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Fault-Tolerant Strong Metric Dimension of Rooted Product Graphs

H. Prathab[1,2], S. Chandrasekaran[3] and Sathish Krishnan[4*]

[1]Department of Mathematics, Saveetha Engineering College, Chennai, 602105, India.

[2]Research Scholar, P.G & Research Department of Mathematics

Pachaiyappa's College, Chennai, 600030, India.

[3]P.G & Research Department of Mathematics, Pachaiyappa's College, Chennai, 600030, India.

[4*]Department of Mathematics and Statistics, Faculty of Science and Humanities

SRM Institute of Science and Technology, Chengalpattu, 603203, India.

*Corresponding Author. e-mail: satiskris@gmail.com sathishk6@srmist.edu.in

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Abstract:

The concept of fault tolerance in graph theory is critical in designing robust networks, ensuring that essential graph properties are preserved despite failures of vertices or edges. In this paper, we investigate the fault-tolerant strong metric dimension of rooted product graphs, a class of graphs derived by attaching multiple copies of a rooted graph to each vertex of a base graph. We extend the notion of a strong metric dimension by considering scenarios where the strong resolving set remains effective even after a specific number of vertices have failed. We provide exact values for specific families of rooted product graphs and demonstrate how the fault-tolerant property varies with the structure of the root and base graphs.

Keywords: Metric dimension, strong metric dimension, fault-tolerant strong metric dimension, rooted product graphs

1. Introduction

The study of graph invariants has long been central to graph theory, offering a framework for understanding the structural and combinatorial properties of networks. Among these invariants, the metric dimension and its variants play a pivotal role in problems related to network navigation, resource allocation, and information retrieval. The metric dimension of a graph quantifies the minimum number of vertices required to uniquely determine the position of any other vertex within the graph using distances. The strong metric dimension strengthens the classical concept of metric dimension by imposing stricter conditions on resolving sets, making it particularly useful in applications where precise and robust vertex identification is essential.

In modern applications, networks are often subject to failures or disruptions, making fault tolerance a crucial consideration. The fault-tolerant strong metric dimension addresses this challenge by identifying resolving sets that retain functionality even when certain vertices or edges fail. This concept is particularly significant in the design of resilient communication and transportation networks, where the ability to maintain operability under adverse conditions is essential.

Rooted product graphs, a construction obtained by combining a base graph with multiple rooted subgraphs, offer a versatile model for various real-world systems. Their hierarchical structure and inherent modularity make them a natural candidate for studying fault-tolerant properties. Despite their relevance, the fault-tolerant strong metric dimension of rooted product graphs remains underexplored in the literature.

In this paper, we investigate the fault-tolerant strong metric dimension of rooted product graphs. The results presented here contribute to the theoretical understanding of fault-tolerant graph properties.

We exclude the definitions of conventional graph-theoretical concepts. These can be read in [1] and other textbooks.

The metric dimension of a graph $G = (V, E)$ is the minimum number of vertices in a subset $R \subseteq V$ such that every pair of distinct vertices in G is uniquely identified by their distances to the vertices in R. A set R with this property is called a resolving set, as it enables a unique identification of vertices based on their distance vectors relative to the elements of R.

The concept of metric dimension was first studied by Slater [13] (independently by Harary and Melter [3]). Another invariant, more restricted than the metric dimension was presented by Sebö and Tannier in [11], and studied further in several articles [4, 5, 7, 8, 9, 10, 14].

A vertex $s \in S$ is said to strongly resolve x and y if $d(x, s) = d(x, y) + d(y, s)$ or $d(y, s) = d(y, x) + d(x, s)$, where $d(x, y)$ denotes a shortest path distance between vertices x and y in G. The strong metric dimension of a graph $G = (V, E)$ is the minimum cardinality of a subset $S \subseteq V$ such that for every pair of distinct vertices $x, y \in V$, there exists a vertex $s \in S$ that strongly resolves x and y.

A fault-tolerant strong resolving set S for a graph G is a set such that for every vertex $s \in S$, the set $S \setminus \{s\}$ remains a strong resolving set for G. The fault-tolerant strong metric dimension of G, denoted $dim_{fs}(G)$, is the smallest cardinality of a fault-tolerant strong resolving set for G [6].

A vertex u of G is said to be maximally distant from v if for every $w \in N(u)$, $d(v, w) \leq d(v, u)$. If u is maximally distant from v and v is maximally distant from u, then u and v are said to be mutually maximally distant.

The following Lemma and Theorem mentioned in [6] is useful in the sequel.

Lemma 1: [6] Let G be a simple connected graph and let S be a fault-tolerant strong resolving set of G. Let $u, v \in V$ be mutually maximally distant in G. Then both u and $v \in S$.

Theorem 1: [6] A strong resolving set S of a graph G is a fault-tolerant strong resolving set if and only if every pair of vertices in G is strongly resolved by at least two vertices of S.

2. Main results

A graph in which one vertex is fixed as a root vertex to distinguish it from other vertices is called a rooted graph. Let G be a graph with n vertices and H be a sequence of n rooted graphs $H_1, H_2 \dots H_n$. The rooted product graph $G(H)$ is obtained from the graphs $G, H_1, H_2 \dots H_n$ by identifying the root vertex of H_i with the i^{th} vertex of G [2].

If H consists of n isomorphic rooted graphs $H_1 \cong H_2 \cong \ldots \cong H_n$, then a particular kind of rooted product graph is generated from G and H, where $H \cong H_i$ for all $i \leq n$. Such a graph is denoted by Go_vH, where v is the root vertex of H [12]. For $V(G) = \{u_1, u_2 \ldots u_n\}$, the vertex set V and edge set E of the rooted product graph Go_vH is defined as $V = V(G) \times V(H)$ and

$$E = \bigcup_{i=1}^{n}\{(u_i,b)(u_i,y) : by \in E(H)\} \cup \{(u_i,v)(u_j,v) : u_iu_j \in E(G)\}.$$

We start by stating the following easily verified theorem.

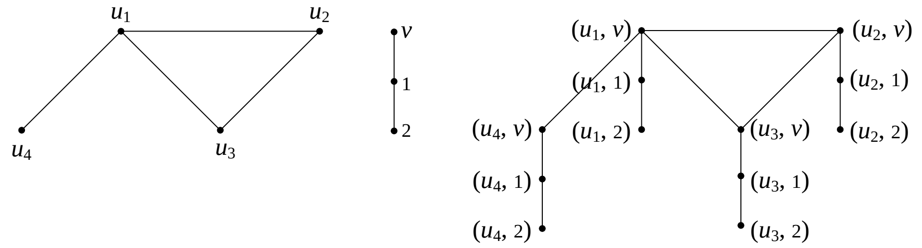
Theorem 2: Let G and H be two graphs of order n_1 and n_2 respectively. Then no two vertices of G are mutually maximally distant in Go_vH.

Proof. Assume the contrary. Suppose that there exist two vertices $(u_i,v),(u_j,v), i \neq j, 1 \leq i,j \leq n_1$ which are mutually maximally distant in Go_vH. Then for every $(u_k,v) \in N\left((u_j,v)\right)$, $d\big((u_i,v),(u_k,v)\big) \leq d\left((u_i,v),(u_j,v)\right)$. Let $d\left((u_i,v),(u_j,v)\right) = l$. Since $(u_j,v),(u_j,w) \in E(Go_vH)$ for some $vw \in E(H_j)$ in Go_vH, $d\left((u_i,v),(u_j,w)\right) = d\left((u_i,v),(u_j,v)\right) + 1 = l + 1$, which contradicts the fact that $d\big((u_i,v),(u_k,v)\big) \leq d\left((u_i,v),(u_j,v)\right)$. Hence vertices (u_i,v) and (u_j,v) are not mutually maximally distant.

By Theorem 2, the following results are obvious.

Lemma 2: Let G be a connected graph of order $n \geq 2$ and H be a connected graph. Let M be the set of all mutually maximally distant vertices in H. Then

1. If $v \in M$, then $dim_{fs}(\mathrm{G}o_v\mathrm{H}) = n|M\backslash\{v\}|$.
2. If $v \notin M$, then $dim_{fs}(Go_v\mathrm{H}) = n|M|$.



(*a*) (*b*) (*c*)

Figure 1 (*a*) Graph G (*b*) P_3 (*c*) $\mathrm{G}o_vP_3$ where v has degree 1.

Consider the rooted product graph Go_vP_m, where $m \geq 2$. The degree of root vertex v of P_m, can be either 1 or 2. Depending on the degree of root vertex in P_m, the different classes of graphs can be generated. If the degree of the root vertex v is 1, then the vertices of i^{th} copy of P_m are labelled as (u_i, j), $1 \leq j \leq m - 1$ in Go_vP_m, except (u_i, v). See Figure 1. If the degree of the root vertex v is 2, then the i^{th} copy of P_m will have two distinct paths $(u_i, 1), (u_i, 2) \ldots (u_i, a) \in P_a$ and $(w_i, 1), (w_i, 2) \ldots (w_i, b) \in P_b$ in Go_vP_m such that $a + b + 1 = m$. See Figure 2.

Theorem 3: Let G be a connected graph of order $n \geq 2$. For $m \geq 2$, $dim_{fs}(\mathrm{G}o_v P_m) = \begin{cases} n\,, & deg(v) = 1; \\ 2n, & deg(v) = 2. \end{cases}$

Proof. We have two cases.

Case 1: $deg(v) = 1$.

Let $(u_i, m-1)$, $1 \leq i \leq n$ be the pendant vertices of $\mathrm{G}o_v P_m$. Clearly the pendant vertices $(u_i, m-1)$ are pairwise mutually maximally distant. Hence by Lemma 1, any fault- tolerant strong resolving set of $\mathrm{G}o_v P_m$ contains the pendant vertices. Define W = $\{(u_i, m-1) : 1 \leq i \leq n\}$.

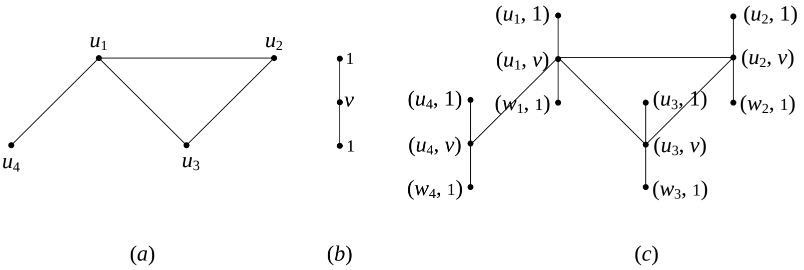


Figure 2 (a) Graph G (b) P_3 (c) $\mathrm{G}o_v P_3$ where v has degree 2.

On the other hand, let $W = \{(u_i, m-1) : 1 \leq i \leq n\}$ and let $(u_i, k_1), (u_j, k_2) \in V \backslash W$. Clearly the vertices (u_i, k_1) and (u_j, k_2) are strongly resolved by both $(u_i, m-1)$ and $(u_j, m-1)$; the shortest path from (u_i, k_1) to $(u_j, m-1)$ contains (u_j, k_2) and similarly the shortest path from (u_j, k_2) to $(u_i, m-1)$ contains (u_i, k_1). Hence by Theorem 2, W is a fault-tolerant strong resolving set.

Case 2: $deg(v) = 2$.

Let (u_i, a) and (w_i, b), $1 \leq i \leq n$, be the pendant vertices of $\mathrm{G}o_v P_m$. Clearly the pendant vertices (u_i, a) and (w_i, b) are pairwise mutually maximally distant. Hence by Lemma 1, any fault-tolerant strong resolving set of $\mathrm{G}o_v P_m$ contains the pendant vertices. Define $W = \{(u_i, a), (w_i, b) : 1 \leq i \leq n\}$.

On the other hand, let $W = \{(u_i, a), (w_i, b) : 1 \leq i \leq n\}$ and let $(u_i, k_1), (u_j, k_2) \in V \backslash W$. Clearly the vertices (u_i, k_1) and (u_j, k_2) are strongly resolved by both $(u_i, a), (w_i, b)$ and $(u_j, a), (w_j, b)$. Hence by Theorem 2, W is a fault- tolerant strong resolving set.

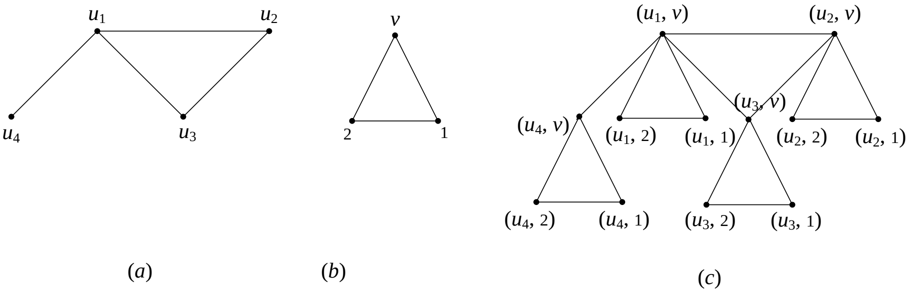


Figure 3 (a) Graph G (b) C_3 (c) $Go_v C_3$ where v has degree 2.

Consider the rooted graph $\mathrm{G}o_vC_m$, where $m \geq 3$. Clearly the degree of root vertex v is 2 and the vertices of i^{th} copy of C_m are labelled as $(u_i, j), 1 \leq i \leq m-1$ in $\mathrm{G}o_vC_m$ except (u_i, v). See Figure 3.

Theorem 4: Let G be a connected graph of order $n \geq 2$. For $m \geq 2$, $dim_{fs}(\mathrm{G}o_vC_m) = n(m-1)$.

Proof. Let $(u_i, v), (u_i, 1), (u_i, 2) \dots (u_i, m-1)$ be the vertices of i^{th} copy of C_m in $\mathrm{G}o_vC_m$. If m is even, then there is exactly one antipodal vertex (u_i, k^*) from $(u_i, k), 1 \leq i \leq n, 1 \leq k \leq m-1$. Hence the vertices $(u_i, k), 1 \leq k \leq m-1$, must belong to any fault-tolerant strong resolving set W.

If m is odd, then there are exactly two antipodal vertices, say (u_i, k_1^*) and (u_i, k_2^*) from $(u_i, k), 1 \leq i \leq n, 1 \leq k \leq m-1$. Hence the vertices $(u_i, k), 1 \leq k \leq m-1$, must belong to any fault-tolerant strong resolving set W. Since this is true for every $i, 1 \leq i \leq n, |W| = n(m-1)$.

Similarly, we have the following results.

Theorem 5: Let G be a connected graph of order $n \geq 2$. For $m \geq 3$, $dim_{fs}(\mathrm{G}o_vK_m) = n(m-1)$.

Theorem 6: Let G be a connected graph of order $n \geq 2$ and T be a tree. Then $dim_{fs}(\mathrm{G}o_vT)$ equals the number of leaves in $\mathrm{G}o_vT$.

A wheel graph $W_{1,m}$, $m \geq 3$, can also be seen as the graph obtained by adding a single vertex (the hub) to a cycle C_m where the hub is connected to all vertices in C_n.

Theorem 7: Let G be a connected graph of order $n \geq 2$. For $m \geq 3$, $dim_{fs}(\mathrm{G}o_vW_m) =$
$$\begin{cases} mn-1, & deg(v) = m; \\ n, & deg(v) = m-1. \end{cases}$$

3. Conclusion

In this paper, we have explored the fault-tolerant strong metric dimension of rooted product graphs, extending the concept of strong metric dimension to account for vertex. We have computed exact values for several families of rooted product graphs, highlighting how the fault-tolerant property depends on the structural characteristics of both the root and base graphs. Future work may involve exploring fault-tolerance in other graph products or generalizing these results to other graph families, with potential applications in areas such as distributed computing, network topology, and communication systems.

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