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# Normal BE-Algebras

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**Abstract:** In this present article we propounded the idea of normal BE-algebras, derived some significant properties of normal BE-algebras and obtained a set of equivalent conditions indicating when a normal BE-algebra assumes the characteristics of an involutory BE-algebra. Some sufficient conditions for a BE-algebra to become a normal BE-algebra are derived. Also, congruence relation is introduced on a normal BE-algebra.
**Keywords**: BE-algebra; congruence; involutory BE-algebra; normal BE-algebra; transitive BE-algebra.

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## 1. Introduction.

In [11], H. S. Kim and Y. H. Kim were introduced the theory of BE-algebras. BE-algebras was made familiar to extend the class of BCK-algebras of K. Iseki and S. Tanaka [10]. In [1], S.S. Ahn and Y. H. Kim studied some properties of filters of BE-algebras and by B. L. Meng in [12]. Some relationships between congruence relations and normal filters of a BE-algebras was discussed by A. Walendziak in [14]. In [13], P. Sun investigated homomorphism theorems via dual ideals of BCK-algebras. In [9], Z. ciloglu and Y. Ceven introduced the notion of commutative and bounded BE-algebras. In [8], R. Borzooei et al. introduced the notion of involutory BE-algebras. In [2], M. Bala Prabhakar, S. K. Vali and M. Sambasiva Rao were introduced the idea of Closed and Dense elements of BE-algebras. Also, these authors were introduced the concepts of Ideals of transitive BE-algebras in [3], Semi Maximal Ideals of BE-algebras in [4], Maximal Ideals of transitive BE-algebras in [5], Prime Ideals of transitive BE-algebras in [6] and Generalized Lower sets of transitive BE-algebras in [7].

In this work, the concept of normal BE-algebras is introduced, derived some properties and equivalent conditions. A congruence is introduced on a normal BE-algebra.

## 2. Preliminary Results.

This section outlines a combination of definitions and results, formerly sourced from existing papers for the readers convenience.

**Definition 1.1.** [11] An algebra $(X,*,1)$of type (2, 0) is called a BE-algebra, if it satisfies the following properties:

(1) $x * x = 1$,

(2) $x * 1 = 1$,

(3) $1 * x = x$,

(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra $X$ is called transitive if $y * z \leq (x * y) * (x * z)$for all $x, y, z \in X$.

Every self-distributive BE-algebra is transitive.

We introduce a relation $\leq$ on a BE-algebra $X$ by $x \leq y$ if and only if $x * y = 1$ for all for all $x, y \in X$.

**Theorem 1.2.** [12] Let $X$ be a transitive BE-algebra and $x, y, z \in X$. Then

(1) $1 \leq x$ implies $x = 1$,

(2) $y \leq z$ implies $x * y \leq x * z$ and $z * x \leq y * x$.

**Definition 1.3.** [11] A non-empty subset $F$ of a BE-algebra $X$ is called a filter of $X$ if, for all $x, y \in X$, it satisfies the following properties:

(1)$1 \in F$,

(2) $x \in F$and $x * y \in F$ imply that $y \in F$.

**Definition 1.4.** [9] A BE-algebra $X$ is called bounded BE-algebra, if there exist an element 0 satisfying $0 \leq x (or\ 0 * x = 1)$ for all $x \in X$.

Define an unary operation $N$ on $X$ by $xN = x * 0$ for all $x \in X$. Clearly, $0N = 1$and $1N = 0$.

**Theorem 1.5.** [9] Let $X$ be a transitive BE-algebra and $x, y, z \in X$. Then

(1) $0N = 1$and $1N = 0$,

(2) $x \leq xNN$,

(3) $x * yN = y * xN$.

**Definition 1.6**. [8] An element $x$ of a BE-algebra $X$ is called involutory element if $xNN = x$. If every element of a BE-algebra $X$ is involutory, then $X$ is called an involutory BE-algebra.

**Definition 1.7**. [2] An element $a$ of a BE-algebra $X$ is called closed element if $aNN = a$.

We denote the set $C(X) = \{a \in X / aNN = a\}$, is the set of all closed elements of a BE-algebra $X$.

**Definition 1.8.** [7] Let $X$ be a bounded BE-algebra. $\emptyset \neq S \subseteq X$ is called a bounded subalgebra if $S$ is closed under the operations $*$ and $N$. In particular $(xN * yN)N \in S$ whenever $x, y \in S$.

**Lemma 1.9.** [3] Let $X$ be a transitive BE-algebra. For any $x, y, z \in X$, we have:

(1) $xNNN \leq xN$,

(2) $x * y \leq yN * xN$,

(3) $x * yN \leq xNN * yN$,

(4) $(x * yNN)NN \leq x * yNN$,

(5) $(xN * yN)NN \leq xN * yN$,

(6) $x \leq y \Rightarrow yN \leq xN$,

(7) $x \leq y \Rightarrow y * zN \leq x * zN$.

**3. Main Results (Normal BE-Algebras).**

In this section, the concept of Normal BE-Algebras is introduced. Some properties of normal BE-algebras are studied. Some sufficient conditions for a BE-algebra to become a normal BE-algebra are derived.

**Definition 2.1.** A bounded BE-algebra $(X, *, 0, 1)$ is said to be a normal BE-algebra, if it satisfies the following properties for all $x, y \in X$:

(N1) $xNNN = xN$,

(N2) $(x * yNN)NN = x * yNN$.

**Proposition 2.2.** Let $(X, *, 0, 1)$ be a normal BE-algebra. For any $x, y \in X$, the following properties hold:

(1) $x * yN = xNN * yN$,

(2) $x * yNN = xNN * yNN$,

(3) $(x * yN)NN = x * yN$.

**Proof.** (1). Let $x, y \in X$. Then $xNN * yN = xNN * (y * 0)$

$$\begin{aligned} &= y * (xNN * 0) \\ &= y * xNNN \\ &= y * xN \\ &= y * (x * 0) \\ &= x * (y * 0) \\ &= x * yN. \end{aligned}$$

(2). Let $x, y \in X$. Then by (1), we have $x * yN = xNN * yN$. Replace $y$ by $yN$, we get

$$x * yNN = x * (yN)N = xNN * yNN.$$

(3). Let $x, y \in X$. Since $X$ is normal, we get that

$$(x * yN)NN = (x * yNNN)NN = x * yNNN = x * yN.$$

**Theorem 2.3.** Let $X$ be a BE-algebra which satisfies the following conditions:

(1) $xNNN = xN$,

(2) $(x * y)NN = xNN * yNN$ for all $x, y \in X$. Then $X$ is a normal BE-algebra.

**Proof.** Let $x, y \in X$. Suppose $X$ is satisfying the above two conditions. Then

$$(x * yNN)NN = (yN * xN)NN = yNNN * xNNN = yN * xN = x * yNN.$$

Therefore, $X$ is a normal BE-algebra.

**Proposition 2.4.** Every involutory BE-algebra is a normal BE-algebra.

**Proof.** Assume that $X$ is an involutory BE-algebra. Let $x \in X$. Then $xNN = x$.

Hence $xNNN = (xNN)N = xN$.

Again, let $x, y \in X$. Then $(x * yNN)NN = (x * y)NN = x * y = x * yNN$.

Therefore, $X$ is a normal BE-algebra.

**Example 2.5**. Let $X = \{1, a, b, c, d, 0\}$ be a set. Define a binary operation $*$ on $X$ as follows:

| * | **1** | ***a*** | ***b*** | ***c*** | ***d*** | **0** |
|---|---|---|---|---|---|---|
| **1** | 1 | *a* | *b* | *c* | *d* | 0 |
| ***a*** | 1 | 1 | *a* | *c* | *c* | *d* |
| ***b*** | 1 | 1 | 1 | *c* | *c* | *c* |
| ***c*** | 1 | *a* | *b* | 1 | *a* | *b* |
| ***d*** | 1 | 1 | *a* | 1 | 1 | *a* |
| **0** | 1 | 1 | 1 | 1 | 1 | 1 |

Clearly, $(X, *, 0, 1)$ is a bounded BE-algebra. It is easy to observe that X is an involutory BE-algebra and a normal BE-algebra too.

The converse of the above proposition is not true. i.e. A normal BE-algebra need not be involutory. For this consider the following example:

**Example 2.6**. Let $X = \{1, a, b, c, 0\}$ be a set. Define a binary operation $*$ on $X$ as follows:

| * | **1** | ***a*** | ***b*** | ***c*** | **0** |
|---|---|---|---|---|---|
| **1** | 1 | *a* | *b* | *c* | 0 |
| ***a*** | 1 | 1 | *b* | *b* | 0 |
| ***b*** | 1 | a | 1 | *a* | 0 |
| ***c*** | 1 | 1 | 1 | 1 | 0 |
| **0** | 1 | 1 | 1 | 1 | 1 |

Clearly, $(X, *, 0, 1)$ is a normal BE-algebra. However, $X$ is not an involutory BE-algebra because of $aNN = 0N = 1, bNN = 0N = 1$ & $cNN = 0N = 1$.

In the following theorem, we derive a set of equivalent conditions for a normal BE-algebra to become an involutory BE-algebra.

**Theorem 2.7**. Let $(X, *, 0, 1)$ is a normal BE-algebra. Then the following conditions are equivalent:

(1) $X$ is involutory;

(2) for any $x, y \in X$, $xN = yN$ implies $x = y$;

(3) for all $x \in X$; $(x * 0) * 0 = (0 * x)x$.

**Proof.** $(1) \Rightarrow (2)$ Assume that $X$ is involutory BE-algebra. Let $x, y \in X$ be such that $xN = yN$. Then $xNN = yNN$ and hence $x = y$.

$(2) \Rightarrow (3)$ Assume the condition (2). Let $x, y \in X$. Since $\big((x * 0) * 0\big)N = (xNN)N = xNNN = xN$.

Then by (2), we get $(x * 0) * 0 = x = 1 * x = (0 * x) * x$.

$(3) \Rightarrow (1)$ Assume the condition (3). Let $x \in X$. Then $(x * 0) * 0 = (0 * x)x$. Hence

$xNN = (xN)N = (x * 0) * 0 = (0 * x)x = 1 * x = x$. Therefore $X$ is involutory.

**Theorem 2.8**. A normal BE-algebra $X$ satisfies the property, $(x * y)NN = xNN * yNN$ if and only if $(yN * xN)NN = (x * y)NN$ for all $x, y \in X$.

**Proof.** Let $X$ be a normal BE-algebra. Assume that $(x * y)NN = xNN * yNN$ for all $x, y \in X$. Then we have

$$\begin{aligned} (yN * xN)NN &= (x * yNN)NN \\ &= x * yNN \\ &= yN * xN \\ &= yN * xNNN \\ &= xNN * yNN \\ &= (x * y)NN. \end{aligned}$$

Conversely, assume the condition $(yN * xN)NN = (x * y)NN$for all $x, y \in X$. For any $x, y \in X$, we get $xNN * yNN = (xNN * yNN)NN = (yN * xNNN)NN = (yN * xN)NN = (x * y)NN$.

In the following theorem, we derive a set of equivalent conditions for a transitive BE-algebra to become normal.

**Proposition 2.9.** Let $X$ be a transitive BE-algebra which satisfies the property, $xNNN = xN$ for all $x, y \in X$. Then the following conditions are equivalent:

(1) $X$ is a normal BE-algebra.

(2) for all $x, y \in X$, $(xNN * yNN)NN = xNN * yNN$;

(3) for all $x, y \in X$, $(x * yNN)NN = xNN * yNN$;

(4) for all $x, y \in X$, $(xN * yN)NN = xN * yN$.

**Proof.** $(1) \Leftrightarrow (2)$ Assume that $X$ is a normal BE-algebra. Let $x, y \in X$. Then by the property (N2), we get $(xNN * yNN)NN = xNN * yNN$.

Conversely, assume the condition (2). Let $x, y \in X$. Then

$$\begin{aligned} &(x * yNN)NN = (yN * xN)NN \\ &= (yN * xNNN)NN \\ &= (xNN * yNN)NN \\ &= xNN * yNN \\ &= yN * xNNN \\ &= yN * xN \\ &= x * yNN. \end{aligned}$$

Hence $X$ is a normal BE-algebra, which proves the condition (1).

$(2) \Rightarrow (3)$ Assume that the condition (2) holds. Let $x, y \in X$. Then

$xNN * yNN = (xNN * yNN)NN = (yN * xNNN)NN = (yN * xN)NN = (x * yNN)NN$

Therefore $(x * yNN)NN = xNN * yNN$ for all $x, y \in X$.

$(3) \Rightarrow (4)$ Assume that the condition (3) holds. Let $x, y \in X$. Then by (3), we get

$$(xN * yN)NN = (y * xNN)NN = yNN * xNN = xN * yNNN = xN * yN.$$

$(4) \Rightarrow (2)$ Assume the condition (4) holds. Let $x, y \in X$. Then by (4), we get $(xNN * yNN)NN = xNN * yNN$. Hence condition (2) is derived.

**Theorem 2.10.** Let $(X, *, 0, 1)$ be a normal BE-algebra. Then for any $x, y \in X$, define a binary relation $\theta$ on $X$ as $(x, y) \in \theta$ if and only if $xN = yN$. Then $\theta$ is an equivalence relation on $X$ and for any $a \in X$, the following are hold:

(1) The element $aNN$ is the greatest element in the class $[a]_\theta$ where$[a]_\theta = \{b \in X/(a, b) \in \theta\}$;

(2) The class $[a]_\theta$ contains just one element from $C(X)$ which is $aNN$.

**Proof.** Clearly $\theta$ is an equivalence relation on $X$.

(1). Let $a \in X$. Since $X$ is normal, we get $aNNN = aN$. Hence $(a, aNN) \in \theta$, which means $aNN \in [a]_\theta$. Let $x \in [a]_\theta$. Then $xN = aN$. Since $x \leq xNN = aNN$, we get that $aNN$ is the greatest element in the class $[a]_\theta$.

(2). Let $b \in C(X)$ such that $b \in [a]_\theta$. Then $bN = aN$. Hence $b = bNN = aNN$.

Therefore, the class $[a]_\theta$ contains one element from $C(X)$ which is $aNN$.

**Theorem 2.11.** Let $(X, *, 0, 1)$ be a normal BE-algebra which satisfies the condition $(x * y)NN = xNN * yNN$ for all $x, y \in X$.Then we get the following:

(1) $\theta$ is a congruence on $X$,

(2) $C(X)$is a retract of $X$.

**Proof.** (1). Let $(x, y) \in \theta$ and $(z, w) \in \theta$ for $x, y, z, w \in X$. Then we get $xN = yN$ and $zN = wN$.

Now $(x * z)NN = xNN * zNN = yNN * wNN = (y * w)NN$.

Hence $(x * z, y * w) \in \theta$. Therefore $\theta$ is a congruence on $X$.

(2). By Proposition 2.9(2), $C(X)$ is a subalgebra of $X$. Define $\varphi: X \to X$ by $\varphi(x) = xNN$ for all$x \in X$. Then we get $\varphi(x) = x$ for all $x \in C(X)$. Clearly $\varphi(x) = xNN \in C(X)$ for all $x \in X - C(X)$. Therefore $C(X)$ is a retract of $X$.

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