

# Using W Transform for Solving Volterra Integro-Differential Equations

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## Abstract:

There are numerous uses for the Volterra integro-differential equation in the fields of mechanics, geometric probability, population dynamics, theory of rejuvenation, facts on particle size and the damping of string vibration, and transmission of heat issues. Finding the approximate or exact solutions to these equations is of interest to many mathematicians and scientists. Our aim of this paper is to explore and figure out the solution of the Volterra integro-differential equation with a convolution kernel. We now introduce the W transform for determining the solution of linear Volterra integro-differential equation of the second kind and their system. The ability of the W Transform to solve these equations is shown by real-world applications. Based on what we discovered, the W Transform is an effective method for locating precise answers to the linear Volterra integro-differential equation of the second kind and their system.

**Keywords:** Volterra Integro-Differential Equation, W Transform, Convolution, Inverse of W Transform.

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## 1 INTRODUCTION:

The study of integral equations possibly considered fundamentally more significant than differential equations since, compared to differential equations, integral equations frequently offer more effective models and less restrictions. Additionally, differentiation in numerical analysis tends to increase error, whereas integration tends to reduce error.

The growing use of integral equations in the literature and in many areas of applied mathematics is evidence of their advantages, as stated that certain issues naturally and directly exhibit their mathematical representation, according to integral equations. additional issues, whose direct form is in terms of differential equations, have integral equations more eloquently and compactly substituted for their auxiliary conditions.

Integro-differential equations are a common mathematical formulation used to describe physical processes, Several domains, including physics, astronomy, potential theory, fluid dynamics, chemical kinetics, and biological models, use these equations.

Multiple techniques have been applied in recent years through certain investigators to solve linear Volterra integro-differential equation of the second kind and their system (LVI-DE of 2<sup>nd</sup> Kind & LSVI-DE of 2<sup>nd</sup> kind). We introduce some numerical and analytical methods for solving VI-DE of 2<sup>nd</sup> Kind & SVI-DE of 2<sup>nd</sup> kind.

Taiye Oyedepo et al., presented a computing methodology to deal with VIDEs using shifted Vieta-Lucas polynomials as the foundational basis functions[1].G.A. Aghayeva, et al., Provides a comparison among the mathematical methods used to address the integral Volterra, and ODEs [2]. Asiya Ansari et al., to solve the linear or non-linear VIDE, the Series Solution Method (SSM) is utilized[3]. Ahmad Issa apply the rebuilding of method of Variational Iteration for finding numerical solution of LSVI-DE [4], Mohamed E.A. Alnair and Ahmed A. Khidir are displays a new novel method for resolving LVI-DEs under border restrictions, the technique is founded on combining the Chebyshev spectral techniques [5]. R.A. Olowe et al., By using the multistep collocation method, trigonometry third derivative matched Simpson's method is created and used in order to get close to the answer of VIDEs[6]. O. A. Uwaheren et al., Utilizing Legendre polynomials as basis functions, Akbari-Ganji's Method (AGM) was utilized to solve VIDE[7]. A. Al-Shimmery et al., By using the sixth-order runge-kutta method, they can solve VIDE numerically[8].

Monali Derle and Dinkar Patil , Convolution theory and dual generalization with the purpose of solving IDE, the Rangaig integral transformation had been employed[9]. S. Aggarwal et al., When dealing with LVI-DE of 2<sup>nd</sup> kind, apply the Sadik transform[10]. S. Al-Ahmad et al., utilize the modified differential transform approach to discover the analytical solution for the LSVI-DE[11]. H. Bozburun and H. A. Peker, apply the Shehu transform to solve the LVI-DE of 2<sup>nd</sup> kind[12]. N. A. Elbhilil et al., solved the Volterra integral and LVI-DE using the Abaoub-Shkheam transform techniques[13]. H. K. Jassim use the Yang-Laplace transform to determine the analytical solutions for LVI-DE within local fractional operators[14]. Z. Rustam and N. Sulaiman are used (Kamal & SEE) transformation methodology for solving LSVI-DE of 2<sup>nd</sup> kind[15, 16]. Marjan Uddin and Musafir Uddin, regarding the Laplace transform-based numerical approximation of the LVI-DE[17].

An investigation and use of a recently developed integral transform, or W transform, were conducted by Ping Wang et al. At this point, It has also been established how the W transform relates to other transforms like (Laplace, Sumudu, Natural transform, Elzaki, Mohand, Aboodh, Sawi, Yang, Emad-Falih, Fareeha and Pourreza transform). To show how successful this transformation is, they have solved the differential and integral equations[18].

The goal of this work is to use the W transform to quickly and simply solve the LVI-DE of 2<sup>nd</sup> Kind & LSVI-DE of 2<sup>nd</sup> kind.

## 2 Basic Definitions

**Definition 1:** The LVI-DE of 2<sup>nd</sup> Kind is provided by [19]:

$$\varphi^{(n)}(\omega) = H(\omega) + \int_0^\omega K(\omega, \tau) \varphi(\tau) d\tau \quad (1)$$

With

$$\varphi^{(m)}(0) = b_m \quad , 0 < m < n - 1$$

In which the unidentified function  $\varphi(\tau)$  will be decided, only show up within the integral sign, whereas the derivative of  $\varphi(\omega)$  usually take place outside of the sign of integral.



4.	nth Derivative	$W\{\varphi^{(k)}(\omega)\} = s^{kn}W(\varphi(\omega)) - s^m \sum_{j=0}^{k-1} s^{n(k-j-1)} \varphi^{(j)}(0), k \geq 1$
5.	Convolution	$W\{\varphi_1(\omega) * \varphi_2(\omega)\} = \frac{1}{s^m} W\{\varphi_1(\omega)\} * W\{\varphi_2(\omega)\}$

**Table 2: W Transform of useful Functions**

S.N	$\varphi(\omega)$	$W\{\varphi(\omega)\} = \mathcal{F}(s)$
1	c	$c \frac{s^m}{s^n}$
2.	$\omega$	$\frac{s^m}{s^{2n}}$
3.	$a\omega^k$	$\frac{a k! s^m}{s^{n(k+1)}}, a \text{ constant}$
4.	$e^{a\omega}$	$\frac{s^m}{s^n - a}$
5.	$\sin(a\omega)$	$\frac{a s^m}{s^{2n} + a^2}$
6.	$\cos(a\omega)$	$\frac{s^{m+n}}{s^{2n} + a^2}$
7.	$\sinh(a\omega)$	$\frac{a s^m}{s^{2n} - a^2}$
8.	$\cosh(a\omega)$	$\frac{s^{m+n}}{s^{2n} - a^2}$

**3 Methodology**

Within this part, We introduce the W transform as a means of solving LVI-DE of 2<sup>nd</sup> Kind and LSVI-DE of 2<sup>nd</sup> kind. In this piece of writing, Assumed to be the kernel  $K(\omega, \tau)$  can be expressed by difference, which is  $(\omega - \tau)$ .

I.Use W Transform to Resolve LVI-DE of 2<sup>nd</sup> Kind:

$$\varphi^{(n)}(\omega) = H(\omega) + \int_0^\omega K(\omega - \tau) \varphi(\tau) d\tau \tag{5}$$

With





$$\left. \begin{aligned}
 \begin{cases} s^{kn}W(\varphi_1(\omega)) \\ -s^{m+n(k-1)}a_{10} \\ -s^{m+n(k-2)}a_{11} \\ \dots -s^m a_{1k-1} \end{cases} &= W\{H_1(\omega)\} + \left[ \begin{aligned} &\frac{1}{s^m}W\{K_{11}(\omega)\}W\{\varphi_1(\omega)\} + \frac{1}{s^m}W\{K_{12}(\omega)\}W\{\varphi_2(\omega)\} \\ &\dots + \frac{1}{s^m}W\{K_{1p}(\omega)\}W\{\varphi_p(\omega)\} \end{aligned} \right] \\
 \begin{cases} s^{kn}W(\varphi_2(\omega)) \\ -s^{m+n(k-1)}a_{20} \\ -s^{m+n(k-2)}a_{21} \\ \dots -s^m a_{2k-1} \end{cases} &= W\{H_2(\omega)\} + \left[ \begin{aligned} &\frac{1}{s^m}W\{K_{21}(\omega)\}W\{\varphi_1(\omega)\} + \frac{1}{s^m}W\{K_{22}(\omega)\}W\{\varphi_2(\omega)\} \\ &\dots + \frac{1}{s^m}W\{K_{2p}(\omega)\}W\{\varphi_p(\omega)\} \end{aligned} \right] \\
 \dots & \\
 \begin{cases} s^{kn}W(\varphi_p(\omega)) \\ -s^{m+n(k-1)}a_{p0} \\ -s^{m+n(k-2)}a_{p1} \\ \dots -s^m a_{pk-1} \end{cases} &= W\{H_p(\omega)\} + \left[ \begin{aligned} &\frac{1}{s^m}W\{K_{p1}(\omega)\}W\{\varphi_1(\omega)\} + \frac{1}{s^m}W\{K_{p2}(\omega)\}W\{\varphi_2(\omega)\} \\ &\dots + \frac{1}{s^m}W\{K_{pp}(\omega)\}W\{\varphi_p(\omega)\} \end{aligned} \right]
 \end{aligned} \right] \quad (13)$$

After simplification system (13), we get

$$\left. \begin{aligned}
 \begin{cases} [s^{kn} - \frac{1}{s^m}W\{K_{11}(\omega)\}]W\{\varphi_1(\omega)\} \\ -\frac{1}{s^m}W\{K_{12}(\omega)\}W\{\varphi_2(\omega)\} \\ \dots -\frac{1}{s^m}W\{K_{1p}(\omega)\}W\{\varphi_p(\omega)\} \end{cases} &= \begin{bmatrix} W\{H_1(\omega)\} + s^{m+n(k-1)}a_{10} \\ +s^{m+n(k-2)}a_{11} \\ +\dots +s^m a_{1k-1} \end{bmatrix} \\
 \begin{cases} -\frac{1}{s^m}W\{K_{21}(\omega)\}W\{\varphi_1(\omega)\} \\ +[s^{kn} - \frac{1}{s^m}W\{K_{22}(\omega)\}]W\{\varphi_2(\omega)\} \\ \dots -\frac{1}{s^m}W\{K_{2p}(\omega)\}W\{\varphi_p(\omega)\} \end{cases} &= \begin{bmatrix} W\{H_2(\omega)\} + s^{m+n(k-1)}a_{20} \\ +s^{m+n(k-2)}a_{21} \\ +\dots +s^m a_{2k-1} \end{bmatrix} \\
 \dots & \\
 \begin{cases} -\frac{1}{s^m}W\{K_{p1}(\omega)\}W\{\varphi_1(\omega)\} \\ -\frac{1}{s^m}W\{K_{p2}(\omega)\}W\{\varphi_2(\omega)\} \\ \dots + [s^{kn} - \frac{1}{s^m}W\{K_{pp}(\omega)\}]W\{\varphi_p(\omega)\} \end{cases} &= \begin{bmatrix} W\{H_p(\omega)\} + s^{m+n(k-1)}a_{p0} \\ +s^{m+n(k-2)}a_{p1} \\ +\dots +s^m a_{pk-1} \end{bmatrix}
 \end{aligned} \right] \quad (14)$$

The solution of system (14) is given as





Utilizing the asset “W transforms of derivatives” in equation (18), We own

$$s^n W\{\varphi(\omega)\} - s^m \varphi(0) = \frac{s^m}{s^n} - \frac{1}{s^m} \left( \frac{s^m}{s^n} \right) W\{\varphi(\omega)\} \quad (19)$$

substituting initial condition (16) in (19), after simplification of equation (19) and operating inverse W transforms, we get the required solution of equation

$$\varphi(\omega) = \sin(\omega) \quad (20)$$

**Problem 2:** Consider the LVI-DE of 2<sup>nd</sup> Kind.

$$\varphi''(\omega) = 1 + \int_0^\omega (\omega - \tau) \varphi(\tau) d\tau \quad (21)$$

$$\text{with } \varphi(0) = 1, \varphi'(0) = 0 \quad (22)$$

Utilizing the W transform to each side of (21), We've got

$$W\{\varphi''(\omega)\} = W\{1\} + W\left\{\int_0^\omega (\omega - \tau) \varphi(\tau) d\tau\right\} \quad (23)$$

as well as employing the convolution theorem on equation (23), We obtain

$$W\{\varphi''(\omega)\} = W\{1\} + \frac{1}{s^m} W\{\omega\} W\{\varphi(\omega)\} \quad (24)$$

Making use of the asset “w transforms of derivatives” on equation (24), We own

$$s^{2n} W\{\varphi(\omega)\} - s^{m+n} \varphi(0) - s^m \varphi'(0) = \frac{s^m}{s^n} + \frac{1}{s^m} \left( \frac{s^m}{s^{2n}} \right) W\{\varphi(\omega)\} \quad (25)$$

substituting initial condition (22) in (25), after simplification of equation (25) and operating inverse W transforms, we get the required solution of equation

$$\varphi(\omega) = \cosh(\omega) \quad (26)$$

**Problem 3:** Consider the LSVI-DE of 2<sup>nd</sup> kind.

$$\left\{ \begin{array}{l} \varphi_1'(\omega) = 2\omega^2 + \int_0^\omega [(\omega - \tau)\varphi_1(\tau) + (\omega - \tau)\varphi_2(\tau)] d\tau \\ \varphi_2'(\omega) = -3\omega^2 - \frac{1}{10}\omega^5 + \int_0^\omega [(\omega - \tau)\varphi_1(\tau) - (\omega - \tau)\varphi_2(\tau)] d\tau \end{array} \right\} \quad (27)$$

$$\text{with } \varphi_1(0) = 1, \varphi_2(0) = 1 \quad (28)$$

Implementing the W transform to both parties of (27), We've got

$$\left\{ \begin{array}{l} W\{\varphi_1'(\omega)\} = W\{2\omega^2\} + W\left\{\int_0^\omega [(\omega - \tau)\varphi_1(\tau) + (\omega - \tau)\varphi_2(\tau)] d\tau\right\} \\ W\{\varphi_2'(\omega)\} = W\{-3\omega^2\} - W\left\{\frac{1}{10}\omega^5\right\} + W\left\{\int_0^\omega [(\omega - \tau)\varphi_1(\tau) - (\omega - \tau)\varphi_2(\tau)] d\tau\right\} \end{array} \right\} \quad (29)$$

and using the convolution theorem of the W transform on equation (29), we get

$$\left\{ \begin{aligned} W\{\varphi_1'(\omega)\} &= W\{2\omega^2\} + \frac{I}{s^m} [ W\{\omega\}W\{\varphi_1(\omega)\} + W\{\omega\}W\{\varphi_2(\omega)\}] \\ W\{\varphi_2'(\omega)\} &= W\{-3\omega^2\} - W\left\{\frac{1}{10}\omega^5\right\} + \frac{I}{s^m} [ W\{\omega\}W\{\varphi_1(\omega)\} - W\{\omega\}W\{\varphi_2(\omega)\}] \end{aligned} \right\} \quad (30)$$

Making use of the asset “W transforms of derivatives” on equation (30), We've got

$$\left\{ \begin{aligned} s^n W\{\varphi_1(\omega)\} - s^m \varphi_1(0) &= 2\left(\frac{2! s^m}{s^{3n}}\right) + \frac{I}{s^m} \left[ \frac{s^m}{s^{2n}} W\{\varphi_1(\omega)\} + \frac{s^m}{s^{2n}} W\{\varphi_2(\omega)\} \right] \\ s^n W\{\varphi_2(\omega)\} - s^m \varphi_2(0) &= -3\left(\frac{2! s^m}{s^{3n}}\right) - \frac{I}{10} \left(\frac{5! s^m}{s^{6n}}\right) + \frac{I}{s^m} \left[ \frac{s^m}{s^{2n}} W\{\varphi_1(\omega)\} - \frac{s^m}{s^{2n}} W\{\varphi_2(\omega)\} \right] \end{aligned} \right\} \quad (31)$$

substituting initial condition (28) in system (31) and after simplification we get

$$\left\{ \begin{aligned} \left(\frac{s^{3n} - I}{s^{2n}}\right) W\{\varphi_1(\omega)\} - \left(\frac{I}{s^{2n}}\right) W\{\varphi_2(\omega)\} &= \frac{4s^m + s^{m+3n}}{s^{3n}} \\ \left(\frac{-I}{s^{2n}}\right) W\{\varphi_1(\omega)\} + \left(\frac{s^{3n} + I}{s^{2n}}\right) W\{\varphi_2(\omega)\} &= \frac{-6s^{m+3n} - 12s^m + s^{m+6n}}{s^{6n}} \end{aligned} \right\} \quad (32)$$

The solution of system (32) is given as

$$W\{\varphi_1(\omega)\} = \frac{\begin{vmatrix} \frac{4s^m + s^{m+3n}}{s^{3n}} & -\frac{I}{s^{2n}} \\ \frac{-6s^{m+3n} - 12s^m + s^{m+6n}}{s^{6n}} & \frac{s^{3n} + I}{s^{2n}} \end{vmatrix}}{\begin{vmatrix} \frac{s^{3n} - I}{s^{2n}} & -\frac{I}{s^{2n}} \\ \frac{-I}{s^{2n}} & \frac{s^{3n} + I}{s^{2n}} \end{vmatrix}}$$

$$W\{\varphi_2(\omega)\} = \frac{\begin{vmatrix} \frac{s^{3n} - I}{s^{2n}} & \frac{4s^m + s^{m+3n}}{s^{3n}} \\ \frac{-I}{s^{2n}} & \frac{-6s^{m+3n} - 12s^m + s^{m+6n}}{s^{6n}} \end{vmatrix}}{\begin{vmatrix} \frac{s^{3n} - I}{s^{2n}} & -\frac{I}{s^{2n}} \\ \frac{-I}{s^{2n}} & \frac{s^{3n} + I}{s^{2n}} \end{vmatrix}}$$

When the aforementioned equations are simplified, we obtain the values of

$$W\{\varphi_1(\omega)\} = \frac{s^m}{s^n} + \frac{6s^m}{s^{4n}}$$

$$W\{\varphi_2(\omega)\} = \frac{s^m}{s^n} - \frac{6s^m}{s^{4n}}$$

operating inverse  $W$  transforms, we get the required solution of equations

$$\varphi_1(\omega) = I + \omega^3$$

$$\varphi_2(\omega) = I - \omega^3$$

**5 Conclusion:** In this research, we have effectively addressed the  $W$  transform for resolving LVI-DE of 2<sup>nd</sup> Kind and LSVI-DE of 2<sup>nd</sup> kind, and we have extensively detailed the process by considering three numerical problems. The answers to these issues show how beneficial and efficient the  $W$  transform in whitening the resolving LVI-DE of 2<sup>nd</sup> Kind and LSVI-DE of 2<sup>nd</sup> kind. The provided applications demonstrate that a precise solution was found in a very short amount of time and with very little processing power.

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