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A Simplified Approach to Solving Balanced Generalized Quadratic Fuzzy Transportation Problems

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Abstract:

The transportation model is a specialized linear programming problem aimed at optimizing the shipment of goods from several sources to different locations. The primary objective is to minimize total transportation costs while ensuring that the supply constraints at each source and the demand requirements at each destination, represented as quadratic fuzzy numbers, are satisfied. This paper introduces a novel approach for solving generalized quadratic fuzzy transportation problems using the Piecewise Quadratic Fuzzy Vogel's Approximation Method (PQFVAM) and the Piecewise Quadratic Fuzzy Modified Distribution Method (PQFMODI)The proposed method is simple, relying on basic arithmetic and logical calculations with generalized quadratic fuzzy numbers, making it efficient and practical for real-world applications.

Keywords: Transportation Problem; Generalized Quadratic Fuzzy Number (GQFN); Piecewise Quadratic Fuzzy Vogel's Approximation Method (PQFVAM); Piecewise Quadratic Fuzzy Modified Distribution Method (PQFMODI);

1. Introduction

In practical decision-making scenarios, data are often incomplete or approximate. To address this, Zadeh [27] introduced fuzzy set theory in 1965, providing a framework for handling uncertainty. In 1987, Dubois and Prade [11] further defined a fuzzy number as a subset of the real line, enabling the mathematical modeling of linguistic variables and uncertain environments. Over time, numerous researchers have contributed to advancing fuzzy transportation problem-solving techniques. Bortolan and Degani explored various ranking methods for ordering fuzzy subsets. Using left and right spreads at particular α-levels, Abbasbandy and Hajjari [1] suggested a ranking method for trapezoidal fuzzy integers. With an emphasis on trapezoidal fuzzy numbers, Venkatachalapathy M. and A. Edward Samuel [25] presented an alternate approach to fuzzy transportation problems utilizing ranking functions. Amit Kumar and Amarpreet Kaur [2] created a novel method for generalized trapezoidal fuzzy numbers in transportation-related issues. Ali Ebrahim [4] and Adrian Ban [3] focused on estimating and simplifying fuzzy numbers for real-world use. Amit Kumar et al. [5] introduced the RM method for ranking generalized trapezoidal fuzzy numbers. Other significant contributions include strategies for handling multi-objective fuzzy transportation problems, such as those proposed by Ammar E.E. and Youness E.A. [6], and solving two-stage problems with symmetric trapezoidal fuzzy numbers by Annie Christi M.S. and Shoba Kumari K. Advancements in Transportation Problem Methods, Hitchcock [15] was among the pioneers in studying the distribution of goods from multiple sources to various destinations. Dharani K. and Selvi D. [11] applied MATLAB to solve intuitionistic fuzzy transportation problems using ranking methods. Edward Samuel A. and Venkatachalapathy M. [13] enhanced the Vogel's Approximation Method for fuzzy transportation problems. Halgurd Namiq Azeez and Abdulqader Othman Hamad Ameen [16] addressed stochastic transportation problems with fuzzy probability distributions using MATLAB. Jain K.K. et al. [17] employed lexicographic goal programming for multi-objective problems. In engineering and management sciences, fuzzy mathematical models were developed by A. Kaufmann and M.M. Gupta [18]. The Vogel's Approximation Method for transportation problems was enhanced by Korukoğlu S. and Balli S. [19]. To get an initial fundamental workable solution for transportation problems, Yusuf Sahin and Kenan Karagul [20] presented a unique approximation method. Genetic algorithms were used by F.T. Lin [21] to solve transportation problems using fuzzy coefficients. T.S. Liou and M.J. Wang [22] created a ranking method for integral-valued fuzzy numbers. Anuradha D. and Pandian P. [23] presented a novel method for resolving solid transportation issues. Multi-objective solid transportation issues were examined by Sifaoui T. and Aïder M. [24] in relation to sustainable development and adaptations of Vogel's Approximation Method for transportation problem solving. Fuzzy numbers provide a flexible representation for uncertain cost parameters, which often cannot be accurately described as random variables. As demonstrated by Zimmermann [29], solutions from fuzzy linear programming consistently deliver efficient results. This has inspired the development of numerous fuzzy optimization methods for transportation problems, including the contributions by Samuel, Venkatachalapathy M. [14], and A. Jayaraja [26], who refined the Modified Vogel's Approximation Method (MVAM) for solving problems involving generalized trapezoidal and orthogonal fuzzy numbers. By leveraging these advancements, this study proposes a robust and efficient solution framework tailored to generalized quadratic fuzzy transportation problems. , Additionally, a study on finding optimal solutions for orthogonal fuzzy numbers using the Modified Vogel's Approximation Method (MVAM) was presented.

The concept of GQFNs has gained traction as researchers sought advanced methods for capturing fuzzy uncertainty in more complex systems. Early research on fuzzy numbers focused on triangular and trapezoidal shapes due to their simplicity. However, as applications grew in complexity (e.g., in engineering, economics, decision-making, and machine learning), the need for more generalized forms, such as GQFNs, became apparent. Studies have demonstrated the effectiveness of GQFNs in modelling nonlinear uncertainties. Applications in fields like financial forecasting, medical diagnostics, and risk assessment have shown that quadratic membership functions can capture subtle variations in data. Comparisons between GQFNs and traditional linear fuzzy numbers reveal that GQFNs often provide better precision and model fit for complex, real-world data.

This paper aims to enhance the balanced generalized quadratic fuzzy transportation problem by introducing modifications that yield an initial basic feasible solution, contributing a refined approach to managing fuzzy data in transportation models. This paper introduces a straight forward and easily understandable method for solving balanced generalized quadratic fuzzy transportation problems. The proposed approach is illustrated with suitable numerical examples and detailed algorithms. The structure of the paper is as follows Section 2, Preliminaries and basic definitions. Sections 3 and 4: Proposed algorithms and numerical examples of balanced generalized piecewise quadratic fuzzy transportation problems, utilizing the PQFVAM and the PQFMODI to formulate transportation problems mathematically and obtain optimal solutions. The conclusion is discussed in section 5.

2. Preliminaries

2.1 Formation of Crisp Balanced Transportation Problems

A Fuzzy balanced transportation problem can be represented into tabular form as shown by

	Fuzzy Cost per unit Distributed						
	Destination						
	1	2	3	.	.	n	Supply $\tilde{S}_i$
1	$\tilde{C}_{11}$	$\tilde{C}_{12}$	$\tilde{C}_{13}$	.	.	$\tilde{C}_{1n}$	$\tilde{S}_1$
2	$\tilde{C}_{21}$	$\tilde{C}_{22}$	$\tilde{C}_{23}$	.	.	$\tilde{C}_{2n}$	$\tilde{S}_2$
3	$\tilde{C}_{31}$	$\tilde{C}_{32}$	$\tilde{C}_{33}$	.	.	$\tilde{C}_{3n}$	$\tilde{S}_3$
.	.	.	.	.	.	.	
.	.	.	.	.	.	.	
m	$\tilde{C}_{m1}$	$\tilde{C}_{m2}$	$\tilde{C}_{m3}$	.	.	$\tilde{C}_{mn}$	$\tilde{S}_m$
Demand $\tilde{D}_j$	$\tilde{D}_1$	$\tilde{D}_2$	$\tilde{D}_3$			$\tilde{D}_4$	

Table .1 Formation of a Fuzzy cost transportation problem

where, (i) *m* represents the number of sources. (ii) *n* represents the number of destinations.

(iii) $\tilde{S}_i$ Represents the *i* th source and $\tilde{D}_j$ represents demand.

2.2 Formulation of Generalized Quadratic Fuzzy Linear Programming Model

The transportation problem is represented by the generalized quadratic fuzzy linear programming model, which is provided as

$$\text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij} .$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} \le \tilde{S}_i \quad ; \quad i = 1, 2, \dots\dots, m$$

$$\sum_{i=1}^{m} x_{ij} \ge \tilde{D}_j \quad ; \quad j = 1, 2, \dots\dots, n$$

$$x_{ij} \ge 0. \quad \text{for all } i \text{ and } j .$$

In mathematical terms the above problem can be expressed as finding a set of x_{ij} 's, $i = 1, 2, \dots\dots, m$; $j = 1, 2, \dots\dots, n$ to

$$\text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij} .$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} = \tilde{S}_i \quad ; \quad i = 1, 2, \dots\dots, m$$

$$\sum_{i=1}^{m} x_{ij} = \tilde{D}_j \quad ; \quad j = 1, 2, \dots\dots, n$$

$$x_{ij} \ge 0 . \quad \text{for all } i \text{ and } j .$$

2.3 Generalized Quadratic Fuzzy Numbers (GQFNs)

Generalized Quadratic Fuzzy Numbers (GQFNs) extend the concept of fuzzy numbers, particularly useful in scenarios with uncertain or imprecise data. Unlike traditional fuzzy numbers, which are linear, GQFNs introduce a quadratic (parabolic) membership function that can handle more complex data structures, offering greater flexibility and precision in representing uncertainty.

2.4 Basic Definition of Generalized Quadratic Fuzzy Numbers

A membership function that specifies the level of belongingness of every element in a universe is commonly used to express a fuzzy number. For GQFNs, the membership function is quadratic, which means it includes a parabolic function in its construction. Membership Function: For a GQFN, the membership function μ(x) is defined as a quadratic function, generally of the form:

$$\tilde{\mu}(x) = \begin{cases} \tilde{a}(x-\tilde{c})^2 + \tilde{b}(x-\tilde{d}) + \tilde{e} & x \in [l, m, n, r] \\ 0 & otherwise \end{cases}$$

where l,m,n,r represent the parameters that define the fuzzy number's "support" and "core" values.

The quadratic expression ensures a parabolic shape, allowing more flexible representation for uncertainty and variability. This parabolic nature of the membership function in GQFNs allows for a more accurate depiction of uncertainties compared to linear membership functions.

2.5 Applications of Generalized Quadratic Fuzzy Numbers

2.5.1 **Engineering**: GQFNs are used in control systems and optimization, where systems may have complex, uncertain inputs that are better captured with quadratic models.

2.5.2 **Decision-Making and Risk Analysis**: GQFNs provide a refined approach in risk assessment, where the uncertain or ambiguous nature of risk factors can be better quantified.

2.5.3 **Medical Diagnosis**: In medical fields, where patient data may not be clear-cut, GQFNs offer an enhanced representation of uncertainty in diagnostic values, enhancing interpretability.

2.5.4 **Economics and Finance**: In financial modelling, GQFNs help in dealing with market uncertainties, providing more robust predictions by capturing nonlinear variations.

2.6 Definition: Piecewise Quadratic Fuzzy Numbers

In essence, a piecewise quadratic fuzzy number $\tilde{A}$ is a fuzzy number that $\tilde{A} = (l_1, l_2, l_3, l_4, l_5)$ is represented by the membership function.

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{1}{2(l_2 - l_1)^2}(x - l_1)^2 & for\, l_1 \le x \le l_2 \\ \frac{1}{2(l_3 - l_2)^2}(x - l_3)^2 + 1 & for\, l_2 \le x \le l_3 \\ \frac{1}{2(l_4 - l_3)^2}(x - l_3)^2 + 1 & for\, l_3 \le x \le l_4 \\ \frac{1}{2(l_5 - l_4)^2}(x - l_5)^2 & for\, l_4 \le x \le l_5 \\ 0 & otherwise \end{cases}$$

The piecewise quadratic fuzzy number is a bell shaped curve symmetric about the line $x = l_3$, possess a supporting interval $[l_1, l_5]$ .Moreover, $l_3 = \frac{1}{2}(l_1 + l_5)$ and $l_3 - l_2 = l_4 - l_3$. The α -cut for $\alpha = \frac{1}{2}$ between the points (l_2, l_4) and they are called cross over points. The interval of confidence at level α is given to be $A_\alpha = \{l_1 + (l_3 - l_1)\alpha, l_5 - (l_5 - l_3)\alpha\}$.

2.7 Arithmetic Operations

The following presents these arithmetic operations as they apply to PQFN.Let $\tilde{A} = (l_1, l_2, l_3, l_4, l_5)$ and $\tilde{B} = (m_1, m_2, m_3, m_4, m_5)$ are two piecewise quadratic fuzzy numbers, Then the arithmetic operations on $\tilde{A}$ and $\tilde{B}$ are given as follows:

(i) **Addition :** $\tilde{A} + \tilde{B} = (l_1 + m_1, l_2 + m_2, l_3 + m_3, l_4 + m_4, l_5 + m_5)$

(ii) **Subtraction :** $\tilde{A} - \tilde{B} = (l_1 - m_5, l_2 - m_4, l_3 - m_3, l_4 - m_2, l_5 - m_1)$

(iii) **Scalar Multiplication**

$$\lambda\tilde{A} = (\lambda l_1, \lambda l_2, \lambda l_3, \lambda l_4, \lambda l_5) \quad for\, \lambda > 0$$
$$\lambda\tilde{A} = (\lambda l_5, \lambda l_4, \lambda l_3, \lambda l_2, \lambda l_1) \quad for\, \lambda < 0$$

(iv) **Multiplication** :

$$\tilde{A}.\tilde{B} = (\tfrac{1}{2}(l_5 m_1 + l_1 m_5), \tfrac{1}{2}(l_4 m_2 + l_2 m_4), l_3 m_3, \tfrac{1}{2}(l_2 m_2 + l_4 m_4), \tfrac{1}{2}(l_1 m_1 + l_5 m_5))$$

(v) Division

$$\frac{\tilde{A}}{\tilde{B}} = \left(\frac{2l_1}{m_1 + m_5}, \frac{2l_2}{m_2 + m_4}, \frac{l_3}{m_3}, \frac{2l_4}{m_2 + m_4}, \frac{2l_5}{m_1 + m_5} \right) \text{(If all m's are non-zero)}$$

$$\frac{\tilde{A}}{\tilde{B}} = \left(\frac{2a_5}{b_1 + b_5}, \frac{2a_4}{b_2 + b_4}, \frac{a_3}{b_3}, \frac{2a_2}{b_2 + b_4}, \frac{2a_1}{b_1 + b_5} \right) \text{ (If } \tilde{B} \text{ is negative and all } m_i \text{'s are non-zero)}$$

3. Algorithms

3.1. Algorithm for *Solving a Balanced or Unbalanced Transportation Problems by* using Piecewise Quadratic VAM

Step 1: Formulate the Transportation Problem

Formulate the transportation problem in matrix form**:** Represent the problem as a matrix, where rows correspond to supply sources and columns correspond to demand destinations. Each element in the matrix represents the transportation cost between a source and a destination.

Check if the problem is balanced**:**

(i) Balanced problem**:** Total supply equals total demand.

(ii) Unbalanced problem**:** Total supply does not equal total demand. If unbalanced, introduce a dummy source or dummy destination (as needed) to balance the problem. The dummy source/ destination will have a cost of zero for all cells involving the dummy.

Step 2: Calculate Penalties

Calculate penalties for each row and column: For each row and column, find the difference between the smallest and the second smallest cost in that row or column.If two or more costs are identical and the smallest, the penalty is zero for that row or column.

Step 3: Select the Highest Penalty Row or Column

Identify the row or column with the highest penalty**:** Look for the row or column with the largest penalty value. If multiple rows or columns have the same penalty, proceed with the following tiebreakers, Select the row or column with the smallest unit cost.If there's still a tie, choose the row or column where the largest allocation can be made (i.e., the one with the highest remaining supply or demand).Allocate to the lowest-cost cell**:** In the row or column with the highest penalty, allocate as much as possible to the cell with the lowest cost.

Step 4: Update the Problem

Remove satisfied rows or columns**:** Once a row or column's supply or demand has been completely allocated, remove it from the matrix (or mark it as satisfied).

Step 5: Repeat Until All Constraints are satisfied

Repeat steps 2 and 3: Continue calculating penalties and allocating resources until all supply and demand constraints are fully satisfied.

Step 6: Obtain the Initial Basic Feasible Solution

Final solution**:** The result of the allocation process in Step 5 gives you the initial basic feasible solution for the transportation problem. This solution is feasible, but it may not necessarily be optimal.

3.2 Numerical Example

A beverage company operates three bottling plants located across a region, with each plant having the following daily production capacities: Plant 1, Plant 2, and Plant 3. The company needs to supply three distribution hubs Hub X, Hub Y, and Hub Z each with specific daily beverage requirements. The transportation cost for shipping one million liters of beverage from each plant to each distribution hub is provided in the table below (in hundreds of rupees). The company aims to determine the optimal shipment quantities from each bottling plant to each distribution hub in order to minimize the total transportation cost, which is represented as a generalized quadratic fuzzy number. The goal is to determine the Initial Basic Feasible Solution (IBFS) for this transportation problem.

	Plant 1	Plant 2	Plant 3	Supply
Hub X	(3,5,6,7,9)	(7,9,10,11,13)	(11,13,14,15,17)	(47,49,50,51,53)
Hub Y	(9,11,12,13,15)	(16,18,19,20,22)	(18,20,21,22,24)	(47,49,50,51,53)
Hub Z	(12,14,15,16,18)	(11,13,14,15,17)	(14,16,17,18,20)	(47,49,50,51,53)
Demand	(27,29,30,31,33)	(37,39,40,41,43)	(52,54,55,56,58)	

Table 2. Generalized Quadratic Fuzzy Transportation table

Check for Balance: Fuzzy Total Supply

=(47,49,50,51,53)+(47,49,50,51,53)+(47,49,50,51,53)=(141,147,150,153,159)

Fuzzy Total Demand =(27,29,30,31,33)+(37,39,40,41,43)+(52,54,55,56,58)

=(116,122,125,128,134)

Since Fuzzy Total Supply ≠ Fuzzy Total Demand, It is called unbalanced.

Step 1: Add a Dummy Column**,** To balance the problem, a dummy Column is added with costs of (0,0,0,0,0) and demand equal to the excess supply:

Step 2: Formulate the Piecewise Quadratic Fuzzy Transportation Problem in Matrix Form

The transportation matrix for costs (in generalized quadratic fuzzy numbers) is:

	Plant 1	Plant 2	Plant 3	Plant 4	Supply	Penalty
Hub X	(3,5,6,7,9)	(7,9,10,11,13)	(11,13,14,15,17)	(0,0,0,0,0)	(47,49,50,51,53)	(3,5,6,7,9)
Hub Y	(9,11,12,13,15)	(16,18,19,20,22)	(18,20,21,22,24)	(0,0,0,0,0)	(47,49,50,51,53)	(9,11,12,13,15)
Hub Z	(12,14,15,16,18)	(11,13,14,15,17)	(14,16,17,18,20)	(0,0,0,0,0)	(47,49,50,51,53)	(12,13,14,15,17)
Demand	(27,29,30,31,33)	(37,39,40,41,43)	(52,54,55,56,58)	(7,19,25,31,43)		
Penalty	(0,4,6,8,12)	(-2,2,4,6,10)	(-3,1,3,5,9)	(0,0,0,0,0)		

Table 3. Balanced Generalized Quadratic Fuzzy Transportation table

Step 3: Calculate Penalties and Allocate Supplies Using the Piecewise Quadratic Fuzzy Vogel's Approximation Method (PQFVAM): To solve the problem, The difference between the smallest and next smallest transportation expenses is used to calculate the penalties for each row and column. After determining which row or column has the largest penalty, the cell with the lowest cost receives the maximum allowable allocation. The supply and demand are modified in accordance with each allocation. This process is repeated until all supply and demand constraints are satisfied.

The optimal allocation for the generalized quadratic fuzzy transportation problem is determined.

(3,5,6,7,9)	***(37,39,40,41,43)*** (7,9,10,11,13)	***(4,8,10,12,16)*** (11,13,14,15,17)	(0,0,0,0,0)
(27,29,30,31,33) (9,11,12,13,15)	(16,18,19,20,22)	***(14,18,20,22,26)*** (18,20,21,22,24)	(0,0,0,0,0)
(12,14,15,16,18)	(11,13,14,15,17)	***(4,18,25,32,46)*** (14,16,17,18,20)	***(7,19,25,31,43)*** (0,0,0,0,0)

Table 4. Optimal Assignment Generalized Quadratic Fuzzy Transportation table

The total cost of the solution is calculated by multiplying the transportation cost per unit in each basic cell by the corresponding number of units allocated, and then summing the results across all allocated cells.

= (7,19,25,31,43)(0,0,0,0,0) + (27,29,30,31,33)(9,11,12,13,15)+
(37, 39, 40, 41, 43)(7,9,10,11,13) + (14,18,20,22,26)(18,20,21,22,24)+
(4, 18, 25, 32, 46)(14,16,17,18,20) + (4,8,10,12,16)(11,13,14,15,17)
= (1628, 1732, 1745, 1758, 1862)

3.2.1 Membership Function

The PQFN is a bell shaped curve symmetric about the line x=1745, possess a supporting interval [1628, 1862]. Moreover, $l_3 = 1745$ and 13=$l_3 - l_2 = l_4 - l_3$. The α-cut for $\alpha = \frac{1}{2}$ between the points (1732, 1758).The interval of confidence at level α is given to be $A_\alpha = \{1686.5, 1803.5\}$.

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x-1628)^2}{21632} & for\ 1628 \le x \le 1732 \\ \dfrac{(x-1732)^2}{338} + 1 & for\ 1732 \le x \le 1745 \\ \dfrac{(x-1745)^2}{338} + 1 & for\ 1745 \le x \le 1758 \\ \dfrac{(x-1862)^2}{21632} & for\ 1758 \le x \le 1862 \\ 0 & otherwise \end{cases}$$

% MATLAB Code to Plot Symmetric Bell-Shaped Fuzzy Curve

Computing the Valuation of Piecewise Quadratic Fuzzy Number

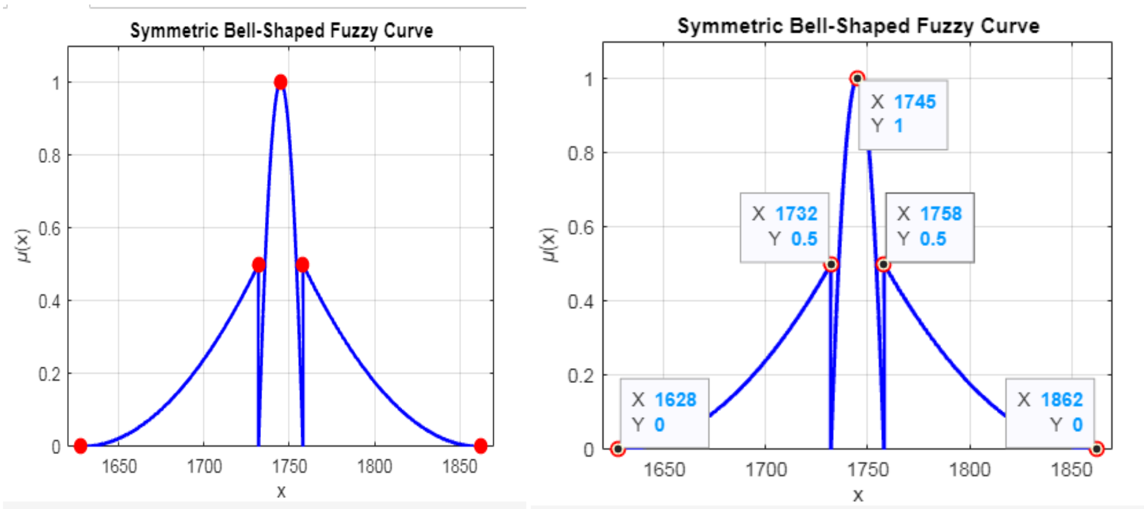

Figure 1. Symmetric Bell shaped fuzzy curve with α-cut

4. Piecewise Quadratic Fuzzy Modified Distribution Method (PQFMODI)

The PQFMODI is an effective technique for verifying the optimality of the initial feasible solution in transportation problems.

4.1 Algorithm

The PQFMODI, also known as the u−v method, offers an efficient approach to finding the minimum-cost solution for transportation problems. Unlike the Stepping Stone Method, which requires evaluating costs for all unoccupied cells by drawing closed paths, the PQFMODI method only needs one closed path for the unoccupied cell with the highest opportunity cost.

Step 1 : Determine Initial Basic Feasible Solution

Start by finding an initial feasible solution using Piecewise Quadratic Fuzzy Vogel's Approximation Method.

Step 2: Calculate Dual Variables u_i and v_j

Determine the values of the dual variables ui and v_j for each row and column.

Step 3: Compute Opportunity Costs

Calculate the opportunity cost for each unoccupied cell.

Step4: Evaluate Opportunity Costs

(i) If all unoccupied cells have opportunity costs that are positive or zero, the solution is optimal.

(ii)If any unoccupied cell has a negative opportunity cost, the solution is not optimal, and further cost reduction is possible.

Step 5: Select Cell with Smallest Negative Opportunity Cost
Choose the unoccupied cell with the most negative opportunity cost for the next iteration.

Step 6: Draw Closed Path for Selected Cell
Draw a closed path or loop starting from the selected cell. Right-angle turns are allowed only at occupied cells and the initially selected unoccupied cell.

Step 7: Assign Signs on Path Cells
Mark the corner cells of the closed path with alternating plus and minus signs, starting with a plus sign on the cell being evaluated.

Step 8: Determine Units to Ship

Determine the maximum number of units that can be shipped to the selected unoccupied cell. This is based on the smallest value at the cells marked with a minus sign. Add this quantity to cells marked with a plus and subtract it from cells marked with a minus to adjust the allocations. The unoccupied cell becomes occupied.

Step 9: Repeat Until Optimal Solution is Achieved

Continue this process until all opportunity costs for unoccupied cells are positive or zero, indicating an optimal solution.

4.2 Degeneracy in Transportation Problems

A transportation problem is considered degenerate when the number of positive allocations (occupied cells) is less than m+n−1, where m represents origins and n represents destinations. Degeneracy can occur in two stages:

1. Initial Solution: When setting up the initial feasible solution.
2. Testing Optimality: When evaluating the current solution for optimality.

To resolve degeneracy, an artificial quantity is assigned to the unoccupied cell with the lowest transportation cost, helping to proceed with the calculations and maintain feasibility.

4.3 Numerical Examples with Piecewise Quadratic Fuzzy MODI method

	Plant 1	Plant 2	Plant 3	Supply
Hub X	(3,5,6,7,9)	(7,9,10,11,13)	(11,13,14,15,17)	(47,49,50,51,53)
Hub Y	(9,11,12,13,15)	(16,18,19,20,22)	(18,20,21,22,24)	(47,49,50,51,53)
Hub Z	(12,14,15,16,18)	(11,13,14,15,17)	(14,16,17,18,20)	(47,49,50,51,53)
Demand	(27,29,30,31,33)	(37,39,40,41,43)	(52,54,55,56,58)	

Table 5. Generalized Quadratic Fuzzy Transportation table

Step 1

	Plant 1	Plant 2	Plant 3	Plant 4	Supply
Hub X	(3,5,6,7,9)	(7,9,10,11,13)	(11,13,14,15,17)	(0,0,0,0,0)	(47,49,50,51,53)

Hub Y	(9,11,12,13,15)	(16,18,19,20,22)	(18,20,21,22,24)	(0,0,0,0,0)	(47,49,50,51,53)
Hub Z	(12,14,15,16,18)	(11,13,14,15,17)	(14,16,17,18,20)	(0,0,0,0,0)	(47,49,50,51,53)
Demand	(27,29,30,31,33)	(37,39,40,41,43)	(52,54,55,56,58)	(22,24,25,26,28)	

Table 6. Balanced Generalized Quadratic Fuzzy Transportation table

Step 2, Initial Basic Feasible Solution Using PQFVAM .

Let us use the PQFVAM to find the initial basic feasible solution. In the previous problems PQFVAM method is used to solve to this problem.

	v_1	v_2	v_3	v_4
u_1	(3,5,6,7,9)	(37,39,40,41,43) (7,9,10,11,13)	(4,8,10,12,16) (11,13,14,15,17)	(0,0,0,0,0)
u_2	(27,29,30,31,33) (9,11,12,13,15)	(16,18,19,20,22)	(14,18,20,22,26) (18,20,21,22,24)	(0,0,0,0,0)
u_3	(12,14,15,16,18)	(11,13,14,15,17)	(4,18,25,32,46) (14,16,17,18,20)	(22,24,25,26,28) (0,0,0,0,0)

Table 7. First Optimal Assignment Generalized Quadratic Fuzzy Transportation table

Now choose the smallest (most) negative value from opportunity cost is $(-16,-8,-4,0,8)$ and draw a closed path. The following table shows the closed path

(3,5,6,7,9)	(37,39,40,41,43) (7,9,10,11,13)	(4,8,10,12,16) (11,13,14,15,17)	(0,0,0,0,0)
(27,29,30,31,33) (9,11,12,13,15)	(16,18,19,20,22)	(18,20,21,22,24) (18,20,21,22,24) **-ve**	(0,0,0,0,0) **+ve**
(12,14,15,16,18)	(11,13,14,15,17)	(4,18,25,32,46) (14,16,17,18,20) **+ve**	(22,24,25,26,28) (0,0,0,0,0) **-ve**

Table 8. Closed Path Optimal Assignment Generalized Quadratic Fuzzy Transportation table

Select the smallest value with a negative location on the closed route from the following values (14, 18, 20, 22, 26). The quantity of units that can be transported to the entering cell is indicated by this amount. Then, deduct this amount from the cells with minus signs and add it to all the cells at the closed path's corner points that are indicated with plus signs. An empty cell becomes populated in this manner.

Calculating u_i and v_j using $u_i + v_j = c_{ij}$ Now finding opportunity cost using $\Delta_{ij} = c_{ij} - \left(u_i + v_j\right)$

Next, create a closed path using the opportunity Fuzzy cost's smallest (most negative) value. Next, choose the smallest value with a negative location on the closed route from the range (2, 4, 5, 6, 8). This will show how many units can be transported to the cell that is entering. Once more, deduct this amount from the cells with minus signs and add it to all the cells at the closed path's corner points that are indicated with plus signs. By doing this, an empty cell is certain to be filled. The solution is now optimal as all of the opportunity costs are non-negative.

	v_1	v_2	v_3	v_4
u_1	**(2,4,5,6,8)** (3,5,6,7,9)	**(37,39,40,41,43)** (7,9,10,11,13)	**(-4,2,5,8,14)** (11,13,14,15,17)	(0,0,0,0,0)
u_2	**(19,23,25,27,31)** (9,11,12,13,15)	(16,18,19,20,22)	(18,20,21,22,24)	**(16,22,25,28,34)** (0,0,0,0,0)
u_3	(12,14,15,16,18)	(11,13,14,15,17)	**(20,40,50,60,80)** (14,16,17,18,20)	(0,0,0,0,0)

Table 9. Closed Path Optimal Assignment Generalized Quadratic Fuzzy Transportation table

The minimum transportation cost is:

= (2,4,5,6,8)(3,5,6,7,9)+(37,39,40,41,43)(7,9,10,11,13)+(-4,2,5,8,14)(11,13,14,15,17)

+(19,23,25,27,31)(9,11,12,13,15)+(16,22,25,28,34)(0,0,0,0,0)+(20,40,50,60,80)

(14,16,17,18,20)

= (1497, 1633, 1650, 1667, 1803)

4.3.1 Membership Function

The piecewise quadratic fuzzy number is a bell shaped curve symmetric about the line x=1745, possess a supporting interval [1497, 1803]. Moreover, $l_3 = 1650$ and $17 = l_3 - l_2 = l_4 - l_3$. The α-cut for $\alpha = \frac{1}{2}$ between the points (1633, 1667) and they are called cross over points. The interval of confidence at level α is given to be $A_\alpha = \{1505.5, 1726.5\}$.

Computing the Valuation of membership function of Piecewise Quadratic Fuzzy Numbers

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-1497)^2}{36992} & for\, 1497 \le x \le 1633 \\ \frac{(x-1650)^2}{578} + 1 & for\, 1633 \le x \le 1650 \\ \frac{(x-1650)^2}{578} + 1 & for\, 1650 \le x \le 1667 \\ \frac{(x-1862)^2}{36992} & for\, 1667\; x \le 1803 \\ 0 & otherwise \end{cases}$$

% MATLAB Code to Plot Symmetric Bell-Shaped Fuzzy Curve

Computing the Valuation of Piecewise Quadratic Fuzzy Number

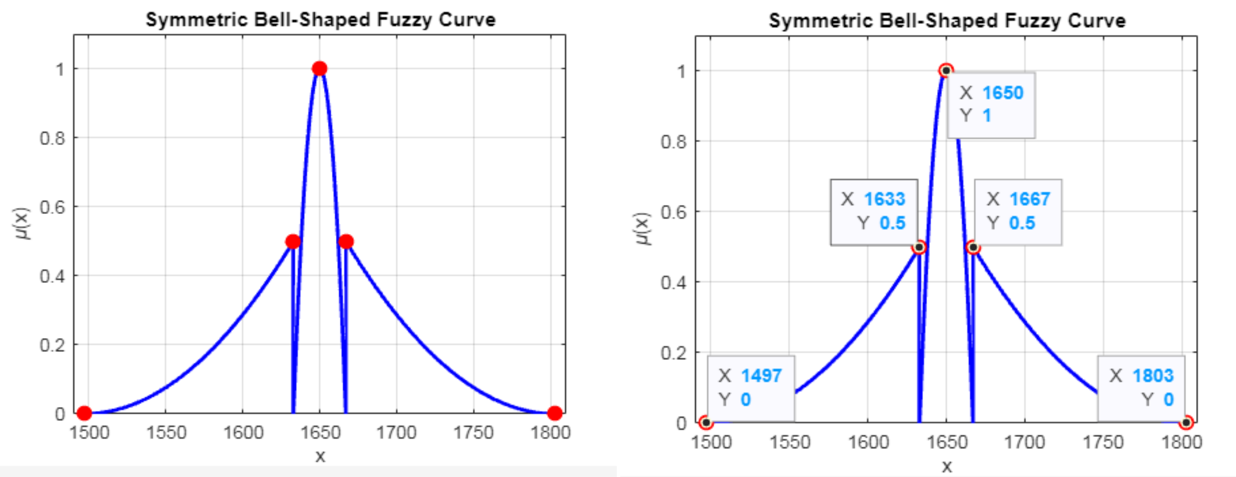

Figure.2. **Symmetric Bell shaped fuzzy curve with α-cut**

The provided image displays a symmetric bell-shaped fuzzy number represented by a membership function μ(x), with x representing the transportation costs or some related variable. Let's break down the explanation of the piecewise quadratic fuzzy numbers illustrated in the graph: Quadratic functions ensure smoothness and gradual changes in the membership values.The symmetric bell-shaped curve effectively models uncertainty and variability in a system. By providing a range of possible values with associated likelihoods, it helps in robust optimization and better decision-making under uncertainty.

5. Conclusion

The PQFMODI method offers a systematic approach through dual variables and opportunity costs, allowing quick identification of cost-saving reallocations. Meanwhile, the Stepping Stone method provides a deeper visual understanding of closed paths and feasible adjustments, which can be valuable in understanding the underlying allocation dynamics.For cases with degeneracy, both methods can handle it by introducing an artificial quantity, ensuring that the solution process remains feasible and accurate. Overall, these methods are essential tools in operations research and optimization, supporting decision-making in logistics, supply chain management, and other resource allocation applications by offering effective ways to minimize transportation costs while ensuring efficient resource distribution. this study tries to make a new method to solve transportation problem more efficiently.

Compliance with Ethical Standards

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Conflicts of Interest

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Research Involving Human Participants and/or Animals

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