

The T-Generalized P - K Wright Function and its Properties

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Abstract:

In this paper, we introduce t-Generalized p-k Wright function $(\text{r}^{\wedge}(t,p))\Psi_s^{\wedge k}$ and discuss about its convergence condition. We obtain functional relation between t-Generalized p-k Wright function, Generalized p-k Wright function and generalized Wright function and some special cases have also been discussed. We obtain integral representation of t-Generalized p-k Wright function and discussed its properties.

Keywords: The t-Generalized p- k Wright function, Generalized p-k Wright function, Generalized k-Wright function, Generalized Wright function, two parameter Pochhammer symbol, two parameter Gamma function.

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1. Introduction

The two parameter Pochhammer symbol is recently introduced by [8,9], equation 13, in the form of the following definitions:

1.1. Definition

Let $x \in C; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Pochhammer symbol (two parameter), ${}_p(x)_{n,k}$ is given by

$${}_p(x)_{n,k} = \left(\frac{xp}{k}\right)\left(\frac{xp}{k} + p\right)\left(\frac{xp}{k} + 2p\right)\dots\dots\dots\left(\frac{xp}{k} + (n-1)p\right). \tag{1}$$

and the two parameter Gamma function is given by [6].

1.2. Definition

For $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Gamma function (two parameter), ${}_p\Gamma_k(x)$ is given as,

$${}_p\Gamma_k(x) = \frac{1}{k} \lim_{n \rightarrow \infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}. \tag{2}$$

or

$${}_p \Gamma_k(x) = \frac{1}{k} \lim_{n \rightarrow \infty} \frac{n! p^{n+1} (np)^{\frac{x-1}{k}}}{({}_p(x)_{n,k})}. \quad (3)$$

The integral representation of p - k Gamma Function is given by,

$${}_p \Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt. \quad (4)$$

Also it is easy to prove following results,

$${}_p \Gamma_k(x) = \left(\frac{p}{k}\right)^{\frac{x}{k}} \Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma\left(\frac{x}{k}\right). \quad (5)$$

$${}_p(x)_{n,k} = \left(\frac{p}{k}\right)^n (x)_{n,k} = (p)^n \left(\frac{x}{k}\right)_n. \quad (6)$$

$${}_p(x)_{n,k} = \frac{{}_p \Gamma_k(x+nk)}{{}_p \Gamma_k(x)}. \quad (7)$$

$${}_p \Gamma_k(x+k) = \frac{xp}{k} {}_p \Gamma_k(x). \quad (8)$$

$${}_p \Gamma_k(x) {}_p \Gamma_k(-x) = \frac{\pi}{xk} \frac{1}{\sin\left(\frac{\pi x}{k}\right)}. \quad (9)$$

$${}_p \Gamma_k(x) {}_p \Gamma_k(k-x) = \frac{p}{k^2} \frac{\pi}{\sin\left(\frac{\pi x}{k}\right)}. \quad (10)$$

$$n {}_p(x)_{n-1,k} = {}_p(x)_{n,k} - {}_p(x-k)_{n,k}. \quad (11)$$

$${}_p(x)_{n+j,k} = {}_p(x)_{j,k} \times {}_p(x+jk)_{n,k}. \quad (12)$$

2. t-Generalized p-k Wright Function

t-Generalized p-k Wright function is denoted by ${}_{r,t,p} \Psi_S^k$ and defined as

$${}_{r,t,p} \Psi_S^k(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r {}_p \Gamma_k(a_i + \alpha_i(n+t)) z^n}{\prod_{j=1}^s {}_p \Gamma_k(b_j + \beta_j n) (n+t)!} \quad (13)$$

where $p, k \in \mathbb{R}^+ - \{0\}$, $t \in \mathbf{N}_0$; $z \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R} (\alpha_i, \beta_j \neq 0; i = 1, 2, \dots, r; j = 1, 2, \dots, s)$

and $(a_i + \alpha_i n), (b_j + \beta_j n) \in \mathbb{C} \setminus k\mathbb{Z}^-$,

We use following notations for describing convergence condition,

$$\Delta = \sum_{j=1}^s \left(\frac{\beta_j}{k}\right) - \sum_{i=1}^r \left(\frac{\alpha_i}{k}\right); \quad \delta = \prod_{i=1}^r \left|\frac{\alpha_i}{k}\right|^{-\frac{\alpha_i}{k}} \prod_{j=1}^s \left|\frac{\beta_j}{k}\right|^{\frac{\beta_j}{k}};$$

$$\mu = \sum_{j=1}^s \left(\frac{b_j}{k}\right) - \sum_{i=1}^r \left(\frac{a_i}{k}\right) + \frac{r-s}{2}.$$

Theorem 1. For $p, k \in \mathbb{R}^+ - \{0\}; t \in \mathbf{N}_0, z \in \mathbb{C}, \alpha_i, \beta_j \in \mathbb{R}(\alpha_i, \beta_j \neq 0; i = 1, 2, \dots, r; j = 1, 2, \dots, s)$

and $(a_i + \alpha_i n), (b_j + \beta_j n) \in \mathbb{C} \setminus k\mathbb{Z}^-$,

(a) If $\Delta > -1$ then series (13) is absolutely convergent for all $z \in \mathbb{C}$ and t-Generalized p-k

Wright function ${}^t_r \Psi_s^k(z)$ is an entire function of z.

(b) If $\Delta = -1$ then series (13) is absolutely convergent for all $|z| < \delta$

and of $|z| = \delta, \operatorname{Re}(\mu) > \frac{1}{2}$.

Proof: Above theorem can prove easily by using results of Diaz and Pariguan [1], Kilbas [10],

K.S. Gehlot [5,6].

3. Special cases

For some particular values of the parameters, we can obtain certain Wright function and Mittag-Leffler function defined earlier.

(i) For $t = 0$ equation (13), reduces in generalized p-k Wright function [4] as

$${}_r \Psi_s^k(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r p\Gamma_k(a_i + \alpha_i n)}{\prod_{j=1}^s p\Gamma_k(b_j + \beta_j n)} \frac{z^n}{n!}.$$

(ii) For $t = 0, p = k$ equation (13), reduces in generalized k-Wright Function [5] as

$${}_r \Psi_s^k(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r k\Gamma_k(a_i + \alpha_i n)}{\prod_{j=1}^s k\Gamma_k(b_j + \beta_j n)} \frac{z^n}{n!} = {}_r \Psi_s^k(z)$$

(iii) For $p = k = 1$ equation (13), reduces in t-Generalized Wright Function as,

$${}_r \Psi_s^1(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r \Gamma_1(a_i + \alpha_i(n+t))}{\prod_{j=1}^s \Gamma_1(b_j + \beta_j n)} \frac{z^n}{(n+t)!} = {}_r \Psi_s^1(z)$$

(iv) For $p = k$ equation (13), reduces in t-generalized k-Wright Function as,

$${}_r \Psi_s^k(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r k\Gamma_k(a_i + \alpha_i(n+t))}{\prod_{j=1}^s k\Gamma_k(b_j + \beta_j n)} \frac{z^n}{(n+t)!}$$

(v) For $t = 0, p = k$ and $k = 1$ equation (13), reduces in generalized Wright Function [10] as

$${}_1r\psi_s^1(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r \Gamma_1(a_i + \alpha_i n)}{\prod_{j=1}^s \Gamma_1(b_j + \beta_j n)} \frac{z^n}{n!} = {}_r\psi_s(z).$$

(vi) For $t = 0$, $r = 1$, $s = 2$; $a_1 = \gamma$, $\alpha_1 = qk$ where $q \in (0,1) \cup N$; $b_1 = \beta$, $\beta_1 = \alpha$; $b_2 = \gamma$, $\beta_2 = 0$, in equation (13) it reduces in p-k Mittag-Leffler function[7,8] as

$${}_1^p\psi_2^k(z) = \sum_{n=0}^{\infty} \frac{{}_p\Gamma_k(\gamma + qkn)}{{}_p\Gamma_k(\beta + \alpha n) {}_p\Gamma_k(\gamma)} \frac{z^n}{n!} = {}_pE_{k,\alpha,\beta}^{\gamma,q}(z).$$

(vii) For $t = j$, $r = 1$, $s = 2$; $a_1 = \gamma$, $\alpha_1 = qk$ where $q \in (0,1) \cup N$; $b_1 = \beta$, $\beta_1 = \alpha$; $b_2 = \gamma$, $\beta_2 = 0$ in equation (13) it reduces in j-generalized p-k Mittag-Leffler function as,

$${}_1^{j,p}\psi_2^k(z) = \sum_{n=0}^{\infty} \frac{{}_p\Gamma_k(\gamma + qk(n+j))}{{}_p\Gamma_k(\beta + \alpha n) {}_p\Gamma_k(\gamma)(n+j)!} z^n$$

Using equation (7) it can be written as

$${}_1^{j,p}\psi_2^k(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)(n+j)q,k}{{}_p\Gamma_k(\beta + \alpha n) (n+j)!} \frac{z^n}{(n+j)!} = {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z).$$

Now it follows all particular cases of j-generalized p-k Mittag-Leffler function given by K.S Gehlot and Anjana Bhandhari .

(viii) For $t = 0$, $r = 1, s = 2$; and $p = k$, $a_1 = \gamma$, $\alpha_1 = qk$ where $q \in (0,1) \cup N$; $b_1 = \beta$, $\beta_1 = \alpha$; $b_2 = \gamma$, $\beta_2 = 0$, in equation (13) it reduces in generalized k- Mittag-Leffler function[3] as

$${}_1^k\psi_2^k(z) = \sum_{n=0}^{\infty} \frac{{}_k\Gamma_k(\gamma + qkn)}{{}_k\Gamma_k(\beta + \alpha n) {}_k\Gamma_k(\gamma)} \frac{z^n}{n!} = GE_{k,\alpha,\beta}^{\gamma,q}(z).$$

(ix) For $t=0$, $r = 1$, $s = 2$ and $p = k$, $a_1 = \gamma$, $\alpha_1 = k$; $b_1 = \beta$, $\beta_1 = \alpha$; $b_2 = \gamma$, $\beta_2 = 0$, in equation (13) it reduces in k-Mittag-Leffler function[2] as

$${}_1^k\psi_2^k(z) = \sum_{n=0}^{\infty} \frac{{}_k\Gamma_k(\gamma + kn)}{{}_k\Gamma_k(\beta + \alpha n) {}_k\Gamma_k(\gamma)} \frac{z^n}{n!} = E_{k,\alpha,\beta}^{\gamma}(z).$$

(x) For $t = 0$, $r = 1$, $s = 2$; and $p = k = 1$, $a_1 = \gamma$, $\alpha_1 = 1$; $b_1 = \beta$, $\beta_1 = \alpha$; $b_2 = \gamma$, $\beta_2 = 0$, in equation (13) it reduces in Mittag-Leffler function[12] as

$${}_1\psi_2^1(z) = \sum_{n=0}^{\infty} \frac{{}_1\Gamma_1(\gamma + n)}{{}_1\Gamma_1(\beta + \alpha n) {}_1\Gamma_1(\gamma)} \frac{z^n}{n!} = E_{\alpha,\beta}^{\gamma}(z).$$

(xi) For $t=0$ $r = 1$, $s = 2$ and $p = k = 1$, $a_1 = 1$, $\alpha_1 = 1$; $b_1 = \beta$, $\beta_1 = \alpha$; $b_2 = 1$, $\beta_2 = 0$, in equation (13) it reduces in Mittag-Leffler Function[13] as

$${}_1\Psi_2^1(z) = \sum_{n=0}^{\infty} \frac{\Gamma_1(1+n)}{\Gamma_1(\beta + \alpha n)\Gamma_1(1)} \frac{z^n}{n!} = E_{\alpha,\beta}(z).$$

(xii) For $t=0$, $r=1$, $s=2$ and $p=k=1$, $\alpha_1=1$, $\alpha_2=1$; $b_1=1$, $\beta_1=\alpha$; $b_2=1$, $\beta_2=0$, in equation (13) it reduces in Mittag-Leffler function [11] as

$${}_1\Psi_2^1(z) = \sum_{n=0}^{\infty} \frac{\Gamma_1(1+n)}{\Gamma_1(1+\alpha n)\Gamma_1(1)} \frac{z^n}{n!} = E_{\alpha}(z).$$

4. Properties of Generalized p-k Wright Function

We evaluate the functional relation between t-Generalized p-k Wright function, Generalized p-k Wright function and Generalized Wright function and we obtain the recurrence relations of t-Generalized p-k Wright function.

Theorem 2. The functional relation between t-Generalized p-k Wright function and Generalized p-k Wright function, is given by,

$${}_r^p\Psi_S^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}; \end{matrix} z \right] = \frac{p^{\sum_{i=1}^r (\frac{\alpha_i + \alpha_i t}{k}) - \sum_{j=1}^s (\frac{\beta_j}{k})}}{k^{r-s}} {}_r^p\Psi_S^k \left[\begin{matrix} (\frac{\alpha_i}{k}, \frac{\alpha_i t}{k})_{l,r}; \\ (\frac{\beta_j}{k}, \frac{\beta_j}{k})_{l,s}; \end{matrix} z p^{\sum_{i=1}^r (\frac{\alpha_i}{k}) - \sum_{j=1}^s (\frac{\beta_j}{k})} \right] \tag{14}$$

Or the counter part,

$${}_r^p\Psi_S^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}; \end{matrix} z \right] = \frac{p^{\sum_{j=1}^s (\beta_j) - \sum_{i=1}^r (\alpha_i + \alpha_i t)}}{k^{s-r}} {}_r^p\Psi_S^k \left[\begin{matrix} (ka_i, k\alpha_i t)_{l,r}; \\ (kb_j, k\beta_j)_{l,s}; \end{matrix} z p^{\sum_{j=1}^s \beta_j - \sum_{i=1}^r \alpha_i} \right] \tag{15}$$

Proof: Consider the right hand side of (14), and using equation (13), we have,

$$A \equiv {}_r^p\Psi_S^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r p^{\Gamma_k(a_i + \alpha_i(n+t))} z^n}{\prod_{j=1}^s p^{\Gamma_k(b_j + \beta_j n)} (n+t)!}$$

Using equation (5), we have, $A \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r \frac{p^{\frac{\alpha_i + \alpha_i(n+t)}{k}} \Gamma(\frac{\alpha_i + \alpha_i(n+t)}{k}) z^n}{\frac{b_j + \beta_j n}{k}}}{\prod_{j=1}^s \frac{p^{\frac{b_j + \beta_j n}{k}} \Gamma(\frac{b_j + \beta_j n}{k}) (n+t)!}$

$$A \equiv \frac{p^{\sum_{i=1}^r \alpha_i + \alpha_i t - \sum_{j=1}^s \beta_j}}{k^{r-s}} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r \Gamma(\frac{\alpha_i + \alpha_i(n+t)}{k}) (z p^{\sum_{i=1}^r \alpha_i - \sum_{j=1}^s \beta_j})^n}{\prod_{j=1}^s \Gamma(\frac{b_j + \beta_j n}{k}) (n+t)!}$$

$$A \equiv \frac{p^{\sum_{i=1}^r (\frac{\alpha_i}{k} + \frac{\alpha_i t}{k}) - \sum_{j=1}^s (\frac{\beta_j}{k})}}{k^{r-s}} {}_r\Psi_s \left[\begin{matrix} (\frac{\alpha_i}{k}, \frac{\alpha_i t}{k})_{l,r}; \\ (\frac{\beta_j}{k}, \frac{\beta_j}{k})_{l,s}; \end{matrix} \quad z p^{\sum_{i=1}^r (\frac{\alpha_i}{k}) - \sum_{j=1}^s (\frac{\beta_j}{k})} \right]$$

Similarly we can Prove counterpart, (15).

Theorem 3. Let $p, k \in R^+ - \{0\}$; $t \in N_0$ $z \in C$, $\alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1, 2, \dots, r; j = 1, 2, \dots, s)$

and $(a_i + \alpha_i n), (b_j + \beta_j n) \in C \setminus kZ^-$, then,

$$\begin{aligned} & {}_r^t p\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}, (b, \beta); \end{matrix} \quad z \right] = \frac{bp}{k} {}_r^t p\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}, (b+k, \beta); \end{matrix} \quad z \right] + \\ & \frac{\beta zp}{k} \frac{d}{dz} {}_r^t p\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}, (b+k, \beta); \end{matrix} \quad z \right] \end{aligned} \tag{16}$$

Proof: Consider the right hand side of (16) and using the definition of t-Generalized p-k Wright function (13), we have,

$$\begin{aligned} B & \equiv \frac{pb}{k} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r p\Gamma_k(a_i + \alpha_i(n+t)) z^n}{\prod_{j=1}^s p\Gamma_k(b_j + \beta_j n) p\Gamma_k(b+k+\beta n) (n+t)!} \\ & + \frac{\beta zp}{k} \frac{d}{dz} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r p\Gamma_k(a_i + \alpha_i(n+t)) z^n}{\prod_{j=1}^s p\Gamma_k(b_j + \beta_j n) p\Gamma_k(b+k+\beta n) (n+t)!} \end{aligned}$$

Using equation (8), we get,

$$\begin{aligned} B & \equiv \sum_{n=0}^{\infty} \frac{p}{k} \frac{\prod_{i=1}^r p\Gamma_k(a_i + \alpha_i t) (b+\beta n) z^n}{\prod_{j=1}^s p\Gamma_k(b_j + \beta_j n) p\Gamma_k(b+k+\beta n) (n+t)!} \\ B & \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r p\Gamma_k(a_i + \alpha_i(n+t)) z^n}{\prod_{j=1}^s p\Gamma_k(b_j + \beta_j n) p\Gamma_k(b+\beta n) (n+t)!} \end{aligned}$$

this immediately leads to, ${}_r^t p\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}, (b, \beta); \end{matrix} \quad z \right]$

Theorem 4. Let $p, k \in R^+ - \{0\}$; $t \in N_0$ $z \in C$, $\alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1, 2, \dots, r; j = 1, 2, \dots, s)$

and $(a_i + \alpha_i n), (b_j + \beta_j n) \in C \setminus kZ^-$, then,

$${}_{r+1}^t p\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}, (a+k, \alpha kt); \\ (b_j, \beta_j)_{l,s}, (a+k, 0); \end{matrix} \quad z \right] - {}_{r+1}^t p\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}, (a, atk); \\ (b_j, \beta_j)_{l,s}, (a, 0); \end{matrix} \quad z \right]$$

$$= \alpha z {}_p(a+k)_{\alpha-l,k} {}_{r+l}^{t,p}\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}, (a + \alpha k, \alpha k t); \\ (b_j, \beta_j)_{l,s}, (a + \alpha k, 0); \end{matrix} z \right] \tag{17}$$

Proof: Consider the right hand side of (17) and using the definition of t-Generalized p-k Wright function (13),

we have,

$$\begin{aligned} & {}_{r+l}^{t,p}\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}, (a + k, \alpha k t); \\ (b_j, \beta_j)_{l,s}, (a + k, 0); \end{matrix} z \right] \\ & \quad - {}_{r+l}^{t,p}\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}, (a, \alpha k t); \\ (b_j, \beta_j)_{l,s}, (a, 0); \end{matrix} z \right] \\ & \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r {}_p\Gamma_k(a_i + \alpha_i(n+t)) {}_p\Gamma_k(a+k+\alpha k(n+t)) z^n}{\prod_{j=1}^s {}_p\Gamma_k(b_j + \beta_j n) {}_p\Gamma_k(a+k)(n+t)!} \end{aligned}$$

Using equation (11) and (12), we get,

$$\begin{aligned} & \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r {}_p\Gamma_k(a_i + \alpha_i(n+t)) z^n}{\prod_{j=1}^s {}_p\Gamma_k(b_j + \beta_j n) (n+t)!} ({}_p(a+k)_{\alpha(n+t),k} - {}_p(a)_{\alpha(n+t),k}) \\ & \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r {}_p\Gamma_k(a_i + \alpha_i(n+t)) z^n}{\prod_{j=1}^s {}_p\Gamma_k(b_j + \beta_j n) (n+t)!} ((n+t)\alpha {}_p(a+k)_{\alpha(n+t)-l,k}) \\ & \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r {}_p\Gamma_k(a_i + \alpha_i(n+1+t)) z^{n+1}}{\prod_{j=1}^s {}_p\Gamma_k(b_j + \beta_j(n+1)) (n+1+t)!} ((n+1+t)\alpha {}_p(a+k)_{\alpha(n+1+t)-l,k}) \\ & \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r {}_p\Gamma_k(a_i + \alpha_i(n+t) + \alpha_i) z^{n+1} \alpha(n+1+t)}{\prod_{j=1}^s {}_p\Gamma_k(b_j + \beta_j n + \beta_j) (n+1+t)!} ({}_p(a+k)_{\alpha-l,k} {}_p(a+k+(\alpha-l)k)_{(n+t)\alpha,k}) \\ & \equiv \alpha z {}_p(a+k)_{\alpha-l,k} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r {}_p\Gamma_k(a_i + \alpha_i(n+t) + \alpha_i) z^n (n+1+t)}{\prod_{j=1}^s {}_p\Gamma_k(b_j + \beta_j n + \beta_j) (n+1+t)!} {}_p(a+k)_{\alpha(n+t),k} \end{aligned}$$

Using equation (13), we have,

$$= \alpha z {}_p(a+k)_{\alpha-l,k} {}_{r+l}^{t,p}\Psi_{s+1}^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}, (a + \alpha k, \alpha k t); \\ (b_j + \beta_j, \beta_j)_{l,s}, (a + \alpha k, 0); \end{matrix} z \right]$$

5. Integral Representation of Generalized p-k Wright Function

Theorem 5. Let $p, k \in R^+ - \{0\}$; $t \in N_0$ $z \in C$, $\alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1, 2, \dots, r; j = 1, 2, \dots, s)$

and $(a_i + \alpha_i n), (b_j + \beta_j n) \in C \setminus kZ^-$, then,

$${}_{r+l}^{t,p}\Psi_{s+1}^k \left[\begin{matrix} (a_i, k\alpha_i t)_{l,r}; \\ (b_j, k\beta_j)_{l,s}; \end{matrix} z \right]$$

$$= \frac{\prod_{i=1}^r p\Gamma_k(a_i)}{\prod_{j=1}^s p\Gamma_k(b_j)} \prod_{l=1}^{\alpha_i} \prod_{m=1}^{\beta_j} \frac{l}{B(\mu_l, \vartheta_m - \mu_l)} \int_0^1 U^{\mu_l - 1} (1 - U)^{\vartheta_m - \mu_l - 1} e^{\frac{(\alpha_i p)^{\alpha_i} U z}{(p\beta_j)^{\beta_j}}} dU \quad (18)$$

Here $\mu_l = \frac{\frac{\alpha_i + t\alpha_i + l - 1}{k}}{\alpha_i}$ and $\vartheta_m = \frac{\frac{b_j + m - 1}{k}}{\beta_j}$.

Proof: Using the definition of t-Generalized p-k Wright function

$$\begin{aligned} {}_{r,t,p}\Psi_s^k \left[\begin{matrix} (a_i, k\alpha_i t)_{l,r}; \\ (b_j, k\beta_j)_{l,s}; \end{matrix} z \right] &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r p\Gamma_k(a_i + k\alpha_i(n+t))}{\prod_{j=1}^s p\Gamma_k(b_j + k\beta_j n)} \frac{z^n}{(n+t)!} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r p\Gamma_k(a_i) p^{(a_i)t\alpha_i,k} p^{(a_i+t\alpha_i k)n\alpha_i,k}}{\prod_{j=1}^s p\Gamma_k(b_j) p^{(b_j)n\beta_j,k} (n+t)!} z^n \\ &= \sum_{n=0}^{\infty} D \frac{p^{(a_i)t\alpha_i,k} p\Gamma_k(a_i)}{p\Gamma_k(b_j) (n+t)!} z^n \quad (19) \end{aligned}$$

Where, $D \equiv \frac{\prod_{i=1}^r p^{(a_i + \alpha_i t k)n\alpha_i,k}}{\prod_{j=1}^s p^{(b_j)n\beta_j,k}}$.

Using equation (6) we have, $D \equiv \frac{\prod_{i=1}^r p^{n(\alpha_i - \beta_j)} (\frac{\alpha_i + t\alpha_i k}{k})n\alpha_i}{\prod_{j=1}^s (\frac{b_j}{k})n\beta_j}$

Also using the relation given by equation equation (7) we have,

$$D \equiv \frac{\prod_{i=1}^r p^{n(\alpha_i - \beta_j)} (\alpha_i)^{\alpha_i n} \prod_{l=1}^{\alpha_i} (\frac{\alpha_i + t\alpha_i + l - 1}{k})_n}{\prod_{j=1}^s (\beta_j)^{\beta_j n} \prod_{m=1}^{\beta_j} (\frac{b_j + m - 1}{k})_n}$$

Let $\frac{\alpha_i + t\alpha_i + l - 1}{k} = \mu_l$ and $\frac{b_j + m - 1}{k} = \vartheta_m$ then,

$$D \equiv \frac{\prod_{i=1}^r p^{n(\alpha_i - \beta_j)} (\alpha_i)^{\alpha_i n}}{\prod_{j=1}^s (\beta_j)^{\beta_j n}} \prod_{l=1}^{\alpha_i} \prod_{m=1}^{\beta_j} \frac{\Gamma(\mu_l + n)}{\Gamma(\mu_l)} \frac{\Gamma(\vartheta_m)}{\Gamma(\vartheta_m + n)} \frac{\Gamma(\vartheta_m - \mu_l)}{\Gamma(\vartheta_m - \mu_l)} \quad (20)$$

Put equation (20) in equation (19) and also using beta function we get,

$$D = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r p^{n(\alpha_i - \beta_j)} p\Gamma_k(a_i) \alpha_i^{n\alpha_i}}{\prod_{j=1}^s p\Gamma_k(b_j) \beta_j^{n\beta_j}} \prod_{l=1}^{\alpha_i} \prod_{m=1}^{\beta_j} \frac{\Gamma(\vartheta_m)}{\Gamma(\vartheta_m - \mu_l) \Gamma(\mu_l)} \int_0^1 U^{\mu_l + n - 1} (1 - U)^{\vartheta_m - \mu_l - 1} dU$$

above equation can be written in the form of beta function as,

$$D = \frac{\prod_{i=1}^r p\Gamma_k(a_i)}{\prod_{j=1}^s p\Gamma_k(b_j)} \prod_{l=1}^{\alpha_i} \prod_{m=1}^{\beta_j} \frac{l}{B(\mu_l, \vartheta_m - \mu_l)} \int_0^1 U^{\mu_l - 1} (1 - U)^{\vartheta_m - \mu_l - 1} e^{\frac{(\alpha_i p)^{\alpha_i} U z}{(p\beta_j)^{\beta_j}}} dU$$

Theorem 6. Let $p, k \in R^+ - \{0\}$; $t \in N_0$ $z \in C$, $\alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1, 2, \dots, r; j = 1, 2, \dots, s)$

and $(a_i + \alpha_i n), (b_j + \beta_j n) \in C \setminus kZ^-$, then,

$$\begin{aligned} & {}_r^t p \Psi_S^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}; \end{matrix} z \right] = \\ & \frac{p^{\sum_{i=1}^r \frac{\alpha_i}{k}}}{k^r} \int_0^\infty e^{-U} U^{\sum_{i=1}^r (\frac{\alpha_i}{k}) - r} {}_0^t p \Psi_S^k \left[\begin{matrix} - ; \\ (b_j, \beta_j)_{l,s}; \end{matrix} z(pU)^{\sum_{i=1}^r (\frac{\alpha_i t}{k})} \right] dU \end{aligned} \tag{21}$$

Proof: Using the definition of t- Generalized p-k Wright function, we have,

$$\begin{aligned} & {}_r^t p \Psi_S^k \left[\begin{matrix} (a_i, \alpha_i t)_{l,r}; \\ (b_j, \beta_j)_{l,s}; \end{matrix} z \right] \\ &= \sum_{n=0}^\infty \frac{\prod_{i=1}^r p \Gamma_k(a_i + \alpha_i(n+t)) z^n}{\prod_{j=1}^s p \Gamma_k(b_j + \beta_j(n+t)) (n+t)!} \\ &= \sum_{n=0}^\infty \frac{\prod_{i=1}^r \frac{p^{\frac{a_i + \alpha_i(n+t)}{k}}}{k} \Gamma(\frac{a_i + \alpha_i(n+t)}{k}) z^n}{\prod_{j=1}^s p \Gamma_k(b_j + \beta_j n) (n+t)!} \\ &= \sum_{n=0}^\infty \frac{\prod_{i=1}^r \frac{p^{\frac{a_i + \alpha_i(n+t)}{k}}}{k} z^n}{\prod_{j=1}^s p \Gamma_k(b_j + \beta_j n) (n+t)!} \int_0^\infty e^{-U} U^{\frac{a_i + \alpha_i(n+t)}{k} - l} dU \\ &= \frac{p^{\sum_{i=1}^r \frac{\alpha_i}{k}}}{k^r} \int_0^\infty e^{-U} U^{\sum_{i=1}^r (\frac{\alpha_i}{k}) - r} {}_0^t p \Psi_S^k \left[\begin{matrix} - ; \\ (b_j, \beta_j)_{l,s}; \end{matrix} z(pU)^{\sum_{i=1}^r (\frac{\alpha_i t}{k})} \right] dU. \end{aligned}$$

References

- [1] Diaz, R. and Pariguan, E. , On hypergeometric functions and k-Pochhammer symbol, Div. Math. 15(2) (2007),179-192.
- [2] Dorrego, G.A. and Cerutti, R.A. The K-Mittag-Leffler Function. Int. J.Contemp. Math. Sciences, Vol. 7 (2012) No.15, 705-716.
- [3] Gehlot, Kuldeep singh. The Generalized K-Mittag-Leffler function. Int. J. Contemp. Math. Sciences, Vol. 7(2012) No. 45,2213-2219.
- [4] Gehlot, K.S. , Chena Ram and Anita , The Generalized p-k Wright Function and its properties, JETIR, Vol. 5, Issue 8, August (2018), PP.25-36.
- [5] Gehlot, K.S. and J.C. Prajapati. Fractional Calculus of Generalized K-Wright Function. Journal of Fractional Calculus and Applications, Vol. 4(2) July 2013, PP. 283-289.
- [6] Gehlot, K.S. and J.C. Prajapati, On Generalization of K-Wright Functions and its Properties. Pacific Journal of Applied Mathematics, Vol.5, Number 2, 81-88.
- [7] Gehlot, K.S. and C. Ram, Integral representation of K-Series, Int. Trans. Math. Sci. Comp., Vol.4 No. 2(2012),387-396.
- [8] Gehlot, K.S. The p-k Mittag-Leffler function, Palestine Journal of Mathematics 7(2),(2018)628-632.
- [9] Gehlot, K.S. Two Parameter Gamma Function and it's Properties, arXiv:1701.01052v1[math.CA] 3 Jan 2017.

- [10] Kilbas, A.A. Fractional Calculus of the Generalized Wright function, FCAA, Vol.(8), NO 2(2005), 113-126.
- [11] Mathai, A.M, Some properties of Mittag Leffler Functions and matrix-variate analogues: A statistical perspective, Frac. Cal. App. Ana, 13(2), (2010).
- [12] T. R. Prabhakar. A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math. J. 19 (1971), 7-15.
- [13] Wiman, A. Uber den fundamental Satz in der Theories der Funktionen $E_{\alpha}(z)$, Acta Math. 29 (1905) 191-201.