \in \mathbb{Z}$ and $a > b$.

To find required n

Since the value d is equal in both the areas, we can equate them and find n as

$$n = \frac{ab(m^2 + l^2)}{ab(m^2 + l^2) - ml(a^2 + b^2)}$$

4.1. Python coding for generation of required kites with sides n and $n-1$

In this section we display python coding to collect kites with sides $n, n-1$, rational diagonals and integer area followed by its output.

Figure 6: Coding 2: Kite with sides $n,\ n-1$

```
#kite with sides n,n-1
import math
from fractions import Fraction as frac
def kite():
 print('\tm\tl\ta\tb\tn\tn-1\td\tArea')
 for l in range(1,k+1):
  for m in range(l+1,k+1):
    for b in range(1,k+1):
      for a in range(b+1,k+1):
          x = (a*b)*(m**2 + l**2)
          y = (a*b)*(m**2+l**2)-(m*l)*(a**2+b**2)
          if y > 0:
            if x % y ==0:
              n = x//y
              d = frac(4*n*m*l,m**2+l**2)
              p = math.sqrt((4*(n**2)*(d**2))-d**4)
              q = math.sqrt((d**2)*(2*n -2)**2- d**4)
              if (p+q)% 4 ==0:
               area = (p+q)//4
               print('\t',m,'\t',l,'\t',a,'\t',b,'\t',n,'\t',n-1,'\t',d,'\t',area)
k=int(input("Enter maximum range:"))
#k is the maximum range for m,l,a,b
kite()
```

Figure 7: Output: Coding 2

```
Enter maximum range:8
        m       l       a       b       n       n-1     d       Area
         2       1       4       3       6       5       48/5    24.0
         2       1       7       4       14      13      112/5   168.0
         2       1       8       6       6       5       48/5    24.0
         3       1       2       1       4       3       24/5    12.0
         3       1       4       2       4       3       24/5    12.0
         3       1       6       3       4       3       24/5    12.0
         3       1       8       4       4       3       24/5    12.0
         3       2       4       3       26      25      48      408.0
         3       2       8       6       26      25      48      408.0
         4       2       4       3       6       5       48/5    24.0
         4       2       7       4       14      13      112/5   168.0
         4       2       8       6       6       5       48/5    24.0
         5       2       7       3       21      20      840/29          420.0
         6       2       4       2       4       3       24/5    12.0
         6       2       6       3       4       3       24/5    12.0
         6       2       8       4       4       3       24/5    12.0
         4       3       7       6       35      34      336/5   504.0
         6       3       4       3       6       5       48/5    24.0
         6       3       7       4       14      13      112/5   168.0
         6       3       8       6       6       5       48/5    24.0
         5       4       6       5       123     122     240     5880.0
         6       4       8       6       26      25      48      408.0
         7       4       6       4       15      14      336/13          168.0
         8       4       7       4       14      13      112/5   168.0
         8       4       8       6       6       5       48/5    24.0
         7       6       8       7       340     339     672     32592.0
         8       6       7       6       35      34      336/5   504.0
>>>
```

4.2. **Experimental Analysis by an Example**

In figure 1, take the diagonals as $BD(=d)$, $AC(=d_1)$ and the sides as $CB = n = CD$ and $AD = n - 1 = AB$. Doing the same procedure as in subsection 3.2, we get

$d_1 = \frac{1}{2}[\sqrt{4n^2 - d^2} + \sqrt{4(n-1)^2 - d^2}\,]$. So the area is $\frac{1}{4}d\left[\sqrt{4n^2 - d^2} + \sqrt{4(n-1)^2 - d^2}\,\right]$.

Now, let us verify this by an example. Take $n = 4$ and $d = \frac{24}{5}$. Then we obtain $d_1 = 5$. This gives the area 12, is same as the one found from the python coding.

5. Kite with Sides $\boldsymbol{n, n+r}$

In this section, we work on kites with sides n and $n+r$, as above. Let us consider a kite ABCD with sides n and $n+r$. In fig 2, take $AB = n = BC$ and $AD = n+r = DC$. Let $AC = d$. Then the kite $ABCD$ can be split into two isosceles triangles ABC and ADC.

Thus,

$$\text{Area of } ABCD = \text{Area of } \Delta\, ABC + \text{Area of } \Delta\, ADC$$

As in the previous sections, we get $\Delta\, ABC$

$$\text{Area of } \Delta\, ABC = \frac{d}{4}\sqrt{4n^2 - d^2} \tag{5}$$

The semi perimeter for $\Delta\, ADC$ is $s = \frac{n+r+n+r+d}{2} = \frac{2n+2r+d}{2}$. Thus by applying Heron's formula, we obtain,

$$\text{Area of } \Delta\, ADC = \frac{d}{4}\sqrt{(2n+2r)^2 - d^2} \tag{6}$$

This implies the fact that the Area of $ABCD \in \mathbb{Z}$, if and only if the Area of $\Delta\, ABD$ and Area of $\Delta\, ADC$ both must be rational numbers.

Area of $\boldsymbol{\Delta\, ABC}$ over $\mathbb{Q}$

``Equation (5)" implies that Area of $\Delta\, ABC \in \mathbb{Q}$ if $t_1 = 2n\left(\frac{m^2-l^2}{m^2+l^2}\right)$ and $d = 4n\left(\frac{ml}{m^2+l^2}\right)$ for some $m, l \in \mathbb{Z}$ and $m > l$.

Area of $\boldsymbol{\Delta\, ADC}$ over $\mathbb{Q}$

``Equation (6)" implies that Area of $\Delta\, ADC \in \mathbb{Q}$ if $(2n+2r)^2 - d^2 = t_2^2$ for some $t_2 \in \mathbb{Q}$. As in the above case, we have $t_2 = (2n+2r)\left(\frac{a^2-b^2}{a^2+b^2}\right)$ and $d = (2n+2r)\left(\frac{2ab}{a^2+b^2}\right)$ for some $a, b \in \mathbb{Z}$ and $a > b$.

To find required $\boldsymbol{n}$

Since the value d is equal in both the areas, we can equate them and find n as

$$n = \frac{rab(m^2+l^2)}{ml(a^2+b^2) - ab(m^2+l^2)}$$

5.1. Python coding for generation of required kites with sides $\boldsymbol{n}$ and $\boldsymbol{n+r}$

In this section we display python coding to collect kites with sides $n, n+r$, rational diagonals and integer area followed by its output.

```
#kite with sides n,n+r
import math
from fractions import Fraction as frac
def kite():
 print('\tm\tl\ta\tb\tn\tn+r\td\tArea')
 for l in range(1,k+1):
  for m in range(l+1,k+1):
    for b in range(1,k+1):
      for a in range(b+1,k+1):
          x = (r*a*b)*(m**2 + l**2)
          y = (m*l)*(a**2+b**2)-(a*b)*(m**2+l**2)
          if y > 0:
            if x % y ==0:
              n = x//y
              d = frac(4*n*m*l,m**2+l**2)
              p = math.sqrt((4*(n**2)*(d**2))-d**4)
              q = math.sqrt((d**2)*(2*n +2*r)**2- d**4)
              if (p+q)% 4 ==0:
               area = (p+q)//4
               print('\t',m,'\t',l,'\t',a,'\t',b,'\t',n,'\t',n+r,'\t',d,'\t',area)
k=int(input("Enter maximum range:"))
r=int(input("Enter the value of r:"))
#k is the maximum range for m,l,a,b
kite()
```

Figure 8: Coding 3: Kite with sides $n, n\ +\ r$

```
Enter maximum range:10
Enter the value of r:10
        m       l       a       b       n       n+r     d       Area
        2       1       3       1       30      40      48      1200.0
        2       1       6       2       30      40      48      1200.0
        2       1       9       3       30      40      48      1200.0
        3       1       9       2       24      34      144/5   720.0
        4       1       9       2       90      100     1440/17         7200.0
        3       2       4       2       65      75      120     4200.0
        3       2       6       3       65      75      120     4200.0
        3       2       7       4       140     150     3360/13         16800.0
        3       2       8       4       65      75      120     4200.0
        3       2       10      5       65      75      120     4200.0
        4       2       6       2       30      40      48      1200.0
        4       2       9       3       30      40      48      1200.0
        6       2       9       2       24      34      144/5   720.0
        8       2       9       2       90      100     1440/17         7200.0
        4       3       6       3       50      60      96      2400.0
        4       3       6       4       250     260     480     40800.0
        4       3       8       4       50      60      96      2400.0
        4       3       10      5       50      60      96      2400.0
        4       3       9       6       250     260     480     40800.0
        5       3       9       5       255     265     450     58500.0
        6       3       9       3       30      40      48      1200.0
        7       3       10      4       200     210     8400/29         42000.0
        5       4       8       6       615     625     1200    186000.0
        6       4       7       4       140     150     3360/13         16800.0
        6       4       8       4       65      75      120     4200.0
        6       4       10      5       65      75      120     4200.0
        7       4       8       4       130     140     224     16800.0
        7       4       10      5       130     140     224     16800.0
        6       5       10      8       1220    1230    2400    588000.0
        7       5       10      7       1480    1490    2800    1386000.0
        8       5       10      6       534     544     960     235200.0
        7       6       8       6       340     350     672     50400.0
        7       6       9       7       510     520     1008    103824.0
        8       6       9       6       250     260     480     40800.0
>>> |
```

Figure 9: Output 3: Kite with sides $n, n\ +\ r$

5.2. Experimental Analysis by an Example

In figure 1, take the diagonals as $BD(=d)$, $AC(=d_1)$ and the sides as $AD = n = AB$ and $BC = n + r = CD$. Doing the same procedure as in subsection 3.2., we get

This gives $d_1 = \frac{1}{2}[\sqrt{4n^2 - d^2} + \sqrt{4(n+r)^2 - d^2}\ ]$.

So the area is $\frac{1}{4}d\left[\sqrt{4n^2 - d^2} + \sqrt{4(n+r)^2 - d^2}\ \right]$.

Now, let us verify this by an example. Take $n = 50$, $r = 10$ and $d = 96$. Then we obtain $d_1 = 50$. This gives the area 2400, is same as the one found from the python coding.

6. Kite with sides $n, n - r$

In this section, we on kites with sides $n - r$, as above. Let us consider a kite $ABCD$ with sides n and $n - r$. In figure 5, take $AB = n = BC$ and $AD = n - r = DC$.

Let $AC = d$. Then the kite $ABCD$ can be split into two isosceles triangles ABC and ADC.

Thus, Area of $ABCD =$ Area of $\Delta ABC +$ Area of ΔADC

As in previous sections, we get Area of $\Delta ABC = \frac{d}{4}\sqrt{4n^2 - d^2}$ (7)

and

Area of $\Delta ADC = \frac{d}{4}\sqrt{(2n + 2r)^2 - d^2}$ (8)

This implies the fact that the Area of $ABCD \in \mathbb{Z}$, if and only if the Area of ΔABD and Area of ΔADC both must be rational numbers.

Area of ΔABC over $\mathbb{Q}$

``Equation (7)" implies that Area of $\Delta ABC \in \mathbb{Q}$ if $t_1 = 2n\left(\frac{m^2-l^2}{m^2+l^2}\right)$ and $d = 4n\left(\frac{ml}{m^2+l^2}\right)$ for some $m, l \in \mathbb{Z}$ and $m > l$.

Area of ΔADC over $\mathbb{Q}$

``Equation (8)" implies that Area of $\Delta ADC \in \mathbb{Q}$ if $(2n - 2r)^2 - d^2 = t_2^2$ for some $t_2 \in \mathbb{Q}$.

As in the above case, we have $t_2 = (2n - 2r)\frac{(a^2-b^2)}{(a^2+b^2)}$ and $d = (2n - 2r)\left(\frac{2ab}{a^2+b^2}\right)$ for some $a, b \in \mathbb{Z}$ and $a > b$.

To find required n

Since the value d is equal in both the areas, we can equate them and find n as

$$n = \frac{rab(m^2 + l^2)}{ab(m^2 + l^2) - ml(a^2 + b^2)}$$

6.1. Python coding for generation of required kites with sides n and $n - r$

In this section we display python coding to collect kites with sides $n,\ n - r$

rational diagonals and integer area followed by its output.

```
#kite with sides n,n-r
import math
from fractions import Fraction as frac
def kite():
 print('\tm\tl\ta\tb\tn\tn-r\td\tArea')
 for l in range(1,k+1):
  for m in range(l+1,k+1):
    for b in range(1,k+1):
      for a in range(b+1,k+1):
         x = (r*a*b)*(m**2 + l**2)
         y = (a*b)*(m**2+l**2)-(m*l)*(a**2+b**2)
         if y > 0:
           if x % y ==0:
             n = x//y
             d = frac(4*n*m*l,m**2+l**2)
             p = math.sqrt((4*(n**2)*(d**2))-d**4)
             q = math.sqrt((d**2)*(2*n -2*r)**2- d**4)
             if (p+q)% 4 ==0:
              area = (p+q)//4
              print('\t',m,'\t',l,'\t',a,'\t',b,'\t',n,'\t',n-r,'\t',d,'\t',area)
k=int(input("Enter maximum range:"))
r=int(input("Enter the value of r:"))
#k is the maximum range for m,l,a,b
kite()
```

Figure 10: Coding 4: Kite with sides $n,\ n-r$

```
Enter maximum range:7
Enter the value of r:6
        m       l       a       b       n       n-r     d       Area
         2       1       3       2       45      39      72      1512.0
         2       1       6       4       45      39      72      1512.0
         2       1       7       4       84      78      672/5   6048.0
         3       1       2       1       24      18      144/5   432.0
         3       1       4       2       24      18      144/5   432.0
         3       1       6       3       24      18      144/5   432.0
         3       1       7       3       35      29      42      1008.0
         3       2       4       3       156     150     288     14688.0
         3       2       7       6       91      85      168     4032.0
         4       2       3       2       45      39      72      1512.0
         4       2       6       4       45      39      72      1512.0
         4       2       7       4       84      78      672/5   6048.0
         5       2       7       3       126     120     5040/29         15120.0
         6       2       4       2       24      18      144/5   432.0
         6       2       6       3       24      18      144/5   432.0
         6       2       7       3       35      29      42      1008.0
         4       3       5       4       375     369     720     66960.0
         4       3       6       5       250     244     480     27360.0
         4       3       7       6       210     204     2016/5          18144.0
         5       3       6       4       136     130     240     13680.0
         5       3       7       6       56      50      1680/17         1680.0
         6       3       6       4       45      39      72      1512.0
         6       3       7       4       84      78      672/5   6048.0
         5       4       6       5       738     732     1440    211680.0
         6       4       7       6       91      85      168     4032.0
         7       4       6       4       90      84      2016/13         6048.0
         6       5       7       6       1281    1275    2520    536760.0
>>>
```

Figure 11: Output 4: Kite with sides $n,\ n-r$

6.2. Experimental Analysis by an Example

In figure 1, take the diagonals as $BD(=d)$, $AC(=d_1)$ and the sides as $CB=n=CD$ and AD$=n-r=AB$. Doing the same procedure as in subsection 3.2., we get

This gives $d_1=\frac{1}{2}\left[\sqrt{4n^2-d^2}+\sqrt{4(n-r)^2-d^2}\,\right]$.

So the area is $\frac{1}{4}d\left[\sqrt{4n^2-d^2}+\sqrt{4(n-r)^2-d^2}\,\right]$.

Now, let us verify this by an example. Take $n = 375$, $r = 6$ and $d = 720$. Then we obtain $d_1 = 186$. This gives the area 66960, is same as the one found from the python coding.

7. Conclusion

In this paper, some particular choices of kites with sides $n, n \pm 1$ and $n, n \pm r$ having integer area and rational diagonal are collected mathematically as well as through Python programming. These types of concepts may be applied in the field of architecture. In future, this thought may extend to other geometrical shapes also.

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