Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 7s (2025)

# Quasi- Laplacian energy of some novel classes of graphs

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***Article History:***

***Received:*** *26-10-2024*

***Revised:****10-11-2024*

***Accepted:****18-12-2024*

**Abstract:**
We formulate the relationship of quasi-Laplacian energy of some novel classes of graphs with their corresponding original graphs. The novel graphs in our discussion are the $\mathcal{S}$-graph, $\mathcal{R}$-graph, $\mathcal{Q}$-graph, total graph, and their join and corona operations graphs. The whole formulation is based on the relationship between quasi-Laplacian energy and the vertex degrees of the novel graph. It is also noted that quasi-Laplacian energy is closely related with the first Zagreb index, number of vertices and edges of the graph. The exact formulas of quasi-Laplacian energy of novel graphs are obtained in terms of the corresponding quasi-Laplacian energies, the first Zagreb indices, and the number of vertices and edges of the original graphs.

**Keywords**: Quasi-Laplacian energy, degree of vertex, Zagreb index, join, corona.

**Mathematics Subject Classification: 05C07, 05C09, 05C50, 05C76**

## 1. Introduction

All graphs discussed in this paper are simple and undirected. Let $G$ be a graph with vertices denoted by $V(G)$ and edges denoted by $E(G)$. Let $|V(G)| = p$ and $|E(G)| = q$.

Let $d_G(v)$ represent the degree of vertices in $G$, where $u \in V(G)$. Let $D(G)$ denote the diagonal matrix and $A(G)$ the adjacency matrix. The quasi Laplacian matrix, represented by $Q(G)$, is defined as $Q(G) = D(G) + A(G)$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ denote the real, symmetric, and positive semi definite eigenvalues of $Q(G)$.

It is known that various graph operations can create novel classes of graphs from the original graphs. Therefore, understanding the relationships between some invariants of such novel graphs and the equivalent invariants of the original graphs is relevant.

Graph energy is one such invariant based on the graph spectrum introduced by Gutman [9]. We discuss here the quasi-Laplacian energy [6] of graph $G$, represented as $E_Q(G)$ and determined by the equation

$$E_Q(G) = \sum_{i=1}^{p} \mu_i^2$$

Yue, Cao and Qi, in the paper [6], defined quasi-Laplacian energy of $G$ is expressed as follows

$$E_Q(G)=\sum_{i=1}^{p} d_G^2(u_i)+\sum_{i=1}^{p} d_G(u_i), \qquad u\in V(G)\ \dots\dots\dots (1.1)$$

Also, equation (1.1) can be represented as follows

$$E_Q(G)=\sum_{i=1}^{p} d_G^2(v_i)+\sum_{i=1}^{p} d_G(v_i)=M_1(G)+2q\ \dots\dots\dots (1.2)$$

where, $M_1$ is called first Zagreb index [10] and second part of equation (1.2) is the handshaking lemma $\sum_{i=1}^{p} d_G(v_i)=2q$. It is noticed that the quasi-Laplacian energy of any graph depends upon the degrees of vertices of that graph.

Based on the above result discussed in the paper [6]. We formulate the relations of the quasi-Laplacian energy of some classes of novel graphs in terms of corresponding original graphs. The subdivision graph $\mathcal{S}(G)$, $\mathcal{R}$-graph $\mathcal{R}(G)$, $\mathcal{Q}$ graph $\mathcal{Q}(G)$, total graph $\mathcal{T}(G)$ [3] and their join and corona operations graphs are the novel graphs for our discussion. Indulal [8] first defined $\mathcal{S}$-vertex and $\mathcal{S}$ edge join and Liu and Zhang [16] determined their spectra. Lu and Miao [13] defined $\mathcal{S}$-vertex corona and edge corona. Liu and Lu [14] defined $\mathcal{S}$-vertex and edge neighbourhood corona. Sun, Shang, and Bu [16] defined $\mathcal{Q}$-vertex and edge join graphs. The $\mathcal{Q}$-vertex and edge corona are defined by Najiya and Chithra [2]. The definition of the corona of the total graph of one regular and another arbitrary graph, and spectra of the corona of the total graph is determined by Zhu, Tian, and Cui [19]. Also, $\mathcal{S}$ vertex- $\mathcal{R}$ vertex join, $\mathcal{S}$ edge- $\mathcal{R}$ edge join, $\mathcal{S}$ vertex- $\mathcal{R}$ edge join and $\mathcal{S}$ edge- $\mathcal{R}$ vertex join are defined by Das and Panigrahi [5, 7]. The $\mathcal{S}$-vertex-vertex-edge join of three $\mathcal{S}$ graphs is defined by Wen, Zhang, and Li [17]. Berberler [1] derived the quasi-Laplacian energy of graphs based on $\mathcal{R}$ graphs [13].

We derive formulas of the quasi-Laplacian energy of $\mathcal{S},\mathcal{Q}$ and $\mathcal{T}$ graphs and further, obtain quasi-Laplacian energy of $\mathcal{S}$-vertex and edge join, $\mathcal{S}$-vertex and edge corona, $\mathcal{S}$-vertex and edge neighbourhood corona in terms of their corresponding original graphs. Similarly, we derive quasi-Laplacian energy of $\mathcal{Q}$-vertex and edge join, $\mathcal{Q}$-vertex and edge corona and $\mathcal{T}$-graph corona. The quasi-Laplacian energy of $\mathcal{S}$ vertex- $\mathcal{R}$ vertex join, $\mathcal{S}$edge- $\mathcal{R}$ edge join, $\mathcal{S}$ vertex- $\mathcal{R}$ edge join, $\mathcal{S}$ edge- $\mathcal{R}$ vertex join and $\mathcal{S}$-vertex-vertex-edge of three graphs join are also derived in terms of their corresponding original graphs.

Let $G$ be the $(p, q)$ graph where sets of old vertices $|V(G)| = p$ and sets of inserted new vertices $I(G)\ |= q$. Let $H_1$ be the $(p_1, q_1)$ graph and $H_2$ be the $(p_2, q_2)$ graph. Consequently, $|V(H_1)| = p_1, |I(H_1)| = q_1, |V(H_2)| = p_2$ and $|I(H_2)| = q_2$.

## 2. $E_Q$ based on $\mathcal{S}$ graphs

The quasi-Laplacian energy of the $\mathcal{S}$ graph, $\mathcal{S}$ join and $\mathcal{S}$ corona graphs are formulated in this section based on the degree of vertices of the $\mathcal{S}$-graph, $\mathcal{S}$-join, and $\mathcal{S}$-corona graphs respectively. Let $u$ be any vertex in $\mathcal{S}(G)$, then the degree of $\mathcal{S}(G)$ is represented by

$$d_{\mathcal{S}(G)}(u_i) = \begin{cases} d_G(u_i), & \text{if } u \in V(G); \\ 2, & \text{if } u \in I(G). \end{cases}$$

**Theorem 2.1.** $E_Q(\mathcal{S}(G)) = E_Q(G) + 6q$

**Proof.** Using definition (1.1) on the $\mathcal{S}(G)$-graph, we have

$$\begin{aligned} E_Q(\mathcal{S}(G)) &= \sum_{i=1}^{|V(\mathcal{S}(G))|} d^2_{\mathcal{S}(G)}(u_i) + \sum_{i=1}^{|V(\mathcal{S}(G))|} d_{\mathcal{S}(G)}(u_i) \\ &= \sum_{i=1}^{p} d^2_{(G)}(u_i) + \sum_{i=1}^{q} 2^2 + \sum_{i=1}^{p} d_{(G)}(u_i) + \sum_{i=1}^{q} 2 \\ &= E_Q(G) + 6q \end{aligned}$$

Now, let $u$ be any vertex in $\mathcal{S}$-vertex join $(H_1 \dot{\vee} H_2)$ of two graphs $H_1$ and $H_2$. Then, the degree of the vertex $u$ in $(H_1 \dot{\vee} H_2)$ is given by

$$d_{H_1 \dot{\vee} H_2}(u_i) = \begin{cases} d_{H_1}(u_i) + p_2, & \text{if } u \in V(H_1), \quad \text{for i} = 1,2, \dots \text{p}_1; \\ 2, & \qquad\qquad \text{if } v \in I(G_1); \\ d_{H_2}(u_j) + p_1, & \text{if } v \in V(H_2), \quad \text{for j} = 1,2, \dots \text{p}_2. \end{cases}$$

and $H_1 \dot{\vee} H_2$ has $p_1 + q_1 + p_2$ vertices. We derive quasi-Laplacian energy of $\mathcal{S}$-vertex join in terms of corresponding original graphs $H_1$ and $H_2$.

**Theorem 2.2.** $E_Q(H_1 \dot{\vee} H_2) = E_Q(H_1) + E_Q(H_2) + 4q_1p_2 + p_1p_2^2 + 6q_1 + 4p_1q_2 + p_1^2p_2 + 2p_1p_2$

**Proof.** Using definition (1.1), quasi-Laplacian energy of $\mathcal{S}$-vertex join is obtained by

$$
\begin{aligned}
E_Q(H_1 \dot{\vee} H_2) &= \sum_{i=1}^{p_1+q_1+p_2} d^2_{H_1\dot{\vee}G_2}(u_i) + \sum_{i=1}^{p_1+q_1+p_2} d_{H_1\dot{\vee}H_2}(u_i) \\
&= \sum_{i=1}^{p_1} d^2_{(H_1)}(u_i) + 2p_2 \sum_{i=1}^{p_1} d_{(H_1)}(u_i) + p_2^2 \sum_{i=1}^{p_1} + 4q_1 \\
&+ \sum_{j=1}^{p_2} d^2_{(H_2)}(u_j) + 2p_1 \sum_{j=1}^{p_2} d_{(H_2)}(u_j) + p_1^2 \sum_{j=1}^{p_2} \\
&+ \sum_{i=1}^{p_1} d_{(H_1)}(u_i) + p_2 \sum_{i=1}^{p_1} + 2q_1 + \sum_{j=1}^{p_2} d_{(H_2)}(u_j) + p_1 \sum_{j=1}^{p_2} \\
&= E_Q(H_1) + E_Q(H_2) + 4q_1p_2 + p_1p_2^2 + 6q_1 + 4p_1q_2 + p_1^2p_2 + 2p_1p_2.
\end{aligned}
$$

Next, we formulate a relation of quasi-Laplacian energy of $H_1 \underline{\vee} H_2$ join.

Let $u$ be any vertex in $\mathcal{S}$-edge join. The degree of vertex $u$ in $H_1 \underline{\vee} H_2$ join is given by

$$
d_{H_1 \underline{\cup} H_2}(u_i) = \begin{cases} d_{H_1}(u_i), & \text{if } u \in V(H_1); \\ 2 + p_2, & \text{if } u \in I(H_1); \\ d_{H_2}(v_i) + q_1, & \text{if } u \in V(H_2). \end{cases}
$$

and $H_1 \underline{\vee} H_2$ has $p_1 + q_1 + p_2$ vertices.

**Theorem 2.3.** $E_Q\left(H_1 \underline{\vee} H_2\right) = E_Q(H_1) + E_Q(H_2) + 6q_1p_2 + q_1p_2^2 + 6q_1 + 4q_1q_2 + q_1^2p_2$

**Proof.** Using equation (1.1), we get

$$
\begin{aligned}
E_Q\left(H_1 \underline{\vee} H_2\right) &= \sum_{i=1}^{p_1+q_1+p_2} d^2_{H_1 \trianglerighteq H_2}(u_i) + \sum_{i=1}^{p_1+q_1+p_2} d_{H_1 \underline{\vee} H_2}(u_i) \\
&= \sum_{i=1}^{p_1} d^2_{H_1}(u_i) + \sum_{i=1}^{q_1} (2+p_2)^2 + \sum_{i=1}^{p_2} \left(d_{(H_2)}(u_i) + q_1\right)^2 \\
&+ \sum_{i=1}^{p_1} d_{(H_1)}(u_i) + \sum_{i=1}^{q_1} 2 + \sum_{i=1}^{p_2} d_{(H_2)}(u_i) + q_1
\end{aligned}
$$

The result can be derived easily.

## 2.1. $E_Q$ of $\mathcal{S}$-vertex and edge corona

Let $u$ be any vertex in $\mathcal{S}$-vertex corona. The degree of vertex of $u$ in $\mathcal{S}$-vertex corona is given by

$$
d_{H_1 \odot H_2}(u_i) = \begin{cases} d_{H_1}(u_i) + p_2, & \text{if } u \in V(H_1); \\ 2, & \text{if } u \in I(H_1); \\ d_{H_2}\left(v_j^i\right) + 1, & \text{if } u = v_j^i, \text{ for } i = 1,2,\ldots p_1, \text{ for } j = 1,2,\ldots p_2. \end{cases}
$$

and $H_1 \odot H_2$ has $p_1 + q_1 + p_1p_2$ vertices. The formula of quasi-Laplacian energy of $\mathcal{S}$-vertex corona is given below

**Theorem2.4.** $E_Q(H_1 \odot H_2) = E_Q(H_1) + p_1E_Q(H_2) + p_1p_2^2 + 4q_1p_2 + 3q_1 + 3p_1p_2 + 4p_1q_2$

**Proof.** By using definition (1.1), we get

$$E_Q(H_1 \odot H_2) = \sum_{i=1}^{p_1+q_1+p_1p_2} d^2_{H_1 \odot H_2}(v_i) + \sum_{i=1}^{p_1+q_1+p_1p_2} d_{H_1 \odot H_2}(u_i)$$

$$= \left. \sum_{i=1}^{p_1} d_{H_1}(u_i) + p_2 \right)^2 + \sum_{i=1}^{q_1} 2^2 + \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} \left(d_{H_2}(v_j^i) + 1\right)^2$$

$$+\sum_{i=1}^{p_1} d_{(H_1)}(u_i) + p_2 + \sum_{i=1}^{q_1} 2 + \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} d_{H_2}(v_j^i) + 1$$

$$= p_2^2 \sum_{i=1}^{p_1} + 2p_2 \sum_{i=1}^{p_1} d_{H_1}(u_i) + \sum_{i=1}^{p_1} d^2_{H_1}(u_i) + 4q_1 + \sum_{i=1}^{p_1} \sum_{j=1}^{p_2}$$

$$+2 \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} d_{H_2}(v_j) + \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} d^2_{H_2}(v_j) + p_1p_2 + \sum_{i=1}^{p_1} d_{H_1}(u_i)$$

$$+2q_1 + \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} + \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} d_{H_2}(v_j)$$

Hence, the result follows.

Let $u$ be any vertex in $\mathcal{S}$-edge corona, then the degree of the vertex $u$ is given by

$$d_{H_1 \ominus H_2}(u_i) = \begin{cases} d_{H_1}(u_i), & \text{if } u \in V(H_1); \\ 2 + p_2, & \text{if } u \in I(H_1); \\ d_{H_2}(v_j^i) + 1, & \text{if } u = v_j^i, \text{for } i = 1,2, \ldots q_1, \text{for } j = 1,2, \ldots p_2. \end{cases}$$

and $H_1 \Theta H_2$ has $p_1 + q_1 + q_1p_2$ vertices. The formula of quasi-Laplacian energy of $\mathcal{S}$-edge corona is given by

**Theorem 2.5.** $E_Q(H_1 \ominus H_2) = E_Q(H_1) + q_1E_Q(H_2) + 6q_1 + 7q_1p_2 + 4q_1q_2$

**Proof.** Using definition (1.1),

$$
\begin{aligned}
E_Q(H_1 \odot H_2) &= \sum_{i=1}^{|V(H_1 \ominus H_2)|} d^2_{H_1 \ominus H_2}(u_i) + \sum_{i=1}^{|V(H_1 \ominus H_2)|} d_{H_1 \ominus H_2}(u_i) \\
&= \sum_{i=1}^{p_1+q_1+q_1p_2} d^2_{H_1 \ominus H_2}(u_i) + \sum_{i=1}^{p_1+q_1+q_1p_2} d_{H_1 \ominus H_2}(u_i) \\
&= \sum_{i=1}^{p_1} d^2_{H_1}(u_i) + \sum_{i=1}^{q_1} (2+p_2)^2 + \sum_{i=1}^{q_1}\sum_{j=1}^{p_2} \left(d_{H_2}(v_j^i)+1\right)^2 \\
&\quad + \sum_{i=1}^{p_1} d_{(H_1)}(u_i) + \sum_{i=1}^{q_1} 2 + \sum_{i=1}^{q_1}\sum_{j=1}^{p_2} d_{H_2}(v_j^i) + 1
\end{aligned}
$$

Hence, the result follows.

## 2.2. $E_Q$ of $\mathcal{S}$-vertex and edge neighbourhood corona

Let $u$ be any vertex in $\mathcal{S}$-vertex neighbourhood corona. Then degree of vertices is given by

$$
d_{H_1 \boxminus H_2}(v_i) = \begin{cases} d_{H_1}(u_i), & \text{if } u \in V(H_1); \\ 2 + 2p_2, & \text{if } u \in I(H_1); \\ d_{H_1}(u_i) + d_{H_2}(v_j), & \text{if } u = v_j^i, \text{ for } i = 1,2, \dots p_1, \text{ for } j = 1,2, \dots p_2. \end{cases}
$$

and $H_1 \boxminus H_2$ has $p_1 + q_1 + p_1p_2$ vertices. The quasi-Laplacian energy of $\mathcal{S}$-vertex neighbourhood corona is given by

**Theorem 2.6.** $E_Q(H_1 \boxminus H_2) = (1+p_2)E_Q(H_1) + p_1E_Q(H_2) + 6q_1 + 10q_1p_2 + 4q_1p_2^2 + 8q_1q_2$

**Proof:** Using definition (1.1), the above Theorem can be easily proved. Also, the degree of any vertex of $u$ in $\mathcal{S}$-edge neighbourhood corona is given by

$$
d_{H_1 \boxminus H_2}(u_i) = \begin{cases} (1+p_2)d_{H_1}(u_i), & \text{if } u \in V(H_1); \\ 2, & \text{if } u \in I(H_1); \\ 2 + d_{H_2}(u_j), & \text{if } u = v_j^i, \text{ for } i = 1,2, \dots q_1, \text{ for } j = 1,2, \dots p_2. \end{cases}
$$

and $H_1 \boxminus H_2$ has $p_1 + q_1 + q_1p_2$ vertices. The quasi-Laplacian energy of $\mathcal{S}$-edge neighbourhood corona is given by

**Theorem 2.7.** $E_Q(H_1 \boxminus H_2) = (1+2p_2)E_Q(H_1) + 2q_1E_Q(H_2) + p_2^2M_1(H_1) + 4q_1 + 2q_1p_2 + 12q_1q_2$, where $M_1$ is the first Zagreb index.

Using definition (1.1), the above Theorem can be easily proved.

## 3. $E_Q$ based on $Q$ graph

The energy of the $Q$ graph, $Q$ join and $Q$ corona graphs are formulated in this section based on the degree of vertices of the $Q$-graph, $Q$-join, and $Q$-corona graphs respectively. Let $u$ be any vertex in $Q(G)$. Then, the degree of vertex is given by

$$d_{\mathcal{S}(G)}(u_i)=\begin{cases} d_G(u_i), & \text{if } u\in V(G);\\ 4, & \text{if } u\in I(G).\end{cases}$$

We derive a formula of energy of $Q(G)$.

**Theorem 3.1.** $E_Q(Q(G))=E_Q(G)+20q$

**Proof.** We follow from definition (1.1),

$$\begin{aligned} E_Q(Q(G)) &==\sum_{i=1}^{p+q} d_{Q(G)}^2(v_i)+\sum_{i=1}^{p+q} d_{Q(G)}(u_i)\\ &=\sum_{i=1}^{p} d_{(G)}^2(u_i)+\sum_{i=1}^{q} 4^2+\sum_{i=1}^{p} d_{(G)}(u_i)+\sum_{i=1}^{q} 4\\ &=E_Q(G)+20q\end{aligned}$$

### 3.1. $E_Q$ of $Q$-vertex and edge join

Let $u$ be any vertex in $H_1\langle v\rangle H_2$. Then, the degree of vertex of $H_1\langle v\rangle H_2$ is given by

$$d_{H_1\langle v\rangle H_2}(u_i)=\begin{cases} d_{H_1}(u_i)+p_2, & \text{if } u\in V(H_1);\\ 4, & \text{if } u\in I(H_1);\\ d_{H_2}(u_i)+p_1, & \text{if } u\in V(H_2).\end{cases}$$

$H_1\langle v\rangle H_2$ has $p_1+q_1+p_2$ vertices. The relation of quasi-Laplacian energy in between $Q$-vertex join with corresponding two original graphs is obtained easily by using definition (1.1) as follows

**Theorem 3.2.** $E_Q(H_1\langle v\rangle H_2)=E_Q(H_1)+E_Q(H_2)+4q_1p_2+p_1p_2^2+20q_1+4p_1q_2+p_1^2p_2+2p_1q_2$

Also, let $u$ be any vertex in $Q$-edge join. The degree of vertex $u$ of $Q$-edge join is given by

$$d_{H_1\langle e\rangle H_2}(u_i)=\begin{cases} d_{H_1}(u_i), & \text{if } u\in V(H_1);\\ 2+p_2, & \text{if } u\in I(H_1)\ ;\\ d_{H_2}(u_i)+q_1, & \text{if } u\in V(H_2).\end{cases}$$

$H_1\langle e\rangle H_2$ has $p_1+q_1+p_2$ vertices. The relation of quasi-Laplacian energy in between $Q$-edge join with corresponding two original graphs is obtained easily by using definition (1.1) as follows

**Theorem 3.3.** $E_Q(H_1\langle e\rangle H_2)=E_Q(H_1)+E_Q(H_2)+12q_1p_2+q_1p_2+20q_1+q_1^2p_2+q_1p_2^2$

### 3.2. $E_Q$ of $Q$-vertex and edge corona

Let $u$ be any vertex in $Q$-vertex corona. Then, the degree of any vertex $u$ in $Q$-vertex corona is given by

$$d_{H_1 \text{o} H_2}(v_i)=\begin{cases} d_{H_1}(u_i)+2, & \text{if } u\in V(H_1);\\ 4, & \text{if } u\in I(H_1)\,;\\ d_{H_2}(v_j^i)+1, & \text{if } u=v_j^i, \text{for } i=1,2,3,\dots p_1, \text{for } j=1,2,3,..p_2.\end{cases}$$

$H_1\text{o}H_2$ has $p_1+q_1+p_1p_2$ vertices. Then, by using equation (1.1) we get

**Theorem 3.4.** $E_Q(H_1oH_2)=E_Q(H_1)+p_1E_Q(H_2)+p_1p_2^2+20q_1+4p_1q_2+3p_1p_2+4q_1p_2$

Next, the degree of any vertex of $u$ in $Q$-edge corona is given by

$$d_{H_1\circledast H_2}(u_i)=\begin{cases} d_{H_1}(u_i), & \text{if } u\in V(H_1);\\ p_2+4, & \text{if } u\in I(H_1);\\ d_{H_2}(v_j^i)+1, & \text{if } u=v_j^i, \text{for } i=1,2,\dots p_1, \text{for } j=1,2,\dots p_2.\end{cases}$$

$G_1\circledast G_2$ has $p_1+q_1+q_1p_2$ vertices. Then, the quasi-Laplacian energy of $Q$-edge corona is easily obtained as follows

**Theorem 3.5.** $E_Q(H_1\circledast H_2)=E_Q(H_1)+p_1E_Q(H_2)+q_1p_2^2+9q_1p_2+20q_1+4q_2p_1+2p_1q_2$

## 4. $E_Q$ based on $\mathcal{T}$-graph

We derive energy of $\mathcal{T}$ graph and $\mathcal{T}$ graph corona based on the degree of vertices of the $\mathcal{T}$-graph and $\mathcal{T}$-corona graphs are determined here.

Let $u$ be any vertex in $\mathcal{T}(G)$, then the degree of vertex $u$ in $\mathcal{T}(G)$ is given by

$$d_{\mathcal{T}(G)}(u_i)=\begin{cases} 2d_G(v_i), & \text{if } u\in V(G);\\ 4, & \text{if } u\in I(G).\end{cases}$$

The vertices of $\mathcal{T}(G)$ is $p+q$. The $Q$ energy of $\mathcal{T}(G)$ is given by

**Theorem 4.1.** $E_Q(\mathcal{T}(G))=2E_Q(G)+20q+2M_1(G)$

Where, $M_1$ is the first Zagreb index.

### 4.1. $E_Q$ of $\mathcal{T}$-graph corona

Let $H_1$ be $r_1$ regular and $H_2$ be any graph. Also, let $u$ be any vertex in $\mathcal{T}(G)$. Then, the degree of vertex of $\mathcal{T}$ - corona is given by

$$d_{H_1 \star H_2}(u_i) = \begin{cases} 2d_{H_1}(u_i) + p_2, & \text{if } u \in V(H_1); \\ 2r_1, & \text{if } u \in I(H_1); \\ d_{H_2}(v_j^i) + 1, & \text{if } u = v_j^i, \text{ for } i = 1,2, \dots p_1, \text{ for } j = 1,2, \dots p_2. \end{cases}$$

$H_1 \star H_2$ has $p_1 + q_1 + p_1p_2$ vertices. The $E_Q$ of $\mathcal{T}$-graph corona is easily obtained by using equation (1.1) as follows

**Theorem 4.2.** $E_Q(H_1 \star H_2) = 2E_Q(H_1) + p_1E_Q(H_2) + 2M_1(H_1) + 5p_1p_2 + p_2^2p_1 + 4r_1^2q_1 + 2r_1q_1 + 4p_1q_2 + 2p_1p_2$
where, $M_1$ is the first Zagreb index.

## 5. $E_Q$ based on $\mathcal{S}$-graph and $\mathcal{R}$-graph join

The $E_Q$ energy of four $\mathcal{S}$-graph and $\mathcal{R}$-graph joins with their corresponding original graphs $H_1$ and $H_2$ are formulated in this part.

### 5.1. $E_Q$ of $\mathcal{S}$-vertex and $\mathcal{R}$-vertex join

First, we formulate the quasi-Laplacian energy of $\mathcal{S}$-vertex and $\mathcal{R}$-vertex join. Let $u$ be any vertex in $\mathcal{S}$-vertex and $\mathcal{R}$-vertex join. Then, the degrre of any vertex in $\mathcal{S}(H_1) \ddot{\vee} \mathcal{R}(H_2)$ is given by

$$d_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i) = \begin{cases} d_{H_1}(u_i) + p_2, & \text{if } u \in V(H_1); \\ 2, & \text{if } u \in I(H_1) \cup I(H_2); \\ 2d_{H_2}(u_i) + p_1, & \text{if } u \in V(H_2). \end{cases}$$

$\mathcal{S}(H_1) \ddot{\vee} \mathcal{R}(H_2)$ has $p_1 + q_1 + p_2 + q_2$ vertices.

**Theorem 5.1**. The quasi-Laplacian energy of $E_Q\left(\mathcal{S}(H_1) \ddot{\vee} \mathcal{R}(H_2)\right) = E_Q(H_1) + 4E_Q(H_2) + p_2^2p_1 + p_1^2p_2 + 4q_1p_2 + 8q_2p_1 + 2q_2 + 6q_1 + 2p_1p_2$

**Proof.** By using equation (1.1), we get

$$
\begin{aligned}
E_Q\big(\mathcal{S}(H_1)\ \ddot{\vee}\ \mathcal{R}(H_2)\big) = &\sum_{i=1}^{p_1+q_1+p_2+q_2} d^2_{\mathcal{S}(H_1)}\ \ddot{\vee}\ \mathcal{R}(H_2)(u_i) + \sum_{i=1}^{p_1+q_1+p_2+q_2} d_{\mathcal{S}(G_1)\ddot{\vee}\mathcal{R}(G_2)}(u_i) \\
&= \sum_{i=1}^{p_1} d^2_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i) + \sum_{i=1}^{q_1} d^2_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i) + \sum_{j=1}^{p_2} d^2_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i) \\
&+ \sum_{i=1}^{q_2} d^2_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i) + \sum_{i=1}^{p_1} d_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i) + \sum_{i=1}^{q_1} d_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i) \\
&+ \sum_{i=1}^{p_2} d_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i) + \sum_{i=1}^{q_2} d_{\mathcal{S}(H_1)\ddot{\vee}\mathcal{R}(H_2)}(u_i)
\end{aligned}
$$

Hence, the result follows.

### 5.2. $E_Q$ of $\mathcal{S}$-edge and $\mathcal{R}$-edge join

Let $u$ be any vertex in $\mathcal{S}$-edge and $\mathcal{R}$-edge join. Then, the degree of any vertex of $u$ in $\mathcal{S}$-edge and $\mathcal{R}$-edge join $\mathcal{S}(H_1)\overline{\overline{V}}\mathcal{R}(H_2)$ is given by

$$
d_{\mathcal{S}(H_1)\overline{\overline{V}}\mathcal{R}(H_2)}(u_i) = \begin{cases} d_{H_1}(u_i), & \text{if } u \in V(H_1); \\ 2 + q_2, & \text{if } v \in I(H_1); \\ 2d_{H_2}(u_i), & \text{if } u \in V(H_2); \\ 2 + q_1, & \text{if } u \in I(H_2). \end{cases}
$$

$\mathcal{S}(H_1)\overline{\overline{V}}\mathcal{R}(H_2)$ has $p_1 + q_1 + p_2 + q_2$ vertices. The $Q$-energy is obtained easily by using equation (1.1).

**Theorem 5.2.** $E_Q\left(\mathcal{S}(H_1)\overline{\overline{V}}\mathcal{R}(H_2)\right) = E_Q(H_1) + 4E_Q(H_2) + 6q_1 + 10q_1q_2 + q_1q_2(q_1 + q_2) + 6q_2$

### 5.3. $E_Q$ of $\mathcal{S}$-vertex and $\mathcal{R}$-edge join

Let $u$ be any vertex in $\mathcal{S}$-vertex and $\mathcal{R}$-edge join. Then, the degree of any vertex of $u$ in $\mathcal{S}$-vertex and $\mathcal{R}$-edge join $\mathcal{S}(H_1)\dot{V}\mathcal{R}(H_2)$ is given by

$$
d_{\mathcal{S}(H_1)\dot{V}\mathcal{R}(H_2)}(u_i) = \begin{cases} d_{H_1}(u_i) + q_2, & \text{if } u \in V(H_1); \\ 2, & \text{if } u \in I(H_1); \\ 2d_{H_2}(u_i), & \text{if } u \in V(H_2); \\ 2 + p_1, & \text{if } u \in I(H_2). \end{cases}
$$

$\mathcal{S}(G_1)\dot{V}\mathcal{R}(G_2)$ has $p_1 + q_1 + p_2 + q_2$ vertices. We get the following result by using equation (1.1).

**Theorem 5.3.** $E_Q\left(\mathcal{S}(H_1)\dot{V}\mathcal{R}(H_2)\right) = E_Q(H_1) + 4E_Q(H_2) + q_2^2p_1 + 4q_1q_2 + 6q_1 + 6q_2p_1 + p_2^2q_2 + 4q_2$

### 5.4. $E_Q$ of $\mathcal{S}$-edge and $\mathcal{R}$-vertex join

Let $u$ be any vertex in $\mathcal{S}$-edge and $\mathcal{R}$-vertex join. Then, the degree of any vertex of $u$ in $\mathcal{S}$-edge and $\mathcal{R}$-vertex join $\mathcal{S}(H_1)\dot{V}\mathcal{R}(H_2)$ is given by

$$d_{\mathcal{S}(H_1)\bar{V}\mathcal{R}(H_2)}(u_i) = \begin{cases} d_{H_1}(u_i), & \text{if } u \in V(H_1); \\ 2 + p_2, & \text{if } u \in I(H_1); \\ 2d_{H_2}(u_i) + q_1, & \text{if } u \in V(H_2); \\ 2, & \text{if } u \in I(H_2). \end{cases}$$

$\mathcal{S}(H_1)\overline{V}\mathcal{R}(H_2)$ has $p_1 + q_1 + p_2 + q_2$ vertices. We easily get the following Theorem by using equation (1.1)

**Theorem 5.4.** $E_Q\left(\mathcal{S}(H_1)\overline{V}\mathcal{R}(H_2)\right) = E_Q(H_1) + 4E_Q(H_2) + 6q_1 + 6q_1p_2 + q_1^2p_2 + q_1p_2^2 + 10q_2$

## 6. $E_Q$ of $\mathcal{S}$-vertex-vertex-edge join of triple graphs

Let $H_3$ be $(p_3, q_3)$ graph. Then $\mathcal{S}$-vertex-vertex-edge join of three $\mathcal{S}$ graphs is denoted by $H_1^S \blacktriangleright (H_2^V \cup H_3^E)$. The degree of vertex $u \in H_1^S \blacktriangleright (H_2^V \cup H_3^E)$ is given by

$$d_{H_1^S \blacktriangleright (H_2^V \cup H_3^E)}(u_i) = \begin{cases} d_{H_1}(u_i) + p_2, & \text{if } u \in V(H_1); \\ 2 + p_1, & \text{if } u \in I(H_1); \\ d_{H_2}(u_j) + p_1, & \text{if } u \in V(H_2), j = 1,2, \ldots p_2; \\ d_{H_3}(w_k), & \text{if } u = w_k \in V(H_3), k = 1,2 \ldots p_3. \end{cases}$$

$H_1^S \blacktriangleright (H_2^V \cup H_3^E)$ has $p_1 + q_1 + p_2 + p_3$ vertices. The following Theorem is easily obtained by using equation (1.1).

**Theorem 6.1.** $E_Q(H_1^S \blacktriangleright (H_2^V \cup H_3^E) = E_Q(H_1) + E_Q(H_2) + E_Q(H_3) + 4q_1p_2 + q_1p_2^2 + 6q_1 + 5q_1p_1 + p_1^2q_1 + 4q_2p_1 + p_1^2p_2 + 4q_1q_3 + q_1^2p_3 + 2p_1p_2 + q_1p_3$.

### Coclusion:

We establish a relation between the quasi-Laplacian energy of few novel graphs and their respective original graphs. The formulation is predicated on the relationships between quasi Laplacian energy and vertex degree in the novel graphs. Moreover, it is directly associated with the first Zagreb index, quantity of vertices and edges of the graph.

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