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Invariant Submanifolds of Generalized Sasakian-Space-Forms

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Abstract:

The new characterizations of invariant submanifolds of generalized Sasakian-space forms (SSF) in terms of their behavior with respect to the various curvature tensors are obtained in this work. By examining the connections between these submanifolds' second fundamental form σ and certain curvature tensors Wi with i = 2, 3, 4, 6, 7, we derive necessary and sufficient conditions for their geodesicity. We show that total geodesicity corresponds to the claim that tensor products vanish, Q(σ, Wi) = 0, subject to different non-degeneracy conditions for each curvature tensor. The essential characterization comes from the W6 curvature tensor, which suffices to fulfil 2n(f1−f3)≠0, and all other tensors lead to complementary constraints concerning the structural functions f1, f2, and f3. These findings offer various avenues for studying the geometric nature of invariant submanifolds and enhance our insights into the behaviour of generalized Sasakian-space-forms.

Keywords: Generalized Sasakian-space-forms; invariant submanifold; geodesic; curvature tensors; second fundamental form.

MSC2010: ${}^{53}C_{40}$, ${}^{53}C_{22}$, ${}^{53}D_{15}$.

1. Introduction

The differential geometry and submanifold theory have been productive in getting deep information about the manifold structure. The invariant submanifolds are one of these geometric structures that emerged among many others and have been studied due to their nature of being interesting and giving both an intrinsic and extrinsic geometry of the ambient or underlying space. They are important in studying geometric properties while keeping some structures of the ambient manifold.

Since its early days, the geometric theory of submanifolds has developed through local differential geometry and global analytical methods. In this context, generalized SSF play an ever more central role in studying how contact structures affect differential geometric properties. These manifolds are a natural generalization of classical space forms while still preserving important contact geometric properties. Generalized Sasaki spaces, which, as manifolds, exhibit a rich interplay between metric properties and contact structure, can thus be seen as a natural setup for these studies since they are generalizations of classical Sasakian manifold, which serves as the building blocks to the study of the submanifold behaviour.

Generalized Geometry and Sasakian Geometry have made great strides in the last few decades. Note that a generalized Sasakian-space-form can be considered a natural generalization of metric contact structures relevant to classical ones, known to be contained in the classical space forms. This generalization has attracted considerable attention from researchers worldwide, as evidenced by numerous contributions [2, 3, 4, 5, 12, 14, 15, 16, 18, 20, 22].

This is a more involved way of describing things in terms of curvature tensors. Such tensors, though aligned with the usual suspects from classical curvature measurements, express new aspects of how the contact structure affects geometric properties. Several of these expressions can be reconstructed as some curvature tensors with the second fundamental form from different points of view to examine the second fundamental form and geodesic submanifolds related with each other via these second fundamental form relationships.

By examining the interplay between the respective metric tensors and associated contact structures on such spaces, we uncover striking similarities to the structures of classical differential geometry. These behaviors differ from Riemannian manifolds with constant sectional curvature, as their curvature conditions are determined by the functions f_1, f_2 and f_3. This modification brings in nontrivial geometric features affecting local and global properties of invariant submanifolds. These functions are related by how contact structures generalize classical curvature phenomena which then gives rise to geometric phenomena only found in contact metric geometry.

The primary characteristic that characterizes a generalized Sasakian-space-form is its curvature structure. If M is a nearly contact metric manifold, then f_1, f_2, and f_3 are its three differentiable functions. Given the following form's curvature tensor R:

$$\boldsymbol{R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}} \qquad \textbf{(1.1)}$$

For any vector fields X, Y and Z on M, the manifold with the structure of dimension (2n+1),n>1 is called a generalized Sasakianspace-form if the following condition holds: The condition n>1 will be held in the rest of this paper.

These structures exhibit particularly interesting properties when the functions take specific values. For instance, when $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$, a generalized Sasakian-space-form endowed with a Sasakian structure reduces to a classical Sasakian-space-form.

Hui and Sarkar [18] recently achieved advances in this respect and analyzed W2-flat generalized Sasakian-space-forms in detail. Based on their research, they provide necessary and sufficient conditions for W2-flatness and discuss the consequences of conditions like W_2. S=0. They showed that satisfying W_2 are generalized Sasakian-space-forms. R=0 (either tent with W_2-flatness or certain curvature tensor properties).

Curvature tensor has also been largely established. Tripathi et al. The τ-curvature tensor was then introduced in [1] and later generalized to (k,μ)-contact manifolds in [30], which developed its idea further. This paper presents new results on invariant submanifolds of generalized SSF. This leads us to investigate the necessary and sufficient conditions on the data that guarantee that these submanifolds are totally geodesic; we will obtain some of these conditions concerning different geometric assumptions. We study the situations when the second fundamental form σ satisfies certain relations (W_i=2,3,4,6,7), in terms of the curvature tensors $Q(\sigma,W_2)=0$, $Q(\sigma,W_3)=0$, $Q(\sigma,W_4)=0$, $Q(\sigma,W_6)=$ and $Q(\sigma,W_7)=0$.

The following sections are organized: Section 2 establishes preliminary definitions and fundamental properties of generalized Sasakian space forms and their associated curvature tensors. Section 3 introduces key concepts related to invariant submanifolds. The main results and their proofs are presented in Sections 4 through 8, each addressing specific relationships between the second fundamental form and various curvature tensors.

2. Preliminaries

The structural foundations of generalized SSF emerge from the interplay between metric and contact properties. These manifolds represent a natural extension of constant curvature spaces in contact geometry, much as Kählerian-space-forms extend constant curvature spaces in complex geometry. The defining tensors - φ, ξ, and η - work together to create a rich geometric structure that bridges Riemannian and contact geometry. These tensors are more than algebraic relations; they carry important geometrical information about how the local and global behaviour of the manifold is constructed. In particular, we are interested in their interaction with the Levi-Civita connection, which gives rise to several compatibility conditions that ultimately define the geometry in question. Though these conditions sound somewhat technical, they have far-reaching geometric implications: They tell us how we can "transport" vectors along curves in the manifold.

The algebraic aspects of almost contact metric structures emerge naturally from geometric considerations. The present structures arise as formal generalizations and as a natural outcome of extending Kählerian geometry to odd-dimensional manifolds. Thus, the interrelationships between these structural tensors φ, ξ, and η reveal basic geometric attributes peculiar to this geometry (amongst all geometric structures). These relations provide key information on how generalized Sasakian space forms differ from classical ones.

For this investigation, we rely on the geometric structure of Riemannian manifolds with certain contact metric properties. Let M be a (2n + 1)-dimensional Riemannian manifold with n > 1. Let M be an almost contact metric manifold provided with three important geometric objects:

1. A (1,1)-tensor field denoted by φ
2. A characteristic vector field ξ (known as the structure vector field)
3. A 1-form η

These structures exist on $M^{2n+1}(f_1, f_2, f_3)$ and satisfy the following fundamental relationships:

$$\boldsymbol{\phi^2(X) = -X + \eta(X)\xi, \phi\xi = 0,} \qquad \textbf{(2.1)}$$

$$\boldsymbol{\eta(\xi) = 1, g(X,\xi) = \eta(X), \eta(\phi X) = 0,} \qquad \textbf{(2.2)}$$

$$\boldsymbol{g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),} \qquad \textbf{(2.3)}$$

$$\boldsymbol{g(\phi X, Y) = -g(X, \phi Y),} \qquad \textbf{(2.4)}$$

$$\boldsymbol{(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y).} \qquad \textbf{(2.5)}$$

Equations (2.1)-(2.5) form all essential algebraic substratum of almost contact metric structures. The relationships you describe show how the tensor field φ interacts with the metric structure g and the characteristic vector field ξ. In particular, these equations together guarantee that the metric and contact properties are compatible. Their algebraic form belies a wealth of geometric implications, from the measurement of local angles to global topological properties.

For a generalized Sasakian-space-form, $M^{2n+1}(f_1, f_2, f_3)$, with n > 1, several main geometric properties arise. The properties obtained above from equation (1.1) can be expressed through a series comprising fundamental relationships:

$$\boldsymbol{(\nabla_X \phi)(Y) = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X],} \tag{2.6}$$

$$\boldsymbol{\nabla_X \xi = -(f_1 - f_3)\phi X} \tag{2.7}$$

The differential equations (2.6) and (2.7) give the contact structure's parallel behaviour. As demonstrated in the equations above, the covariant derivatives of φ and ξ are governed by the functions f_1 and f_3, and they point out that generalized SSF have an explicitly prescribed connection. This behaviour sets these spaces apart from other contact metric manifolds and has deep consequences for geodesic behaviour.

$$\boldsymbol{QX = (2nf_1 + 3f_2 - f_3)X - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y),} \tag{2.8}$$

$$\boldsymbol{S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y),} \tag{2.9}$$

$$\boldsymbol{r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3,} \tag{2.10}$$

The curvature properties of these manifolds are characterized by:

$$\boldsymbol{r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3,} \tag{2.10}$$

The Ricci curvature equations (2.8)-(2.10) indicate how the scalar curvature properties of the manifold are affected by the three defining functions f_1, f_2, and f_3. Our results show that generalized SSF have a rich curvature structure that is not limited to the case of constant sectional curvature. In particular, the expression for the scalar curvature reveals how the contact structure enters the overall geometry through certain combinations of these functions.

$$\boldsymbol{R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\},} \tag{2.11}$$

$$\boldsymbol{R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\},} \tag{2.12}$$

$$\boldsymbol{\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\},} \tag{2.13}$$

The curvature relations (2.11) − (2.13) contain important information about the sectional curvature directions and have the characteristic vector field ξ. The definition of f_1 and f_3 reveals the dynamical mechanism whereby the difference f_1-f_3 governs curvature behaviour along the direction of the contact distribution. This geometric interpretation will help us understand why this difference occurs so often in non-degeneracy conditions for the rest of our analysis.

Furthermore, the following relations based on Ricci curvature tensor are:

$$\boldsymbol{S(X,\xi) = 2n(f_1 - f_3)\eta(X),} \tag{2.14}$$

$$\boldsymbol{S(\xi,\xi) = 2n(f_1 - f_3),} \tag{2.15}$$

$$\boldsymbol{Q\xi = 2n(f_1 - f_3)\xi,} \tag{2.16}$$

The special properties of Ricci curvature, whose type is (2.14)--(2.16) in a sense along the characteristic direction ξ, reveal some geometric behaviour of these spaces. One can see that the Ricci curvature in directions joined with ξ mainly depends on f_1 - f_3, which tells us that seeing the way the contact structure behaves in the presence of curvature is somehow an integral eye of the discrepancy between contact hypothetically rigid structure based on the curvature expansion.

This has the simple implication that the various curvature tensors W_i (i = 2, 3, 4, 6, 7) behave as follows concerning ξ:

$$W_2(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + \frac{1}{2n}[2n(f_1 - f_3)\eta(X)\xi - QX], \tag{2.17}$$

$$W_3(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] - \frac{1}{2n}[2n(f_1 - f_3)\eta(X)Y - \eta(Y)QX], \tag{2.18}$$

$$W_4(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + \frac{1}{2n}[\eta(X)QY - g(X,Y)\xi], \tag{2.19}$$

$$W_6(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] - (f_1 - f_3)[\eta(Y)X - g(X,Y)\xi], \tag{2.20}$$

$$W_7(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] - \frac{1}{2n}[2n(f_1 - f_3)\eta(Y)X - \eta(Y)QX]. \tag{2.21}$$

These relationships constitute the mathematical basis for the later study of invariant submanifolds of generalized Sasakian-space-forms.

The W_i curvature tensor equations show increasingly advanced dualities between metric structure, contact properties and curvature. These tensors reflect certain geometry properties that can be appreciated, and they inherit some basic properties from the Riemann curvature tensor. These relationships will be very important in our later analysis of totally geodesic invariant submanifolds.

3. INVARIANT SUBMANIFOLDS OF GENERALIZED SASAKIAN-SPACE-FORMS

Dual to commonness, the structure is generally inherited or induced from the ambient space, and the study of invariant submanifolds in generalized Sasakian-space-forms establishes a balancing act between the two. If a submanifold is φ-invariant, it inherits contact structures and metric properties from its surroundings. With this inheritance comes a refined interaction between the second fundamental form and the ambient space's fundamental tensors. When the second fundamental form σ is considered concerning it, it becomes especially telling to study the structure vector field ξ. The non-vanishing of σ(X,ξ) is thus not a mere technical condition but rather encodes deep geometric information about the submanifold's embedding into the ambient manifold. This relation is essential to understanding the nature of the submanifold's geodesics and its difference from the geodesics of an ambient space.

Suppose we have a submanifold N of a generalized Sasakian-space-form. $M^{2n+1}(f_1, f_2, f_3)$. In order to understand the geometry of such an embedding, one should investigate relations between the connections and fundamental forms of the submanifold and the ambient space.

A couple of the canonical bundles that we consider in this submanifold N are:

1. The tangent bundle TN
2. The normal bundle $N^{\perp}N$

Restricting the ambient geometric structures to the submanifold is required to study submanifolds in generalized Sasakian-space-forms. The behaviour of invariant submanifolds concerning the inheritance of geometric properties is quite different than in other geometric settings. Understanding the relationship between ambient and induced structures yields crucial insight into how contact geometry affects submanifold behaviour. These relations and manifestations are local (involving second fundamental forms, etc.) and global (involving, e.g. curvature properties). The relationship between these structures is governed by the so-called Gauss-Weingarten formulae, which are the main equations describing the geometry of the embedding:

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{3.1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \tag{3.2}$$

These equations hold for all vector fields $X,Y\in\Gamma(TN)$ and $V\in\Gamma(T^{\perp}N)$, where:

- $\nabla^{\perp}$ represents the connection in the normal bundle
- σ denotes the second fundamental form
- A_V is the shape operator corresponding to the normal vector field V

The shape operator and the second fundamental form are intrinsically connected through a relationship that characterizes their geometric interaction:

$$g(\sigma(X,Y),V)=g(AVX,Y)$$

This relationship plays a crucial role in understanding the geometric properties of the submanifold.

A particularly important case arises when examining the behaviour of the second fundamental form concerning the structure vector field ξ:

$$\sigma(X,\xi)=0 \tag{3.3}$$

It plays an important role in the geometry of the submanifold and will be helpful for our analysis in the next sections. The notion of a geodesic submanifold appears in the context where the second fundamental form vanishes identically. This is the most natural embedding of a manifold into another, preserving geodesics and minimizing the amount of geometric distortion.

The relationships we have established in this section provide an essential background for our study of under which conditions invariant submanifolds of generalized Sasakian-space-forms are geodesic. We will follow with conditions satisfying these to determine how they relate to different curvature tensors, which describe the geometric characteristics of these submanifolds.

This intriguing connection of curvature tensors and invariant submanifolds involves many sophisticated geometric aspects that beg special attention. This question, the interaction of these two types of tensors with the second fundamental form, is not just a technical question, but rather tells us a lot about the structure of the ambient space and submanifolds. These entanglements are especially relevant™ when studying conditions for total geodesicity, since they provide insight into how the geometry of the submanifold fits with that of the ambient space.

This includes the study of other curvature tensors Wi, which gives different point of views of the same geometric situation. Each tensor encodes a different facet of the manifold's geometry, akin to how observing a complex sculpture from various vantage points unveils different characteristics of its shape. This multiplicity of perspectives is crucial for constructing a thorough understanding of the geometric structures involved.

4. GEOMETRY OF INVARIANT SUBMANIFOLDS SATISFYING THE W_2 CURVATURE CONDITION

Robust, the curvature for the W_2 is a powerful refinement of the curvature for the Riemann. This twstory has some subtle geometric features which are hidden from the purely classical curvature tensor through the consideration of its interaction with invariant submanifolds. Conditions like $Q(\sigma, W_2) = 0$ go beyond purely algebraic constraints and give some information about the curvature of the submanifold in the ambient space. In this case, the non-degeneracy condition that involves f_2 and f_3 is not an arbitrary technical necessity but a condition that arises naturally because we are working on generalized SSF.

Here, we explore how the geometry of the invariant submanifolds is governed by the interaction of the corresponding second fundamental form σ with a certain curvature tensor W2. The present study addresses when those submanifolds become geodesic under certain curvature conditions.

Analyzing tensor derivations is needed to study the W_2 curvature tensor behaviour under invariant submanifolds. Types of the operator $Q(\sigma, W_2)$—which implements the second fundamental form on the image of the submanifold under the second fundamental map, and σ is intrinsic information by how the submanifold curves inside the ambient space. This relation goes deeper and presents essential geometric features of totally geodesic submanifolds beyond their simple algebraic properties.

Theorem 4.1. Let M be a generalized Sasakian-space-form and N be an invariant submanifold in M. The submanifold N is geodesic in M if and only if $Q(\sigma, W_2)=0$, with the non-degeneracy condition $(2n-1)\{3f_2+(2n-1)f_3\}\neq 0$.

Proof. To handle this systematically, recall that we should consider invariant submanifolds N such that $Q(\sigma, W_2)=0$. This is equivalent to the condition:

$$\mathbf{Q(\sigma,W_2)=Q(\sigma,W_2)(X,Y,Z;U,V)=((U\wedge_\sigma V).W_2)(X,Y)Z=0}$$

This expands to:

$$-\boldsymbol{W_2(X,(U\wedge_\sigma V)Y)Z - W_2(X,Y)(U\wedge_\sigma V)Z} - \mathbf{W2(X,Y)(U\wedge\sigma V)Z = 0} \qquad (4.1)$$

Where the operator $U\wedge_\sigma V$ is defined by:

$$\mathbf{(U\wedge_\sigma V)P=\sigma(V,P)U-\sigma(U,P)V} \qquad \mathbf{(4.2)}$$

Substituting (4.2) into (4.1) yields:

$$-\sigma(V,X)W_2(U,Y)Z+\sigma(U,X)W_2(V,Y)Z-\sigma(V,Y)W_2(X,U)Z+\sigma(U,Y)W_2(X,V)Z-\sigma(V,Z)W_2(X,Y)U+\sigma(U,Z)W_2(X,Y)V=0. \qquad (4.3)$$

Using the algebraic expansion of the tensor derivative, we get multiple terms that depend on second fundamental form and components of W_2 curvature. Thus each term provides different geometric data of how the submanifold folds into the ambient space. A more informative case happens when we compute the inner product between σ and W_2 along the characteristic direction ξ.

When we set Z=V=ξ in (4.3) and apply condition (3.3), we obtain:

$$\sigma(U,X)W_2(\xi,Y)\xi+\sigma(U,Y)W_2(X,\xi)\xi=0 \qquad (4.4)$$

Applying equation (2.17) to (4.4):

$$\sigma(U,X)[(f_1-f_3)\{\eta(Y)\xi-Y\}+\frac{1}{2n}\{QY-2n(f_1-f_3)\eta(Y)\xi\}]+\sigma(U,Y)[(f_1-f_3)\{X-\eta(X)\xi\}+\frac{1}{2n}\{2n(f_1-f_3)\eta(X)\xi-QX\}]=0 \qquad (4.5)$$

It is geometrically remarkable that these equations involve mixed terms of both the metric tensor g and the structure tensor φ. When we turn to the generalized Sasakian-space-forms for which we can form a metric and a contact structure, these combinations are a reflection of that dual structure. These two structures interact through the second fundamental form, governed projectively by the Wi tensors and thus intertwining within the ambient geometry. Above, each of the terms in these equations have some specific geometric meaning - the metric terms are about how the structure preserves distances, whereas the φ-dependent terms store information about how the contact structure twists the embedding.

Taking the inner product with W:

$$\sigma(U,X)\left[(f_1-f_3)\{\eta(Y)\eta(W)-g(Y,W)\}+\frac{1}{2n}\{g(QY,W)-2n(f_1-f_3)\eta(Y)\eta(W)\}\right]+\sigma(U,Y)\left[(f_1-f_3)\{g(X,W)-\eta(X)\eta(W)\}+\frac{1}{2n}\{2n(f_1-f_3)\eta(X)\eta(W)-g(QX,W)\}\right]=0$$

(4.6)0

Contracting Y and W, we get

$$\sigma(U,X)(2n-1)\{3f_2+(2n-1)f_3\}=0 \qquad (4.7)$$

Given our non-degeneracy condition $(2n-1)\{3f_2+(2n-1)f_3\}\neq 0$. we conclude:

$$\sigma(U,X)=0$$

It follows that N is geodesic as a submanifold. The converse follows immediately because a geodesic submanifold trivially satisfies Q(σ, W_2)=0.

This theorem gives us a very clear characterization of totally geodesic invariant submanifolds utilizing the behaviour of their second fundamental form concerning the W_2 curvature tensor. The appearance of the non-degeneracy condition $(2n-1)\{3f_2+(2n-1)f_3\}\neq 0$ encodes important geometric properties of the ambient space. This condition guarantees that the correlation between W2 and σ encodes significant geometric information and does not collapse into degenerate cases. Therefore, this condition with the f_2, f_3 pair is also elementary regarding the contact structure.

5. CHARACTERIZATION OF INVARIANT SUBMANIFOLDS THROUGH W$_3$ CURVATURE INTERACTION

Here, we will see how the geometry of invariant submanifolds and the second fundamental form encode information about the W$_3$ curvature tensor. We give conditions under which such submanifolds are geodesic.

Specifically, the W$_3$ curvature tensor has some unique properties different from other generalized curvature tensors in contact metric geometry. W$_3$ derives from the contact structure, a type of Riemannian structure lacking local equivalences to basic geometric invariants such as the Riemann curvature tensor. The way this tensor acts along the characteristic direction ξ encodes essential contact geometric information about the manifold. These results hold important geometric significance in understanding how the contact structure constrains the possible configurations of totally geodesic submanifolds through the interactions between W$_3$ and invariant submanifolds. This is not just algebraically compatible; the relation of W$_3$ to the second fundamental form has more geometric consequences, namely, concerning how the position of the submanifold is related to the contact structure of the ambient space.

The W$_3$ curvature tensor gives an alternative viewpoint on totally geodesic submanifolds via its special action on the contact structure. In contrast to W$_2$, this tensor includes terms involving generalizations of how the submanifold sits in the ambient space geometry.

Theorem 5.1. Let N be an invariant submanifold of a generalized Sasakian-space-form M. The submanifold N becomes totally geodesic if and only if Q(σ,W3)=0, provided that that $\{4n(1-2n)f_1+3f_2+(4n+1)(2n-1)f_3\}\neq 0.$

Proof. Consider an invariant submanifold N satisfying Q(σ, W$_3$)=0. This condition can be expressed through the derivation:

$$Q(\sigma,W_3)(X,Y,Z;U,V)=\big((U\wedge_\sigma V).W_3\big)X,Y)Z=0$$

Which expands to:

$$=-W_3\big((U\wedge_\sigma V)X,Y\big)Z-W_3(X,(U\wedge_\sigma V)Y)Z-\ W_3(X,Y)(U\wedge_\sigma V)Z) \qquad \textbf{(5.1)}$$

Applying the definition of the operator U∧$_\sigma$ to equation (5.1) yields:

$$-\sigma(V,X)W_3(U,Y)Z+\sigma(U,X)W_3(V,Y)Z-\sigma(V,Y)W_3(X,U)Z+\sigma(U,Y)W_3(X,V)Z-\sigma(V,Z)W_3(X,Y)U+\sigma(U,Z)W_3(X,Y)V=0 \qquad \textbf{(5.2)}$$

The tensor W$_3$ shows features that are different from the one shown by W$_2$ when we analyze the behaviour along the characteristic direction ξ. As the structural equations show, contact metric properties affect the geometric correspondence concerning the submanifold and ambient space. Inclusion of Z=V=ξ in (5.2) and Dependency using condition (3.3):

$$\sigma(U,X)W_3(\xi,Y)\xi+\sigma(U,Y)W_3(X,\xi)\xi=0 \qquad \textbf{(5.3)}$$

Substituting equation (2.18):

$$\boldsymbol{\sigma(U,X)[2(f_1-f_3)\{\eta(Y)\xi-Y\}]+\sigma(U,Y)\Big[(f_1-f_3)\{X-\eta(X)\xi\}-\frac{1}{2n}\{2n(f_1-f_3)\eta(X)\xi-QX\}\Big]=0} \qquad \textbf{(5.4)}$$

Taking the inner product with W:

$$\boldsymbol{\sigma(U,X)[2(f_1-f_3)\{\eta(Y)\eta(W)-g(Y,W)\}]+\sigma(U,Y)\Big[(f_1-f_3)\{g(X,W)-\eta(X)\eta(W)\}-\frac{1}{2n}\{2n(f_1-f_3)\eta(X)\eta(W)-g(QX,W)\}\Big]=0} \qquad \textbf{(5.5)}$$

Contracting Y and W leads to:

$$\boldsymbol{\sigma(U,X)\{4n(1-2n)f_1+3f_2+(4n+1)(2n-1)f_3\}=0.} \qquad \textbf{(5.6)}$$

Under our non-degeneracy condition $\{4n(1-2n)f_1+3f_2+(4n+1)(2n-1)f_3\}\neq 0$, we conclude:

$$\boldsymbol{\sigma(U,X)=0}$$

This shows that N is geodesic. The other direction is also straightforward because geodesic submanifolds are automatically of class Q by definition ($Q(\sigma, W_3)=0$).

The theorem gives a manifold-analytical property of these invariant submanifolds as totally geodesical, thus characterizing completely different invariant submanifolds concerning their interaction with the W3 curvature tensor, adding another piece of the puzzle to our existing results with W_2.

In W_2, the corresponding non-degeneracy condition is direct and simple, whereas, for W_3, we will see that it is much more interesting and stranger and is the way that we can solve W_3 and allows us to a determinant function on f_1, f_2 and f_3. This allows the geometrical information contained in W_3 to have a sensible meaning , previously used to characterize geodesic submanifolds.

The basic framework from W_2 to W_4 shows an improvement of geometry interlacing results in generalized Sasakian-space-forms. The higher tensor is built on the new information gained from the previous one and explores the new geometric nature. This progressive technique of knowing the geometry creates more and more refined instruments for studying the submanifold behaviour. Instead, the varying complexities of non-degeneracy conditions associated to each tensor carry the implication (which is already known to hold) that nature has a preference for several complementary descriptions of fundamental geometric features.

6. INVARIANT SUBMANIFOLDS AND THE W_4 CURVATURE RELATIONSHIP

In this sense, this section presents a new characterization of totally geodesic submanifolds in terms of their relations with the W4 curvature tensor. We set out necessary and sufficient conditions under which this interaction determines the geometric structure of the submanifold.

More specifically, while natural numbers up to 4 are certain types of W_4 curvature tensor and while W_2 and W_3 have properties that extend to W_4, there are properties exhibited exclusively by the W_4 curvature tensor itself, properties that have to do with how it interacts with the contact structure. It encodes geometric data relevant to studying the interaction between a contact metric structure on the ambient space and the invariant submanifolds. The result W_4 encompasses terms with metric tensor and structure tensor φ, allowing a balanced view of the manifold's geometric and contact aspects. Such a balance contributes a lot when looking at totally geodesic submanifolds, as it demonstrates how the contact structure affects the alignment of geodesics in the submanifold with those in the ambient space.

The novelty here lies in the different geometric interactions of the W_4 curvature tensor underneath the underlying metric and contact structures. Behaviour along the characteristic direction illuminates complementary properties relative to what we learned from W_2 and W_3.

Theorem 6.1. Let M be a generalized Sasakian-space-form and N∈Mbe an invariant submanifold. The following conditions are equivalent:

1. N is totally geodesic
2. $Q(\sigma,W_4)=0$

This equivalence holds under the condition $\{f_1+3f_2+2(n-1)f_3\}\neq 0$

Proof. Let M be a generalized Sasakian-space-form, and let N be an invariant submanifold of M, then the following statements are equivalent:

$$\boldsymbol{Q(\sigma,W_4)=Q(\sigma,W_4)(X,Y,Z;U,V)=\big((U\wedge_\sigma V).W_4\big)(X,Y)Z=0}$$

This expands to:

$$\boldsymbol{-W_4\big((U\wedge_\sigma V)X,Y\big)Z-W_4(X,(U\wedge_\sigma V)Y)Z-W_4(X,Y)(U\wedge_\sigma V)Z} \qquad \textbf{(6.1)}$$

Applying the fundamental relationship:

$$\boldsymbol{-\sigma(V,X)W_4(U,Y)Z+\sigma(U,X)W_4(V,Y)Z-\sigma(V,Y)W_4(X,U)Z+\sigma(U,Y)W_4(X,V)Z-\sigma(V,Z)W_4(X,Y)U+\sigma(U,Z)W_4(X,Y)V=0} \qquad \textbf{(6.2)}$$

Setting Z=V=ξ and utilizing (3.3):

$$\boldsymbol{\sigma(U,X)W_4(\xi,Y)\xi+\sigma(U,Y)W_4(X,\xi)\xi=0} \qquad \textbf{(6.3)}$$

The internal structure of W_4 gives rise to relationships between σ and the ambient geometry that are very different from those seen with earlier tensors. These connections offer additional geometric insight into the nature of totally geodesic submanifolds.

Substituting equation (2.19):

$$\boldsymbol{\sigma(U,X)[(f_1-f_3)\{\eta(Y)\xi-Y\}+\frac{1}{2n}\{QY-2n(f_1-f_3)\eta(Y)\xi]}$$
$$\boldsymbol{+\sigma(U,Y)(f_1-f_3)\{X-\eta(X)\xi\}=0} \qquad \textbf{(6.4)}$$

Taking the inner product with W:

$$\boldsymbol{\sigma(U,X)[(f_1-f_3)\{\eta(Y)\eta(W)-g(Y,W)\}+\frac{1}{2n}\{g(QY,W)-2n(f_1-f_3)\eta(Y)\eta(W)]}$$
$$\boldsymbol{+\sigma(U,Y)(f_1-f_3)\{g(X,W)-\eta(X)\eta(W)\}=0} \qquad \textbf{(6.5)}$$

Contracting Y and W leads to:

$$\boldsymbol{\sigma(U,X)\{f_1+3f_2+2(n-1)f_3\}=0.} \tag{6.6}$$

Given our non-degeneracy condition $\{f_1+3f_2+2(n-1)f_3\}\neq 0$, we conclude:

$$\boldsymbol{\sigma(U,X)=0}$$

This shows that N is completely geodesic. The converse is simple: if N is totally geodesic, we have $\sigma=0$ identically, hence $Q(\sigma, W_4)=0$.

This theorem establishes a strong relationship between the W_4 curvature tensor and the geometric properties of invariant submanifolds, leading to a novel criterion for total geodesicity that enhances our previous descriptions in terms of W_2 and W_3.

7. TOTAL GEODESICITY THROUGH W_6 CURVATURE INTERACTIONS

In this section, we study a different definition of totally geodesic submanifolds. We analyze geodesic submanifolds regarding their interaction with the W_6 curvature tensor. The conditions we obtain afford a simpler test than those of earlier sections, underscoring the uniqueness of the W_6 tensor.

Of all the curvature tensors considered here, W_6 has strikingly simple properties in terms of its interaction with invariant submanifolds. The simplification of conditions on W_6 at the level of W_4 is not an accident but reflects underlying geometric information about generalized SSF. The action of this tensor offers a view of the impact of the contact structure on the geometry of invariant submanifolds. The second fundamental form and its geometric integration in W_2 is found in W_6, a primary source that keeps necessary information about the submanifold, including its contact metric properties with that of the ambient space. The non-degeneracy condition associated with W_6 is relatively simple, which could lead to the conclusion that this tensor is particularly suitable for practical applications of contact geometry.

No curvature tensor studied to date interacts with invariant submanifolds like the W_6 tensor; it is simply an elegant animal. Such decomposition manifests in a simple algebraic structure, providing clearer geometric interpretations while encoding essential information about the geodesic properties of the manifold.

Theorem 7.1. Suppose $2n(f1-f3)\neq 0$, then an invariant submanifold N of the generalized Sasakian-space-form M is geodesic if and only if $Q(\sigma, W_6)=0$.

Proof. Consider an invariant submanifold N where $Q(\sigma, W_6)=0$. We can express this condition through:

$$\boldsymbol{Q(\sigma,\ W_6)=Q(\sigma,\ W_6)(X,Y,Z;U,V)=((U\wedge_\sigma V).W_6)(X,Y)Z=0}$$

This leads to:

$$\boldsymbol{-W_6\big((U\wedge_\sigma V)X,Y\big)Z-W_6(X,(U\wedge_\sigma V)Y)Z-W_6(X,Y)(U\wedge_\sigma V)Z} \tag{7.1}$$

Applying the structural operator and expanding:

$$\boldsymbol{-\sigma(V,X)W_6(U,Y)Z+\sigma(U,X)W_6(V,Y)Z-\sigma(V,Y)W_6(X,U)Z+\sigma(U,Y)W_6(X,V)Z-\sigma(V,Z)W_6(X,Y)U+\sigma(U,Z)W_6(X,Y)V=0} \tag{7.2}$$

Setting $Z=V=\xi$ and using the fundamental condition (3.3):

$$\boldsymbol{\sigma(U,X)W_6(\xi,Y)\xi+\sigma(U,Y)W_6(X,\xi)\xi=0} \tag{7.3}$$

This simple nature of W6 points suggests that this tensor plays a special role in the submanifold-geometry description. In contrast, while the characterization of other tensors requires non-degeneracy conditions of vast complexity, W6 enjoys an exquisite simplicity of requirements. This reduction means that W6 might be especially useful for practical implementations and theoretical analyses that require reducing computational complexity, while preserving geometric insight.

Substituting equation (2.20):

$$\boldsymbol{\sigma(U,X)[(f_1-f_3)\{\eta(Y)\xi-Y\}]=0} \tag{7.4}$$

Taking the inner product with W:

$$\boldsymbol{\sigma(U,X)[(f_1-f_3)\{\eta(Y)\eta(W)-g(Y,W)\}]=0} \tag{7.5}$$

Contracting Y and W yields:

$$\boldsymbol{\sigma(U,X)\{2n(f_1-f_3)\}=0.} \tag{7.6}$$

Under our non-degeneracy condition $2n(f_1-f_3)\neq0$, we obtain: **σ(U,X)=0**

It follows that N is geodesic. Conversely, any geodesic submanifold satisfies $Q(\sigma, W_6)=0$ by definition; thus, the converse holds trivially.

As such, this theorem gives a remarkably beautiful characterization of totally geodesic invariant submanifolds in terms of the W6 curvature tensor. The geometrical structure of these fold submanifolds might be analyzed particularly well within the family of orbits of the W6 representation thanks to the non-degeneracy condition $2n(f_1-f_3)\neq0$ itself being extremely easy to check.

So, the very simple non-degeneracy condition $2n(f_1 - f_3) \neq 0$ indicates that W_6 encodes fundamental geometric properties more directly than any other curvature tensors. This simplification is not just algebraic but is rooted in the basic geometric properties.

8. CHARACTERIZATION THROUGH W_7 CURVATURE INTERACTIONS

We conclude this section with our main characterization of totally geodesic submanifolds in terms of the W_7 curvature tensor. The relations we obtain scale our thorough study between relations of curvature tensors in generalized SSF.

Our work, as the previously selected paragraphs and sentence describes, is an extension of the aforementioned one with the introduction of the W_7 curvature tensor to fully complete our knowledge about the problem in the context of generalized Sasakian-social forms. This tensor has some elements found in previously explored curvature tensors but also has some peculiarities. Particularly, the nature of W_7 in the characteristic direction is key to understanding the contact structure's influence over the geometries of invariant submanifolds. The relationship between W_7 and the second fundamental form has shown subtle geometric properties, which hold in addition to the victory from another curvature tensor. Moreover, while the non-degeneracy condition for W_7 is far more complicated than that of W_6, it then picks up essential geometric information that other curvature tensors alone cannot see.

There is a complete geometric characterization of totally geodesic properties with the last curvature tensor W_7. Unlike previous tensors, its structure combines elements seen in previous tensors but with distinct features that capture different geometric information.

Theorem 8.1. For an invariant submanifold N of a generalized Sasakian-space-form M, N is geodesic if and only if $(\sigma, W_7) = 0$, provided that $\{2n(1-2n)f_1 + 3f_2 - (2n+1)(2n-1)f_3\} \neq 0$.

Proof. Consider an invariant submanifold N satisfying Q(σ, W_7)=0. We begin by expressing:

$$\boldsymbol{Q(\sigma, W_7) = Q(\sigma, W_7)(X, Y, Z; U, V) = \big((U \wedge_\sigma V).W_7\big)(X, Y)Z = 0}$$

This expands to:

$$\boldsymbol{-W_7\big((U \wedge_\sigma V)X, Y\big)Z - W_7(X, (U \wedge_\sigma V)Y)Z - W_7(X, Y)(U \wedge_\sigma V)Z} \qquad \textbf{(8.1)}$$

Using the fundamental operator relationship:

$$\boldsymbol{-\sigma(V, X)W_7(U, Y)Z + \sigma(U, X)W_7(V, Y)Z - \sigma(V, Y)W_7(X, U)Z + \sigma(U, Y)W_7(X, V)Z - \sigma(V, Z)W_7(X, Y)U + \sigma(U, Z)W_7(X, Y)V = 0} \qquad \textbf{(8.2)}$$

Setting Z=V=ξ and applying condition (3.3):

$$\boldsymbol{\sigma(U, X)W_7(\xi, Y)\xi + \sigma(U, Y)W_7(X, \xi)\xi = 0} \qquad \textbf{(8.3)}$$

The behaviour of W_7 along the characteristic direction reveals subtle geometric properties that complement our understanding of other curvature tensors. These properties prove essential in establishing a comprehensive characterization of totally geodesic submanifolds.

Substituting equation (2.21):

$$\boldsymbol{\sigma(U, X)[(f_1 - f_3)\{\eta(Y)\xi - Y\}] + \sigma(U, Y)[QY - 2n(f_1 - f_3)\eta(X)\xi] = 0} \qquad \textbf{(8.4)}$$

Taking the inner product with W:

$$\boldsymbol{\sigma(U, X)[(f_1 - f_3)\{\eta(Y)\eta(W) - g(Y, W)\}] + \sigma(U, Y)[g(QY, W) - 2n(f_1 - f_3)\eta(X)\eta(W)] = 0} \qquad \textbf{(8.5)}$$

Contracting Y and W:

$$\boldsymbol{\sigma(U, X)\{2n(1 - 2n)f_1 + 3f_2 - (2n + 1)(2n - 1)f_3\} = 0.} \qquad \textbf{(8.6)}$$

Given our non-degeneracy condition $\{2n(1-2n)f_1 + 3f_2 - (2n+1)(2n-1)f_3\} \neq 0$. we conclude:

$$\boldsymbol{\sigma(U,X)=0}$$

This establishes that N is geodesic. The converse follows naturally as any geodesic submanifold satisfies Q(σ, W_7)=0.

The non-degeneracy condition derived in Proposition W_7 illuminates how the contact structure (defined by a one-form η) offers intertwining metric or geometric properties interlacing that guarantee the most interesting characterization by geometry. Although W_6 is more complicated than this, this condition encodes some important features that are not easily seen through other tensors.

This last theory finalizes our study concerning the relations between curvature tensors and geodesic invariant submanifolds of generalized Sasakian-space-forms. Along with our previous results, we succinctly delineate five different yet interrelated characterizations of totally geodesic submanifolds via their interactions with the curvature tensors W_2 to W_7.

Despite its apparent simplicity, the previously defined invariants provide different characteristic views of the geometric structure of invariant submanifolds, as we will see in some detail: the advantages of each are its computational simplicity in settings of interest and its geometrical insight into the nature of the adjoint configuration. The different non-degeneracy conditions we obtained indicate that the analysis in a given space form will require different curvature tensors depending on the new functions f_1, f_2, and f_3 for each wave source.

The study of these five curvature tensors provides a holistic view of how the contact metric structures affect the geometry of invariant submanifolds. While each tensor is novel in its own right, all three provide a framework for understanding geodesic submanifolds of generalized SSF. Non-degeneracy conditions vary in complexity, so that different tensors may be better suited for analysis in different geometries. The implications of this hierarchic relation between curvature tensors and geometric characteristics in a submanifold lead to further research in the context of broader classes within contact metric geometry.

Summary of Key Results

Our investigation has revealed five distinct characterizations of totally geodesic invariant submanifolds in generalized Sasakian-space-forms, each involving different curvature tensors:

1. **W_2 Characterization**: Total geodesicity is equivalent to $Q(\sigma,W_2)=0$ when $(2n-1)\{3f_2+(2n-1)f_3\}\neq0$
2. **W_3 Characterization**: The condition $Q(\sigma,W_3)=0$ ensures total geodesicity under $\{4n(1-2n)f_1+3f_2+(4n+1)(2n-1)f_3\}\neq0$
3. **W_4 Characterization**: Total geodesicity is characterized by $Q(\sigma,W_4)=0$ when $\{f_1+3f_2+2(n-1)f_3\}\neq0$
4. **W_6 Characterization**: The simplest condition appears with $Q(\sigma,W_6)=0$ under $2n(f_1-f_3)\neq0$
5. **W_7 Characterization**: The final characterization uses $Q(\sigma,W_7)=0$ with $\{2n(1-2n)f_1+3f_2-(2n+1)(2n-1)f_3\}\neq0$

These five curvature tensors represent more than their individual characterizations; their total impact is summed with these five. They offer a complete account of the geometry of generalized Sasakian-space-forms. That the conditions upon which each tensor gives its characterization are so diverse suggests that these spaces carry a rich geometric structure, one that certainly cannot be ascertained through only one perspective. On one hand, this variety of descriptions extends our insight and on the other hand, it offers technical versatility to study particular geometrical contexts.

Conclusion

The current investigation has provided a general framework to understand the geodesic invariant submanifolds of generalized Sasakian-space-forms via their relationships with different curvature tensors. We show that the geometry of these submanifolds can be described in several equivalent ways, with each one highlighting some characteristic aspect of their geometry.

The dictates of the second fundamental form σ relate to the curvature tensors W_i (i = 2, 3, 4, 6, 7) and show different aspects of the same geometric property - total geodesicity. Of the various characterizations, the W6 curvature tensor produces the neatest condition, with minimal non-degeneracy condition required, and thus seems especially suited to practical applications.

These results generalize known facts about generalized Sasakian space forms and introduce new tools to study their submanifolds. Geometers now have a range of available conditions that they can use as the most suitable characterization according to the setting of their study. In addition, the distinct non-degenerate conditions associated with these different characterizations suggest that the results may be complementary and potentially useful in different regions of the manifold for varying f1, f2, and f3 values.

Future research directions could focus on studying similar characterizations for other kinds of submanifolds, understanding the geometric significance of the different non-degeneracy conditions, and finally, considering how these results can be generalized to other generalized geometrical structures. Moreover, the interplay of these characterizations with other geometric features of SSF could also lead to profitable attributions.

Our results expand the lexicon of differential geometry with several equivalent geometric conditions for a submanifold to satisfy being geodesic, each of which sheds light on the underlying geometric structure of generalized SSF.

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