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Some more Extensions on Product Cordial Graphs

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Abstract:

A mapping function $f: \alpha(G)\ to\ \{0,1\}$, causes an labeling of edge $f: \beta(G) \rightarrow \{0,1\}$ as specified by $f*(ty) = f(t) * f(y)$. This type of labeling is called product cordial labeling if it satisfies certain circumstances regarding the differences in the number of vertices and edges labeled with 0 and 1. In this investigation, we prove that the copies of concentric n-sunlet graphs, star of concentric n-sunlet graph and copies of shell graph satisfies these conditions, eventually they are product cordial graphs.

Keywords: Graph labeling, Product cordial labeling, Concentric n-sunlet graph, Star of concentric n-sunlet graph, Shell graph.

1. Introduction

Graph labeling is a fundamental concept in graph theory, involving the assignment of integers, typically 0 and 1, to the vertices or edges of a graph. This area of study has diverse applications across various domains, including network modeling, coding theory, circuit layout optimization, database management, protocol design, and resource allocation.

Cordial labeling, introduced by I. Cahit [2] in 1958, is a specific type of graph labeling where the labels assigned to the graph of vertices or edges aim to balance the frequencies of adjacent labels. Specifically, the differences in the frequencies of labels among adjacent elements (vertices or edges) should be either 0 or 1. This concept has significantly contributed to the theoretical understanding of graph theory and offers a framework for analyzing the structural properties of graphs.

The concept of product cordial labeling was introduced by Sundaram et al. [11] as an extension of the cordial labeling theory. Gallian [4] introduced a dynamic survey of graph labeling. Vaidya and Dani [14] demonstrated that joining apex vertices of k copies of stars, shells, and wheels to a new vertex results in product cordial graphs. Some graphs in the context of product cordial labeling proved by prajapati et al. [7]. Srivastav and Gupta [10] divisor 3-Equitable Labeling of Graphs. Vaidya and Kanani [15] focused on cycle-related graphs, while Vaidya and Barasara [13] proved that cycles with chords, friendship graphs, and middle graphs of paths admit product cordial labeling, and also investigated graph operations. Mathew [5] demonstrated that vector switching of signed graphs, Vaidya and Vyas [12] explored tensor products of graphs. Acharya and Kureethara [1] focused on dragon graph. Chartrand et al. [3] introduced on subset labelings of trees. Raja et al. [8] proved that eccentric completion of a graph. Cordial labeling for different types of shell graph demonstrated by Meena et al. [6]. Sadawarte and Srivastav [9] on Sum 3-Equitable Labeling of Some Graphs. These contributions enhance the understanding and applicability of product cordial labeling in various domains of graph theory.

In this concern research, we focus on product cordial graph of specific types of graphs, such as copies of the concentric n-sunlet graph, the star of the concentric n-sunlet graph, and copies of the shell graph. These graphs exhibit unique structural properties and complexities, making them ideal subjects for studying labeling properties. By examining their product cordial labeling, we aim to gain valuable insights into their characteristics and relationships, further advancing the field of graph theory and its applications.

2. Terminology And Notation

DEFINITION 2.1 : The definition of a graph labeling, It explains that graph labeling involves giving the vertices, edges, or maybe both numeric values, based on the specific demands of the problem or application. The labeling is called vertex labeling (or edge labeling) when the set of vertices (or edges) serves as the domain for the mapping

DEFINITION 2.2 : A product cordial of a graph G involves a labelling of vertex $g: \alpha \rightarrow \{0,1\}$, which induces an labelling of edge $g: \beta(G) \rightarrow \{0,1\}$, where $g * (ty) = g(t)g(y)$. For the graph G to be considered a product cordial graph, the labeling must meet the conditions: the difference between the number of vertices labeled 0 and those labeled 1, $|\alpha_g(0) - \alpha_g(1)|$, must be less than or equal to 1, and similarly, the distinction between the number of edges labeled 0 and those labeled 1, $|\beta_g(0) - \beta_g(1)|$,must also be less than or equal to 1.

DEFINITION 2.3 : A star graph $S_{1,n}$ is a tree graph consisting of one central node α_1 and n leaf nodes $\alpha_2, \alpha_3, \dots, \alpha_{m+1}$, where each leaf node is connected directly to the central node v_1 say (α_1, α_i) for $i = 2,3, \dots, m + 1$. There are no edges between the leaf nodes themselves.

DEFINITION 2.4 : The n-sunlet graph, indicated as S_n, consisting of $2n$ vertices, formed by connecting n pendant edges to the cycle C_n.

DEFINITION 2.5 : The G graph created by connecting three replicas of the n-sunlet graph is 3-concentric n-sunlet graph denoted by $3 * S_n$. Inner most copy of n-Sunlet graph as $\alpha_1, \alpha_2, \dots \alpha_n$ and its spokes denoted as $\alpha_1', \alpha_2', \dots, \alpha_n'$, the spokes of inner most sunlet graph is fused with the vertices of middle graph ,the vertices of middle copy of n-sunlet graph is now denoted as $\alpha_1', \alpha_2', \dots, \alpha_n'$ and its spokes as $x_1', x_2', \dots . x_3'$ and again its spokes are fused with vertices of outer sunlet graph. The outer most copy is denoted by $x_1', x', \dots . x_3'$ and spokes are denoted as $w_1', w_2', \dots . , w_n'$. In below Figure 1 we have considered 3*S4 as an example.

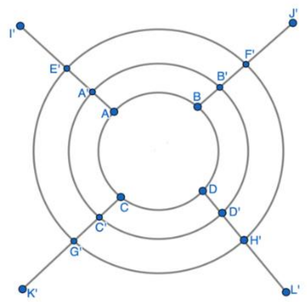

Figure 1: $3 * S_4$

DEFINITION 2.6 : The shell graph $C_{(n,n-3)}$, also known as the fan graph f_{n-1}, is constructed by selecting $n - 3$ simultaneous chords in a graph of cycle C_n , the vertices where every chords intersect referred to as the apex vertex. The shell S_n is also known as the fan f_{n-1} . Therefore, we can express this as $S_n = f_{n-1} = P_{n-1} + K_1$.

3. Main Results

Theorem 3.1. A graph acquired of two copies by joining the 3- concentric n-sunlet graph ,∀ n is product cordial.

Proof : The graph obtained of two copies by joining the 3-concentric n-sunlet graph. In this graph we consider inner most copy of n-Sunlet graph as $\alpha_1, \alpha_2, \dots, \alpha_n$ and its spokes denoted as $\alpha_1', \alpha_2', \dots, \alpha_n'$ the spokes of inner most sunlet graph is fused with the vertices of middle graph ,now the vertices of middle copy of n-sunlet graph is denoted as $\alpha_1', \alpha_2', \dots, \alpha_n'$ and its spokes as $x_1', x_2', \dots, x_n'$ and again its spokes are fused with outer vertices of the graph , then the outer most copy is denoted by $x_1', x_2', \dots, x_n'$ and spokes are denoted as $w_1', w_2', \dots, w_n'$. The two graphs are joined by a vertex obtained by fusing the outer vertices of both the graphs.

To define $g : \alpha\,(3 * S_n) \to \{0, 1\}$, we examine the following cases:

$$G(t) = \begin{cases} 1; t = \alpha_i, i = 1,2,..,n \\ 1; t = \alpha_i', i = 1,2,..,n \\ 0\,; t = x_i', i = 1,2,..,n \\ 0; t = w_i', i = 1,2,..,n \end{cases}$$

With the above labeling pattern satisfies the conditions $|\alpha_g(0) - \alpha_g(1)| \le 1$ and $|\beta_g(0) - \beta_g(1)| \le 1$. i.e., This means G admits a Product Cordial labeling.

To better understand this labeling pattern, we have considered below example in Figure 2 of even number of vertices and Figure 3 of odd number of vertices.

1. **n=8, i.e. $3*S_8$.**

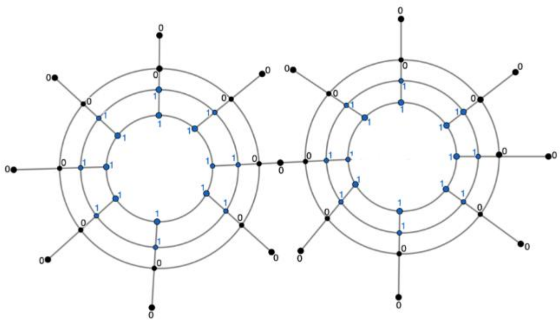

Figure 2: Two copies of $3 * S_8$

2. **n=7, $3*S_7$**

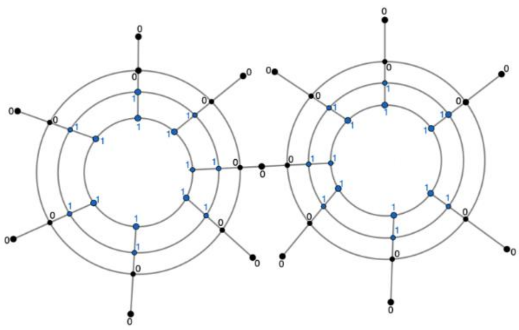

Figure 3: Two copies of $3 * S_7$

Theorem 3.2. Star of 3- concentric n-sunlet graph ,∀n is product cordial graph.

Proof : Let $x_1, x_2, \ldots, x_n$ be a successive apex vertices of inner most 3- concentric n-sunlet graph. Its spokes denoted by $x_1', x_2', \ldots x_n'$ and immediate next spokes denoted by $x_1'', x_2'', \ldots x_n''$ and $x_1''', x_2''', \ldots x_n'''$ respectively. The outer spokes of apex vertices are attached with another 3 - concentric n – sunlet graph for all the outer graphs inner most circle labelled as $\alpha_1, \alpha_2, \ldots, \alpha_n$ and immediate successors are $\alpha_1', \alpha_2', \ldots, \alpha_n'$, $\alpha_1'', \alpha_2'', \ldots, \alpha_n''$ and $\alpha_1''', \alpha_2''', \ldots, \alpha_n'''$ respectively.

To define the function we consider the following cases for apex graph and outer graphs. Type equation here.

$$g(t) = \begin{cases} 1; t = x_i''', i = 1,2,..,n \\ 1; t = x_i'', i = 1,2,..,n \\ 0; t = x_i', i = 1,2,..,n \\ 0; t = x_i, i = 1,2,..,n \end{cases}$$

$$g(t) = \begin{cases} 1; t = \alpha_{ij}''', i = 1,2,..,n, j = 1,2,..,n \\ 1; t = \alpha_{ij}'', i = 1,2,..,n, j = 1,2,..,n \\ 0; t = \alpha_{ij}', i = 1,2,..,n, j = 1,2,..,n \\ 0; t = \alpha_{ij}, i = 1,2,..,n, j = 1,2,..,n \end{cases}$$

With the above the label pattern satisfies the conditions $|\alpha_g(0) - \alpha_g(1)| \leq 1$ and $|\beta_g(0) - \beta_g(1)| \leq 1$. i.e., This means G admits a Product Cordial labeling.

To better understand this labeling pattern, we have considered below example in Figure 4 of even number of vertices and Figure 5 of odd number of vertices.

1. $n = 8, 3 * S_8^*$

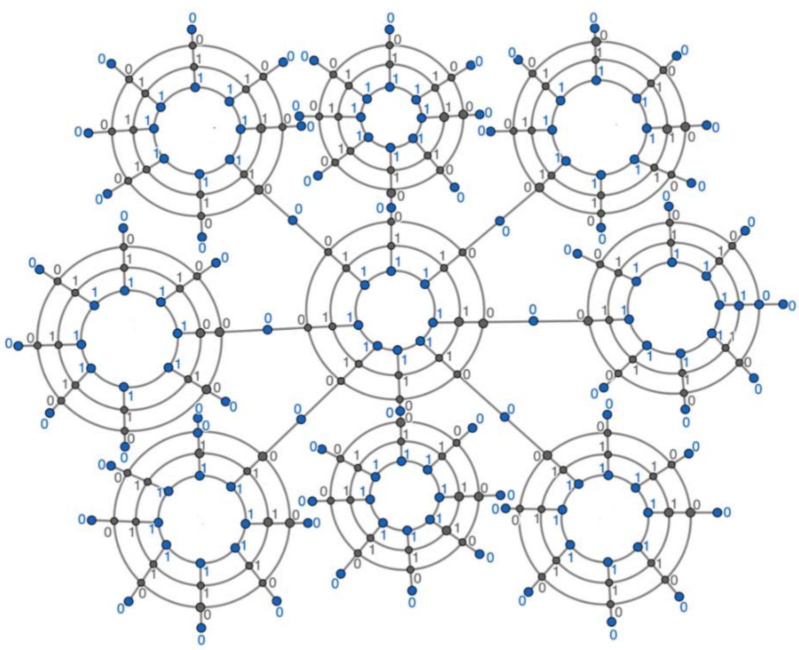

Figure 4: $3 * S_8^*$

2. $\boldsymbol{n = 7, 3 * S_7{}^*}$

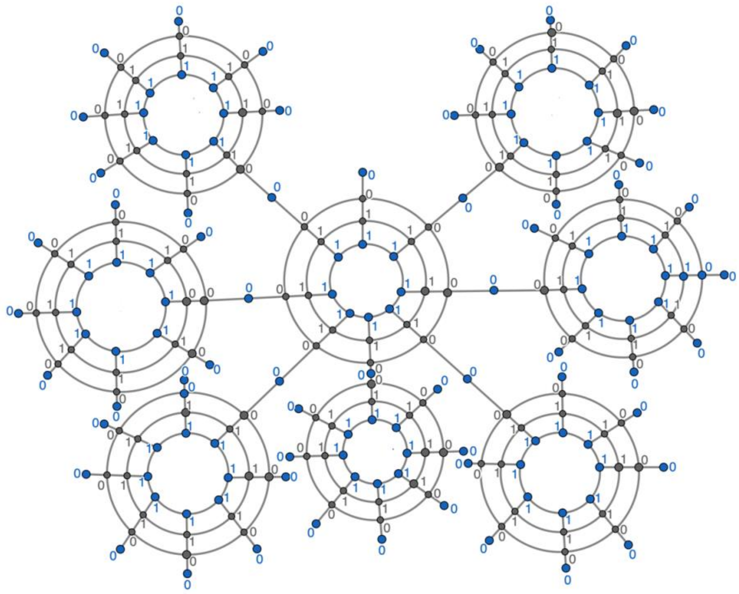

Figure 5: $3 * S_8{}^*$

Theorem 3.3. A graph obtained by joining two copies of shell graph 2*$C_{(n,n-3)}$,$\forall odd\ n$ is product cordial graph.

Proof : The labeling created by participating two replicas by path of shell graph. In this graph we consider 1st copy of shell graph as $\alpha_1, \alpha_2, \dots, \alpha_n$ and the vertices of 2nd copy of shell graph is $x_1, x_2, \dots, x_n$.

To define $g : \alpha\ (C_{(n,n-3)}) \rightarrow \{0, 1\}$, we consider the following case.

$$g(t) = \begin{cases} 1; t = \alpha_i, i = 1,2,..,n \\ 0; t = x_i, i = 1,2,..,n \end{cases}$$

The conditions are satisfied with the above labeling pattern $|\alpha_g(0) - \alpha_g(1)| \leq 1$ and $|\beta_g(0) - \beta_g(1)| \leq 1$. i.e., This shows G admits a Product Cordial labeling.

To better understand this labeling pattern, we have considered below example in Figure 6 and Figure 7 of odd number of vertices.

1. n=11

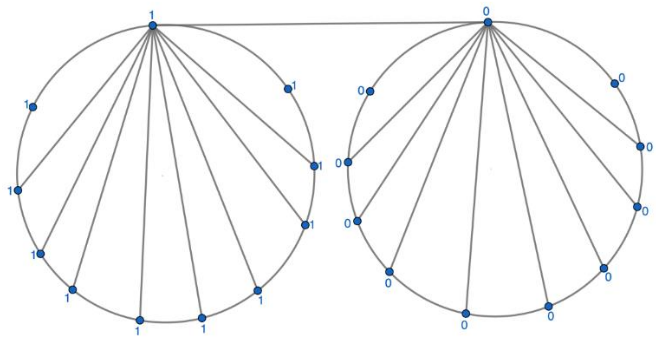

Figure 6: Two copies of $C_{(11,8)}$

2. n=13

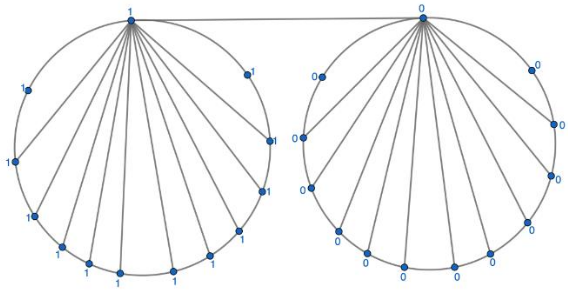

Figure 7: Two copies of $C_{(13,10)}$

4. Concluding Remarks

Product cordial labeling in graph theory, involves labeling operations on graphs such as duplication, fusion, or reconstruction. These operations are used to manipulate graphs in various ways to potentially enhance outcomes or achieve specific research goals.

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