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Exploring Advanced Stability of Higher-Order Functional Equations in Neutrosophic Normed Spaces via Hyers-Ulam Methodologies

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Abstract:

In this article, focuses on examining the stability of higher-order functional equations within the framework of neutrosophic normed spaces, which incorporate elements of uncertainty and indeterminacy. By utilizing the Hyers-Ulam method, the study investigates how small perturbations in the functional equations impact their solutions. The research extends classical stability theories, such as Hyers-Ulam stability, into the neutrosophic normed space context, providing a broader understanding of how functional equations behave under uncertainty. The findings offer significant contributions to the field of functional equations and neutrosophic mathematics, opening up new pathways for applications in areas that require the handling of imprecise data.

Keywords: Duodecic, Tridecic Functional Equations, Generalized Hyers - Ulam Stability.

1. Introduction

The Ulam-Hyers stability, introduced by Stanislaw Ulam and Donald Hyers in the 1940s and 1941s respectively [1, 2], is a fundamental concept in functional analysis and mathematical stability theory. It examines the behavior of solutions to functional equations when subjected to small perturbations. Functional equations, which establish specific relationships between the values of functions, are prevalent across various fields, including mathematics, physics, engineering, and economics. The stability of such equations focuses on how slight changes in inputs or parameters influence their solutions.

The Ulam-Hyers stability theory establishes conditions under which solutions to functional equations remain close to the original solutions despite small perturbations. It addresses the existence and uniqueness of solutions while analyzing their dependence on initial conditions and parameters. This concept is significant due to its extensive applicability and relevance to practical problems. It enhances the robustness and reliability of mathematical models and numerical algorithms, offering valuable insights into the behavior and predictability of dynamic systems. Ongoing research continues to expand the theory, uncovering connections with other mathematical domains and exploring its potential applications in various disciplines ([3, 4, 5, 6, 7]).

Lotfi A. Zadeh, a renowned mathematician and computer scientist, introduced the concept of fuzzy sets in his seminal 1965 paper titled "Fuzzy Sets" [8]. Zadeh developed this concept to overcome the constraints of classical set theory, which depends on sharp, well-defined boundaries for membership. Building upon this foundation, Atanassov proposed the notion of intuitionistic fuzzy sets (IFS) in 1983 [9, 10]. IFS extended fuzzy sets to incorporate a more nuanced framework for addressing uncertainty, vagueness, and hesitation. Further advancing these ideas, neutrosophic sets were introduced as an extension of classical set theory, providing a robust mechanism for managing indeterminacy, uncertainty, and incomplete information [11, 12].

Fuzzy normed spaces (FNS) are mathematical structures that generalize classical normed spaces by incorporating fuzzy numbers. Katsaras first introduced the concept of FNS, defining them as vector spaces equipped with a fuzzy norm, where the norm values are represented as fuzzy numbers rather than real numbers [13]. In 2006, the concept of intuitionistic fuzzy normed spaces (IFNS) was introduced [14]. IFNS merge elements of fuzzy mathematics, intuitionistic fuzzy sets, and normed spaces, offering a flexible framework for addressing uncertainty and imprecision in mathematical modeling and analysis.

Neutrosophic normed linear spaces [15, 16] extend neutrosophic set theory to linear algebraic structures, enabling a more comprehensive representation of uncertainty within vector spaces. Neutrosophic concepts have been widely applied in various branches of mathematics, including groups and subgroups [17], vector spaces [18], ring homomorphisms [19, 20], linear transformations [21], number theory [22], graph theory [23], measure theory, integral theory, and probability theory [24]. Additionally, neutrosophic normed linear spaces find practical applications in diverse fields such as decision-making, control systems, optimization, image processing, pattern recognition, medical diagnosis, finance and risk management, information retrieval, and artificial intelligence [25, 26, 27, 28, 29, 30, 31]. Agilan et al. have introduced novel functional equations and established the Hyers-Ulam stability of these equations across various normed spaces ([32, 33, 34, 35, 36, 37, 38]).

This article introduces a novel mixed duodecic-tridecic functional equation and investigates its Ulam-Hyers stability within neutrosophic normed linear spaces (NNLS). Classical approaches are employed for the stability analysis in the newly proposed equation. Given the unique properties of neutrosophic normed spaces and their broad potential applications, the stability analysis of this equation is of considerable importance. Notably, this study marks the first instance in the literature where the stability of a functional equation is examined within the framework of neutrosophic normed spaces, underscoring the distinctiveness and significance of the research.

This study sets out to accomplish the following key objectives:

(i) To expand and advance the existing research on neutrosophic normed linear spaces.

(ii) To establish the uniqueness of solutions for the newly introduced Higher-order functional equation.

(ii) To investigate the Hyers-Ulam stability of the proposed equation within neutrosophic normed linear spaces, employing the direct method.

$$\mathfrak{F}(11\mathfrak{Z}) = 1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z}) - 1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}) \qquad (1)$$

The above equation having solution $\mathfrak{F}(\mathfrak{Z}) = \mathcal{A}_1\, \mathfrak{Z}^{13} + \mathcal{A}_2\, \mathfrak{Z}^{12}$.

Remark 1.1 *Let us take an odd mapping* $\mathfrak{F}: \mathcal{X}_M \to \mathcal{Y}_M$ *which satisfies the FE (1) then it is tridecic*

$$\mathfrak{F}(11\mathfrak{Z}) = 96{,}88{,}90{,}10{,}407\mathfrak{F}(\mathfrak{Z}) = 11^{13}\mathfrak{F}(\mathfrak{Z})$$

for all $\mathfrak{Z} \in \mathcal{X}_M$

Remark 1.2 *Let us take an even mapping* $\mathfrak{F}: \mathcal{X}_M \to \mathcal{Y}_M$ *which satisfies the FE (1) then it is duodecic*

$$\mathfrak{F}(11\mathfrak{Z}) = 13{,}84{,}12{,}87{,}201\mathfrak{F}(\mathfrak{Z}) = 11^{12}\mathfrak{F}(\mathfrak{Z})$$

for all $\mathfrak{Z} \in \mathcal{X}_M$

2 Definition of Neutrosophic normed spaces

Definition 2.1 *The Seven-tuple* $(\mathbb{A}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 *, \diamond, \oslash)$ *is said to be a neutrosophic normed space (for short, NNS) if* $\mathbb{A}$ *is a vector space,* $*$ *is a continuous* κ*-norm,* $\diamond$ *and* $\oslash$ *is a continuous* $\kappa -$ *conorm, and* $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ *are fuzzy sets on* $\mathbb{A} \times (0, \infty)$ *satisfying the following conditions. For every* $p, q \in \mathbb{A}$ *and* $s, \kappa > 0$*,*

(ϕ_1) $\mathcal{A}_1(p,\kappa) + \mathcal{A}_2(p,\kappa) + \mathcal{A}_3(p,\kappa) \leq 3$,

(ϕ_2) $0 \leq \mathcal{A}_1(p,\kappa) \leq 1, 0 \leq \mathcal{A}_2(p,\kappa) \leq 1, 0 \leq \mathcal{A}_3(p,\kappa) \leq 1$,

(ϕ_3) $\mathcal{A}_1(p,\kappa) > 0$,

(ϕ_4) $\mathcal{A}_1(p,\kappa) = 1$, if and only if $p = 0$.

(ϕ_5) $\mathcal{A}_1(\alpha p,\kappa) = \mathcal{A}_1\left(p, \frac{\kappa}{|\alpha|}\right)$ for each $\alpha \neq 0$,

(ϕ_6) $\mathcal{A}_1(p,\kappa) * \mathcal{A}_1(q,s) \leq \mathcal{A}_1(p+q, \kappa+s)$,

(ϕ_7) $\mathcal{A}_1(p,\cdot): (0,\infty) \to [0,1]$ is continuous,

(ϕ_8) $\lim\limits_{\kappa\to\infty} \mathcal{A}_1(p,\kappa) = 1$ and $\lim\limits_{\kappa\to 0} \mathcal{A}_1(p,\kappa) = 0$,

(ϕ_9) $\mathcal{A}_2(p,\kappa) < 1$,

(ϕ_{10}) $\mathcal{A}_2(p,\kappa) = 0$, if and only if $p = 0$.

(ϕ_{11}) $\mathcal{A}_2(\alpha p,\kappa) = \mathcal{A}_2\left(p, \frac{\kappa}{|\alpha|}\right)$ for each $\alpha \neq 0$,

(ϕ_{12}) $\mathcal{A}_2(p,\kappa) \diamond \mathcal{A}_2(q,s) \geq \mathcal{A}_2(p+q, \kappa+s)$,

(ϕ_{13}) $\mathcal{A}_2(p,\cdot): (0,\infty) \to [0,1]$ is continuous,

(ϕ_{14}) $\lim\limits_{\kappa\to\infty} \mathcal{A}_2(p,\kappa) = 0$ and $\lim\limits_{\kappa\to 0} \mathcal{A}_2(p,\kappa) = 1$

(ϕ_{15}) $\mathcal{A}_3(p,\kappa) < 1$,

(ϕ_{16}) $\mathcal{A}_3(p,\kappa) = 0$, if and only if $p = 0$.

(ϕ_{17}) $\mathcal{A}_3(\alpha p,\kappa) = \mathcal{A}_3\left(p, \frac{\kappa}{|\alpha|}\right)$ for each $\alpha \neq 0$,

(ϕ_{18}) $\mathcal{A}_3(p,\kappa) \oslash \mathcal{A}_3(q,s) \geq \mathcal{A}_3(p+q, \kappa+s)$,

(ϕ_{19}) $\mathcal{A}_3(p,\cdot):(0,\infty)\to[0,1]$ is continuous,

(ϕ_{20}) $\lim\limits_{\kappa\to\infty}\mathcal{A}_3(p,\kappa)=0$ and $\lim\limits_{\kappa\to 0}\mathcal{A}_3(p,\kappa)=1$.

3 Stability Results in neutrosophic normed space: Hyers Classical Direct Method

Theorem 3.1 *Assume that* $\mathcal{X}_M$ *is a LS,* $(\mathcal{Z}_m,\mathcal{A}_1',\mathcal{A}_2',\mathcal{A}_3')$ *is a NNS and* $(\mathcal{Y}_M,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$ *an NBS . Let* $\eth:\mathcal{X}_M\longrightarrow\mathcal{Z}_m$ *be a function such that for some* $0<\left(\frac{\mathfrak{B}}{11^{13}}\right)^F<1$ *with* $F\in\{1,-1\}$.

$$\left.\begin{aligned}&\mathcal{A}_1'(\eth(11^{nF}\mathfrak{z}),\mathcal{M})\ge\mathcal{A}_1'(\mathfrak{B}^{nF}\eth(\mathfrak{z}),\mathcal{M})\\&\mathcal{A}_2'(\eth(11^{nF}\mathfrak{z}),\mathcal{M})\le\mathcal{A}_2'(\mathfrak{B}^{nF}\eth(\mathfrak{z}),\mathcal{M})\\&\mathcal{A}_3'(\eth(11^{nF}\mathfrak{z}),\mathcal{M})\le\mathcal{A}_3'(\mathfrak{B}^{nF}\eth(\mathfrak{z}),\mathcal{M})\end{aligned}\right\}\qquad(1)$$

for all $\mathfrak{z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$ and

$$\left.\begin{aligned}&\lim_{n\to\infty}\mathcal{A}_1'(\eth(11^{Fn}\mathfrak{z}),11^{13Fn}\mathcal{M})=1\\&\lim_{n\to\infty}\mathcal{A}_2'(\eth(11^{Fn}\mathfrak{z}),11^{13Fn}\mathcal{M})=0\\&\lim_{n\to\infty}\mathcal{A}_3'(\eth(11^{Fn}\mathfrak{z}),11^{13Fn}\mathcal{M})=0\end{aligned}\right\}\qquad(2)$$

for all $\mathfrak{z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$. Let an odd function $\mathfrak{F}:\mathcal{X}_M\longrightarrow\mathcal{Y}_M$ satisfying

$$\left.\begin{aligned}&\mathcal{A}_1(\mathfrak{F}(11\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{z}),\mathcal{M})\ge\mathcal{A}_1'(\eth(\mathfrak{z}),\mathcal{M})\\&\mathcal{A}_2(\mathfrak{F}(11\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{z}),\mathcal{M})\le\mathcal{A}_2'(\eth(\mathfrak{z}),\mathcal{M})\\&\mathcal{A}_3(\mathfrak{F}(11\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{z}),\mathcal{M})\le\mathcal{A}_3'(\eth(\mathfrak{z}),\mathcal{M})\end{aligned}\right\}\qquad(3)$$

for all $\mathfrak{z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$. Then there exists a unique tridecic mapping $\mathcal{T}:\mathcal{X}_M\longrightarrow\mathcal{Y}_M$ satisfying (1) and

$$\left.\begin{aligned}&\mathcal{A}_1(\mathfrak{F}(\mathfrak{z})-\mathcal{T}(\mathfrak{z}),\mathcal{M})\ge\mathcal{A}_1'(\eth(\mathfrak{z}),\mathcal{M}|11^{13}-\mathfrak{B}|)\\&\mathcal{A}_2(\mathfrak{F}(\mathfrak{z})-\mathcal{T}(\mathfrak{z}),\mathcal{M})\le\mathcal{A}_2'(\eth(\mathfrak{z}),\mathcal{M}|11^{13}-\mathfrak{B}|)\\&\mathcal{A}_3(\mathfrak{F}(\mathfrak{z})-\mathcal{T}(\mathfrak{z}),\mathcal{M})\le\mathcal{A}_3'(\eth(\mathfrak{z}),\mathcal{M}|11^{13}-\mathfrak{B}|)\end{aligned}\right\}\qquad(4)$$

for all $\mathfrak{z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$.

Proof. For the first case $F=1$. Using oddness of $\mathfrak{F}$ in in (3), we obtain

$$\left.\begin{aligned}
&\mathcal{A}_1(\mathfrak{F}(11\mathfrak{Z}) - 11^{13}\mathfrak{F}(\mathfrak{Z}), \mathcal{M}) \geq \mathcal{A}_1{}'(\eth(\mathfrak{Z}), \mathcal{M})\\
&\mathcal{A}_2(\mathfrak{F}(11\mathfrak{Z}) - 11^{13}\mathfrak{F}(\mathfrak{Z}), \mathcal{M}) \leq \mathcal{A}_2{}'(\eth(\mathfrak{Z}), \mathcal{M})\\
&\mathcal{A}_3(\mathfrak{F}(11\mathfrak{Z}) - 11^{13}\mathfrak{F}(\mathfrak{Z}), \mathcal{M}) \leq \mathcal{A}_3{}'(\eth(\mathfrak{Z}), \mathcal{M})
\end{aligned}\right\} \tag{5}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Using (NNS5), (NNS11) and (NNS17) in (5), we have

$$\left.\begin{aligned}
&\mathcal{A}_1(\frac{\mathfrak{F}(11\mathfrak{Z})}{11^{13}} - \mathfrak{F}(\mathfrak{Z}), \frac{\mathcal{M}}{11^{13}}) \geq \mathcal{A}_1{}'(\eth(\mathfrak{Z}), \mathcal{M})\\
&\mathcal{A}_2(\frac{\mathfrak{F}(11\mathfrak{Z})}{11^{13}} - \mathfrak{F}(\mathfrak{Z}), \frac{\mathcal{M}}{11^{13}}) \leq \mathcal{A}_2{}'(\eth(\mathfrak{Z}), \mathcal{M})\\
&\mathcal{A}_3(\frac{\mathfrak{F}(11\mathfrak{Z})}{11^{13}} - \mathfrak{F}(\mathfrak{Z}), \frac{\mathcal{M}}{11^{13}}) \leq \mathcal{A}_3{}'(\eth(\mathfrak{Z}), \mathcal{M})
\end{aligned}\right\} \tag{6}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Let us take $\mathfrak{Z}$ by $11^n\mathfrak{Z}$ in (6), we arrive

$$\left.\begin{aligned}
&\mathcal{A}_1(\frac{\mathfrak{F}(11^{(n+1)}\mathfrak{Z})}{11^{13}} - \mathfrak{F}(11^n\mathfrak{Z}), \frac{\mathcal{M}}{11^{13}}) \geq \mathcal{A}_1{}'(\eth(11^n\mathfrak{Z}), \mathcal{M})\\
&\mathcal{A}_2(\frac{\mathfrak{F}(11^{(n+1)}\mathfrak{Z})}{11^{13}} - \mathfrak{F}(11^n\mathfrak{Z}), \frac{\mathcal{M}}{11^{13}}) \leq \mathcal{A}_2{}'(\eth(11^n\mathfrak{Z}), \mathcal{M})\\
&\mathcal{A}_3(\frac{\mathfrak{F}(11^{(n+1)}\mathfrak{Z})}{11^{13}} - \mathfrak{F}(11^n\mathfrak{Z}), \frac{\mathcal{M}}{11^{13}}) \leq \mathcal{A}_3{}'(\eth(11^n\mathfrak{Z}), \mathcal{M})
\end{aligned}\right\} \tag{7}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. It is simple to confirm that (7) and using (1), (NNS5), (NNS11) and (NNS17) that

$$\left.\begin{aligned}
&\mathcal{A}_1(\frac{\mathfrak{F}(11^{(n+1)}\mathfrak{Z})}{11^{13(n+1)}} - \frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}}, \frac{\mathcal{M}}{11^{13}\cdot 11^{13n}}) \geq \mathcal{A}_1{}'\left(\eth(\mathfrak{Z}), \frac{\mathcal{M}}{\mathfrak{V}^n}\right)\\
&\mathcal{A}_2(\frac{\mathfrak{F}(11^{n+1}\mathfrak{Z})}{11^{13(n+1)}} - \frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}}, \frac{\mathcal{M}}{11^{13}\cdot 11^{13n}}) \leq \mathcal{A}_2{}'\left(\eth(\mathfrak{Z}), \frac{\mathcal{M}}{\mathfrak{V}^n}\right)\\
&\mathcal{A}_3(\frac{\mathfrak{F}(11^{(n+1)}\mathfrak{Z})}{11^{13(n+1)}} - \frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}}, \frac{\mathcal{M}}{11^{13}\cdot 11^{13n}}) \leq \mathcal{A}_3{}'\left(\eth(\mathfrak{Z}), \frac{\mathcal{M}}{\mathfrak{V}^n}\right)
\end{aligned}\right\} \tag{8}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Swapping $\mathcal{M}$ into $\mathfrak{V}^n\mathcal{M}$ in (8), we have

$$\left.\begin{aligned}
&\mathcal{A}_1(\frac{\mathfrak{F}(11^{n+1}\mathfrak{Z})}{11^{13(n+1)}} - \frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}}, \frac{\mathcal{M}\cdot\mathfrak{V}^n}{11^{13}\cdot 11^{13n}}) \geq \mathcal{A}_1{}'(\eth(\mathfrak{Z}), \mathcal{M})\\
&\mathcal{A}_2(\frac{\mathfrak{F}(11^{n+1}\mathfrak{Z})}{11^{13(n+1)}} - \frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}}, \frac{\mathcal{M}\cdot\mathfrak{V}^n}{11^{13}\cdot 11^{13n}}) \leq \mathcal{A}_2{}'(\eth(\mathfrak{Z}), \mathcal{M})\\
&\mathcal{A}_3(\frac{\mathfrak{F}(11^{n+1}\mathfrak{Z})}{11^{13(n+1)}} - \frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}}, \frac{\mathcal{M}\cdot\mathfrak{V}^n}{11^{13}\cdot 11^{13n}}) \leq \mathcal{A}_3{}'(\eth(\mathfrak{Z}), \mathcal{M})
\end{aligned}\right\} \tag{9}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. It is simple to observe that

$$\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}) = \sum_{i=0}^{n-1} \frac{\mathfrak{F}(11^{i+1}\mathfrak{Z})}{11^{13(i+1)}} - \frac{\mathfrak{F}(11^{i}\mathfrak{Z})}{11^{13i}} \tag{10}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$. It follows from (9) and (10), we get

$$\left.\begin{aligned}
\mathcal{A}_1\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &= \mathcal{A}_1\left(\sum_{i=0}^{n-1} \frac{\mathfrak{F}(11^{i+1}\mathfrak{Z})}{11^{13(i+1)}} - \frac{\mathfrak{F}(11^{i}\mathfrak{Z})}{11^{13i}}, \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right)\\
\mathcal{A}_2\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &= \mathcal{A}_2\left(\sum_{i=0}^{n-1} \frac{\mathfrak{F}(11^{i+1}\mathfrak{Z})}{11^{13(i+1)}} - \frac{\mathfrak{F}(11^{i}\mathfrak{Z})}{11^{i}}, \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right)\\
\mathcal{A}_3\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &= \mathcal{A}_3\left(\sum_{i=0}^{n-1} \frac{\mathfrak{F}(11^{i+1}\mathfrak{Z})}{11^{13(i+1)}} - \frac{\mathfrak{F}(11^{i}\mathfrak{Z})}{11^{13i}}, \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right)
\end{aligned}\right\} \tag{11}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Using (NNS5), (NNS11) and (NNS17) in (11), we have

$$\left.\begin{aligned}
\mathcal{A}_1\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &\geq \prod_{i=0}^{n-1} \mathcal{A}_1\left(\frac{\mathfrak{F}(11^{i+1}\mathfrak{Z})}{11^{13(i+1)}} - \frac{\mathfrak{F}(11^{i}\mathfrak{Z})}{11^{13i}}, \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right)\\
\mathcal{A}_2\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &\leq \prod_{i=0}^{n-1} \mathcal{A}_2\left(\frac{\mathfrak{F}(11^{i+1}\mathfrak{Z})}{11^{13(i+1)}} - \frac{\mathfrak{F}(11^{i}\mathfrak{Z})}{11^{13i}}, \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right)\\
\mathcal{A}_3\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &\leq \coprod_{i=0}^{n-1} \mathcal{A}_3\left(\frac{\mathfrak{F}(11^{i+1}\mathfrak{Z})}{11^{13(i+1)}} - \frac{\mathfrak{F}(11^{i}\mathfrak{Z})}{11^{13i}}, \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right)
\end{aligned}\right\} \tag{12}$$

where

$$\prod_{\mathcal{J}=0}^{n-1} Q_j = Q_1 * Q_2 * \cdots * Q_n \quad and \quad \coprod_{\mathcal{J}=0}^{n-1} R_j = R_1 \diamond R_2 \diamond \cdots \diamond R_n and \quad \coprod_{\mathcal{J}=0}^{n-1} S_j = S_1 \oslash S_2 \oslash \cdots \oslash S_n$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Hence, from (12) and (9), we arrive

$$\left.\begin{aligned}
\mathcal{A}_1\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &\geq \prod_{i=0}^{n-1} \mathcal{A}_1'(\eth(\mathfrak{Z}), \mathcal{M}) = \mathcal{A}_1'(\eth(\mathfrak{Z}), \mathcal{M})\\
\mathcal{A}_2\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &\leq \prod_{i=0}^{n-1} \mathcal{A}_2'(\eth(\mathfrak{Z}), \mathcal{M}) = \mathcal{A}_2'(\eth(\mathfrak{Z}), \mathcal{M})\\
\mathcal{A}_3\left(\frac{\mathfrak{F}(11^n\mathfrak{Z})}{11^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathfrak{V}^i\,\mathcal{M}}{11^{13}\cdot 11^{13i}}\right) &\leq \prod_{i=0}^{n-1} \mathcal{A}_1'(\eth(\mathfrak{Z}), \mathcal{M}) = \mathcal{A}_3'(\eth(\mathfrak{Z}), \mathcal{M})
\end{aligned}\right\} \tag{13}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Replacing $\mathfrak{Z}$ by $11^m\mathfrak{Z}$ in (13) and using (1), (NNS5), (NNS11) and (NNS17), we obtain

$$\left.\begin{aligned}
&\mathcal{A}_1\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\sum_{i=0}^{n-1}\frac{\mathfrak{V}^{i}\,\mathcal{M}}{11^{13}\cdot 11^{13(i+m)}}\right)\geq \mathcal{A}_1{}'(\eth(11^{m}\mathfrak{Z}),\mathcal{M})=\mathcal{A}_1{}'\left(\eth(\mathfrak{Z}),\frac{\mathcal{M}}{\mathfrak{V}^{m}}\right)\\
&\mathcal{A}_2\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\sum_{i=0}^{n-1}\frac{\mathfrak{V}^{i}\,\mathcal{M}}{11^{13}\cdot 11^{13(i+m)}}\right)\leq \mathcal{A}_2{}'(\eth(11^{m}\mathfrak{Z}),\mathcal{M})=\mathcal{A}_2{}'\left(\eth(\mathfrak{Z}),\frac{\mathcal{M}}{\mathfrak{V}^{m}}\right)\\
&\mathcal{A}_3\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\sum_{i=0}^{n-1}\frac{\mathfrak{V}^{i}\,\mathcal{M}}{11^{13}\cdot 11^{13(i+m)}}\right)\leq \mathcal{A}_3{}'(\eth(11^{m}\mathfrak{Z}),\mathcal{M})=\mathcal{A}_1{}'\left(\eth(\mathfrak{Z}),\frac{\mathcal{M}}{\mathfrak{V}^{m}}\right)
\end{aligned}\right\}(14)$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$ also m,n are positive numbers. Changing $\mathcal{M}$ by $\mathfrak{V}^{m}\mathcal{M}$ in (14), we get

$$\left.\begin{aligned}
&\mathcal{A}_1\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\sum_{i=0}^{n-1}\frac{\mathfrak{V}^{i+m}\,\mathcal{M}}{11^{13}\cdot 11^{13(i+m)}}\right)\geq \mathcal{A}_1{}'(\eth(\mathfrak{Z}),\mathcal{M})\\
&\mathcal{A}_2\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\sum_{i=0}^{n-1}\frac{\mathfrak{V}^{i+m}\,\mathcal{M}}{11^{13}\cdot 11^{13(i+m)}}\right)\leq \mathcal{A}_2{}'(\eth(\mathfrak{Z}),\mathcal{M})\\
&\mathcal{A}_3\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\sum_{i=0}^{n-1}\frac{\mathfrak{V}^{i+m}\,\mathcal{M}}{11^{13}\cdot 11^{13(i+m)}}\right)\leq \mathcal{A}_3{}'(\eth(\mathfrak{Z}),\mathcal{M})
\end{aligned}\right\}\qquad(15)$$

which implies

$$\left.\begin{aligned}
&\mathcal{A}_1\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\mathcal{M}\right)\geq \mathcal{A}_1{}'\left(\eth(\mathfrak{Z}),\frac{\mathcal{M}}{\sum_{i=m}^{n-1}\frac{\mathfrak{V}^{i}}{11^{13}\cdot 11^{13i}}}\right)\\
&\mathcal{A}_2\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\mathcal{M}\right)\leq \mathcal{A}_2{}'\left(\eth(\mathfrak{Z}),\frac{\mathcal{M}}{\sum_{i=m}^{n-1}\frac{\mathfrak{V}^{i}}{11^{13}\cdot 11^{13i}}}\right)\\
&\mathcal{A}_3\left(\frac{\mathfrak{F}(11^{n+m}\mathfrak{Z})}{11^{13(n+m)}}-\frac{\mathfrak{F}(11^{m}\mathfrak{Z})}{11^{13m}},\mathcal{M}\right)\leq \mathcal{A}_3{}'\left(\eth(\mathfrak{Z}),\frac{\mathcal{M}}{\sum_{i=m}^{n-1}\frac{\mathfrak{V}^{i}}{11^{13}\cdot 11^{13i}}}\right)
\end{aligned}\right\}\qquad(16)$$

Here $\left\{\frac{\mathfrak{F}(11^{n}\mathfrak{Z})}{11^{13n}}\right\}$ is a Cauchy sequence in $(\mathcal{Y}_M,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$ also a complete NNS-space is $(\mathcal{Y}_M,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$ then this sequence converges to a particular point $\mathcal{T}(\mathfrak{Z})\in Y$.

$$\lim_{n\to\infty}\mathcal{A}_1\left(\frac{\mathfrak{F}(11^{n}\mathfrak{Z})}{11^{13n}}-\mathcal{T}(\mathfrak{Z}),\mathcal{M}\right)=1,$$

$$\lim_{n\to\infty}\mathcal{A}_2\left(\frac{\mathfrak{F}(11^{n}\mathfrak{Z})}{11^{13n}}-\mathcal{T}(\mathfrak{Z}),\mathcal{M}\right)=0$$

$$\lim_{n\to\infty}\mathcal{A}_3\left(\frac{\mathfrak{F}(11^{n}\mathfrak{Z})}{11^{13n}}-\mathcal{T}(\mathfrak{Z}),\mathcal{M}\right)=0$$

and

$$\frac{\mathfrak{F}(11^{n}\mathfrak{Z})}{11^{13n}}\xrightarrow{NNS}\mathcal{T}(\mathfrak{Z}),\quad as\quad n\to\infty.$$

Taking $m=0$ in (15), we reach

$$\left.\begin{aligned}
\mathcal{A}_1\left(\frac{\mathfrak{F}(11^n\mathfrak{z})}{11^{13n}}-\mathfrak{F}(\mathfrak{z}),\mathcal{M}\right) &\geq \mathcal{A}_1{}'\left(\Psi(\mathfrak{z}),\frac{\mathcal{M}}{\sum_{i=0}^{n-1}\frac{\mathfrak{V}^i}{11^{13}\cdot 11^{13i}}}\right)\\
\mathcal{A}_2\left(\frac{\mathfrak{F}(11^n\mathfrak{z})}{11^{13n}}-\mathfrak{F}(\mathfrak{z}),\mathcal{M}\right) &\leq \mathcal{A}_2{}'\left(\Psi(\mathfrak{z}),\frac{\mathcal{M}}{\sum_{i=0}^{n-1}\frac{\mathfrak{V}^i}{11^{13}\cdot 11^{13i}}}\right)\\
\mathcal{A}_3\left(\frac{\mathfrak{F}(11^n\mathfrak{z})}{11^{13n}}-\mathfrak{F}(\mathfrak{z}),\mathcal{M}\right) &\leq \mathcal{A}_3{}'\left(\Psi(\mathfrak{z}),\frac{\mathcal{M}}{\sum_{i=0}^{n-1}\frac{\mathfrak{V}^i}{11^{13}\cdot 11^{13i}}}\right)
\end{aligned}\right\} \quad (17)$$

for all $\mathfrak{z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Considering $n \to \infty$ in (17), we arrive

$$\left.\begin{aligned}
\mathcal{A}_1(\mathcal{T}(\mathfrak{z})-\mathfrak{F}(\mathfrak{z}),\mathcal{M}) &\geq \mathcal{A}_1{}'(\eth(\mathfrak{z}),\mathcal{M}|11^{13}-\mathfrak{V}|)\\
\mathcal{A}_2(\mathcal{T}(\mathfrak{z})-\mathfrak{F}(\mathfrak{z}),\mathcal{M}) &\leq \mathcal{A}_2{}'(\eth(\mathfrak{z}),\mathcal{M}|11^{13}-\mathfrak{V}|)\\
\mathcal{A}_3(\mathcal{T}(\mathfrak{z})-\mathfrak{F}(\mathfrak{z}),\mathcal{M}) &\leq \mathcal{A}_3{}'(\eth(\mathfrak{z}),\mathcal{M}|11^{13}-\mathfrak{V}|)
\end{aligned}\right\} \quad (18)$$

for all $\mathfrak{z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Next, we have to show $\mathfrak{F}$ satisfies (1), letting $\mathfrak{z}$ by $11^n\mathfrak{z}$ in (3) respectively, we have

$$\left.\begin{aligned}
&\mathcal{A}_1\left(\frac{1}{11^{13n}}[\mathfrak{F}(3\cdot 11^n\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(11^n\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-11^n\mathfrak{z})],\mathcal{M}\right)\\
&\qquad\qquad\geq \mathcal{A}_1{}'(\eth(11^n\mathfrak{z}),11^{13n}\mathcal{M})\\
&\mathcal{A}_2\left(\frac{1}{11^{13n}}[\mathfrak{F}(3\cdot 11^n\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(11^n\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-11^n\mathfrak{z})],\mathcal{M}\right)\\
&\qquad\qquad\leq \mathcal{A}_2{}'(\eth(11^n\mathfrak{z}),11^{13n}\mathcal{M})\\
&\mathcal{A}_3\left(\frac{1}{11^{13n}}[\mathfrak{F}(3\cdot 11^n\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(11^n\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-11^n\mathfrak{z})],\mathcal{M}\right)\\
&\qquad\qquad\leq \mathcal{A}_3{}'(\eth(11^n\mathfrak{z}),11^{13n}\mathcal{M})
\end{aligned}\right\} \quad (19)$$

for all $\mathfrak{z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Now,

$$\begin{aligned}
&\mathcal{A}_1(\mathcal{T}(11\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathcal{T}(\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathcal{T}(-\mathfrak{z}),\mathcal{M})\\
&\geq \mathcal{A}_1(\mathcal{T}(11\mathfrak{z})-\frac{1}{11^{13n}}\mathfrak{F}(11\mathfrak{z}),\frac{\mathcal{M}}{4})\\
&\quad *\mathcal{A}_1(-1{,}88{,}30{,}57{,}02{,}60{,}326\mathcal{T}(\mathfrak{z})+1{,}88{,}30{,}57{,}02{,}60{,}326\frac{1}{11^{13n}}\mathfrak{F}(\mathfrak{z}),\frac{\mathcal{M}}{4})\\
&\quad *\mathcal{A}_1(1{,}56{,}92{,}14{,}18{,}83{,}605\mathcal{T}(-\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\frac{1}{11^{13n}}\mathfrak{F}(-\mathfrak{z}),\frac{\mathcal{M}}{4})\\
&\quad *\mathcal{A}_1(\frac{1}{11^{13n}}\mathfrak{F}(11\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\frac{1}{11^{13n}}\mathfrak{F}(\mathfrak{z})+\\
&1{,}56{,}92{,}14{,}18{,}83{,}605\frac{1}{11^{13n}}\mathfrak{F}(-\mathfrak{z}),\frac{\mathcal{M}}{4})
\end{aligned} \quad (20)$$

$$\mathcal{A}_2(\mathcal{T}(11\mathfrak{Z}) - 1{,}88{,}30{,}57{,}02{,}60{,}326\mathcal{T}(\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\mathcal{T}(-\mathfrak{Z}), \mathcal{M})$$

$$\geq \mathcal{A}_1(\mathcal{T}(11\mathfrak{Z}) - \tfrac{1}{11^{13n}}\mathfrak{F}(11\mathfrak{Z}), \tfrac{\mathcal{M}}{4})$$

$$\diamond\, \mathcal{A}_2(-1{,}88{,}30{,}57{,}02{,}60{,}326\mathcal{T}(\mathfrak{Z}) + 1{,}88{,}30{,}57{,}02{,}60{,}326\tfrac{1}{11^{13n}}\mathfrak{F}(\mathfrak{Z}), \tfrac{\mathcal{M}}{4})$$

$$\diamond\, \mathcal{A}_2(-1{,}56{,}92{,}14{,}18{,}83{,}605\mathcal{T}(-\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\tfrac{1}{11^{13n}}\mathfrak{F}(-\mathfrak{Z}), \tfrac{\mathcal{M}}{4})$$

$$\diamond\, \mathcal{A}_2(\tfrac{1}{11^{13n}}\mathfrak{F}(11\mathfrak{Z}) - 1{,}88{,}30{,}57{,}02{,}60{,}326\tfrac{1}{11^{13n}}\mathfrak{F}(\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\tfrac{1}{11^{13n}}\mathfrak{F}(-\mathfrak{Z}), \tfrac{\mathcal{M}}{4}) \qquad (21)$$

and

$$\mathcal{A}_3(\mathcal{T}(11\mathfrak{Z}) - 1{,}88{,}30{,}57{,}02{,}60{,}326\mathcal{T}(\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\mathcal{T}(-\mathfrak{Z}), \mathcal{M})$$

$$\geq \mathcal{A}_3(\mathcal{T}(11\mathfrak{Z}) - \tfrac{1}{11^{13n}}\mathfrak{F}(11\mathfrak{Z}), \tfrac{\mathcal{M}}{4})$$

$$\oslash\, \mathcal{A}_3(-1{,}88{,}30{,}57{,}02{,}60{,}326\mathcal{T}(\mathfrak{Z}) + 1{,}88{,}30{,}57{,}02{,}60{,}326\tfrac{1}{11^{13n}}\mathfrak{F}(\mathfrak{Z}), \tfrac{\mathcal{M}}{4})$$

$$\oslash\, \mathcal{A}_3(1{,}56{,}92{,}14{,}18{,}83{,}605\mathcal{T}(-\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\tfrac{1}{11^{13n}}\mathfrak{F}(-\mathfrak{Z}), \tfrac{\mathcal{M}}{4})$$

$$\oslash\, \mathcal{A}_1(\tfrac{1}{11^{13n}}\mathfrak{F}(11\mathfrak{Z}) - 1{,}88{,}30{,}57{,}02{,}60{,}326\tfrac{1}{11^{13n}}\mathfrak{F}(\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\tfrac{1}{11^{13n}}\mathfrak{F}(-\mathfrak{Z}), \tfrac{\mathcal{M}}{4}) \qquad (22)$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Also,

$$\left.\begin{array}{l}
\lim\limits_{n\to\infty} \mathcal{A}_1(\frac{1}{11^{13n}}[\mathfrak{F}(3\cdot 11^n\mathfrak{Z}) - 1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(11^n\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-11^n\mathfrak{Z})], \frac{\mathcal{M}}{4}) = 1 \\
\lim\limits_{n\to\infty} \mathcal{A}_2(\frac{1}{11^{13n}}[\mathfrak{F}(3\cdot 11^n\mathfrak{Z}) - 1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(11^n\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-11^n\mathfrak{Z})], \frac{\mathcal{M}}{4}) = 0 \\
\lim\limits_{n\to\infty} \mathcal{A}_3(\frac{1}{11^{13n}}[\mathfrak{F}(3\cdot 11^n\mathfrak{Z}) - 1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(11^n\mathfrak{Z}) + 1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-11^n\mathfrak{Z})], \frac{\mathcal{M}}{4}) = 0
\end{array}\right\} \qquad (23)$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$.

By Taking $n \to \infty$ in (21), (22) and using (23), we proved that $\mathcal{T}$ satisfies (1). Therefore, $\mathcal{T}$ is a tridecic mapping.

Next, we need to prove $\mathcal{T}(\mathfrak{Z})$ is unique,

let $\mathcal{T}'(\mathfrak{Z})$ be another tridecic FE satisfying (1) and (4), Then

$$\mathcal{A}_1(\mathcal{T}(\mathfrak{Z}) - \mathcal{T}'(\mathfrak{Z}), \mathcal{M})$$

$$\geq \mathcal{A}_1\left(\mathcal{T}(11^n\mathfrak{Z}) - \mathfrak{F}(11^n\mathfrak{Z}), \frac{\mathcal{M}.11^{13n}}{2}\right) * \mathcal{A}_1\left(\mathfrak{F}(11^n\mathfrak{Z}) - \mathcal{T}'(11^n\mathfrak{Z}), \frac{\mathcal{M}.11^{13n}}{2}\right)$$

$$\geq \mathcal{A}_1{}'\left(\eth(11^n\mathfrak{Z}), \frac{11^{13n}\ \mathcal{M}\ |11^{13} - \mathfrak{V}|}{2}\right) \geq \mathcal{A}_1{}'\left(\eth(\mathfrak{Z}), \frac{11^{13n}\ \mathcal{M}\ |11^{13} - \mathfrak{V}|}{2\cdot\mathfrak{V}^n}\right)$$

$$\mathcal{A}_2(\mathcal{T}(\mathfrak{Z}) - \mathcal{T}'(\mathfrak{Z}), \mathcal{M})$$

$$\leq \mathcal{A}_2\left(\mathcal{T}(11^n\mathfrak{Z}) - \mathfrak{F}(11^n\mathfrak{Z}), \frac{\mathcal{M}.11^{13n}}{2}\right) \diamond \mathcal{A}_2\left(\mathfrak{F}(11^n\mathfrak{Z}) - \mathcal{T}'(11^n\mathfrak{Z}), \frac{\mathcal{M}.11^{13n}}{2}\right)$$

$$\leq \mathcal{A}_2{}'\left(\eth(11^n\mathfrak{Z}), \frac{11^{13n}\mathcal{M}\ |11^{13} - \mathfrak{V}|}{2}\right) \leq \mathcal{A}_2{}'\left(\eth(\mathfrak{Z}), \frac{11^{13n}\ \mathcal{M}\ |11^{13} - \mathfrak{V}|}{2\cdot\mathfrak{V}^n}\right)$$

$$\begin{aligned}&\mathcal{A}_3(\mathcal{T}(\mathfrak{Z})-\mathcal{T}'(\mathfrak{Z}),\mathcal{M})\\ &\leq \mathcal{A}_3\left(\mathcal{T}(11^n\mathfrak{Z})-\mathfrak{F}(11^n\mathfrak{Z}),\frac{\mathcal{M}.11^{13n}}{2}\right)*\mathcal{A}_3\left(\mathfrak{F}(11^n\mathfrak{Z})-\mathcal{T}'(11^n\mathfrak{Z}),\frac{\mathcal{M}.11^{13n}}{2}\right)\\ &\leq \mathcal{A}_3{}'\left(\eth(11^n\mathfrak{Z}),\frac{11^{13n}\,\mathcal{M}\,|11^{13}-\mathfrak{V}|}{2}\right)\leq \mathcal{A}_3{}'\left(\eth(\mathfrak{Z}),\frac{11^{13n}\,\mathcal{M}\,|11^{13}-\mathfrak{V}|}{2\cdot\mathfrak{V}^n}\right)\end{aligned}$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$. Since $\lim_{n\to\infty}\frac{11^{13n}\,\mathcal{M}\,|11^{13}-\mathfrak{V}|}{2\,\mathfrak{V}^n}=\infty$, we obtain

$$\left.\begin{aligned}&\lim_{n\to\infty}\mathcal{A}_1{}'\left(\eth(\mathfrak{Z}),\frac{11^{13n}\,\mathcal{M}\,|11^{13}-\mathfrak{V}|}{2\cdot\mathfrak{V}^n}\right)=1\\ &\lim_{n\to\infty}\mathcal{A}_2{}'\left(\eth(\mathfrak{Z}),\frac{11^{13n}\,\mathcal{M}\,|11^{13}-\mathfrak{V}|}{2\cdot\mathfrak{V}^n}\right)=0\\ &\lim_{n\to\infty}\mathcal{A}_3{}'\left(\eth(\mathfrak{Z}),\frac{11^{13n}\,\mathcal{M}\,|11^{13}-\mathfrak{V}|}{2\cdot\mathfrak{V}^n}\right)=0\end{aligned}\right\}$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$. Thus

$$\left.\begin{aligned}&\mathcal{A}_1(\mathcal{T}(\mathfrak{Z})-\mathcal{T}'(\mathfrak{Z}),\mathcal{M})=1\\ &\mathcal{A}_2(\mathcal{T}(\mathfrak{Z})-\mathcal{T}'(\mathfrak{Z}),\mathcal{M})=0\\ &\mathcal{A}_3(\mathcal{T}(\mathfrak{Z})-\mathcal{T}'(\mathfrak{Z}),\mathcal{M})=0\end{aligned}\right\}$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$.

Hence, $\mathcal{T}(\mathfrak{Z})=\mathcal{T}'(\mathfrak{Z})$. Therefore, $\mathcal{T}(\mathfrak{Z})$ is unique.

For second case, we have to take $F=-1$. Considering $\mathfrak{Z}$ by $\frac{\mathfrak{Z}}{13}$ in (5), we have

$$\left.\begin{aligned}&\mathcal{A}_1\left(\mathfrak{F}(\mathfrak{Z})-11^{13}\mathfrak{F}\left(\frac{\mathfrak{Z}}{13}\right),\mathcal{M}\right)\geq\mathcal{A}_1{}'\left(\eth\left(\frac{\mathfrak{Z}}{13}\right),\mathcal{M}\right)\\ &\mathcal{A}_2\left(\mathfrak{F}(\mathfrak{Z})-11^{13}\mathfrak{F}\left(\frac{\mathfrak{Z}}{13}\right),\mathcal{M}\right)\leq\mathcal{A}_2{}'\left(\eth\left(\frac{\mathfrak{Z}}{13}\right),\mathcal{M}\right)\\ &\mathcal{A}_3\left(\mathfrak{F}(\mathfrak{Z})-11^{13}\mathfrak{F}\left(\frac{\mathfrak{Z}}{13}\right),\mathcal{M}\right)\leq\mathcal{A}_3{}'\left(\eth\left(\frac{\mathfrak{Z}}{13}\right),\mathcal{M}\right)\end{aligned}\right\}\tag{24}$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$.

Corollary 3.2 *Let $\mathcal{L},r$ are constants, with $\mathcal{L}>0$ and $r\neq 12$ and let an odd function $\mathfrak{F}:\mathcal{X}_M\longrightarrow\mathcal{Y}_M$ satisfies*

$$\left.\begin{aligned}&\mathcal{A}_1(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M})\geq\begin{cases}\mathcal{A}_1{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_1{}'(\mathcal{L}(||\mathfrak{Z}||^r),\mathcal{M}),\end{cases}\\ &\mathcal{A}_2(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M})\leq\begin{cases}\mathcal{A}_2{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_2{}'(\mathcal{L}(||\mathfrak{Z}||^r),\mathcal{M}),\end{cases}\\ &\mathcal{A}_3(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M})\leq\begin{cases}\mathcal{A}_3{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_3{}'(\mathcal{L}(||\mathfrak{Z}||^r),\mathcal{M}),\end{cases}\end{aligned}\right\}\tag{25}$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$, Then there exists a unique tridecic function $\mathcal{T}:\mathcal{X}_M\longrightarrow\mathcal{Y}_M$ such that

$$\left.\begin{array}{l}
\mathcal{A}_1(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M}) \geq \begin{cases}\mathcal{A}_1{}'(\mathcal{L},|11^{13}-1|\mathcal{M}),\\ \mathcal{A}_1{}'(\mathcal{L}||\mathfrak{Z}||^{r},|11^{13}-11^{r}|\mathcal{M}),\\ \end{cases}\\[2em]
\mathcal{A}_2(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M}) \leq \begin{cases}\mathcal{A}_2{}'(\mathcal{L},|11^{13}-1|\mathcal{M}),\\ \mathcal{A}_2{}'(\mathcal{L}||\mathfrak{Z}||^{r},|11^{13}-11^{r}|\mathcal{M}),\\ \end{cases}\\[2em]
\mathcal{A}_3(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M}) \leq \begin{cases}\mathcal{A}_3{}'(\mathcal{L},|11^{13}-1|\mathcal{M}),\\ \mathcal{A}_3{}'(\mathcal{L}||\mathfrak{Z}||^{r},|11^{13}-11^{r}|\mathcal{M}),\\ \end{cases}
\end{array}\right\} \tag{26}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$.

Theorem 3.3 *Assume that* $\mathcal{X}_M$ *is a LS,* $(\mathcal{Z}_m,\mathcal{A}_1{}',\mathcal{A}_2{}',\mathcal{A}_3{}')$ *is a NNS and* $(\mathcal{Y}_M,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$ *an NBS . Let* $\eth:\mathcal{X}_M \longrightarrow \mathcal{Z}_m$ *be a function such that for some* $0<\left(\frac{\mathfrak{B}}{11^{12}}\right)^{F}<1$ *with* $F \in \{1,-1\}$.

$$\left.\begin{array}{l}
\mathcal{A}_1{}'(\eth(11^{nF}\mathfrak{Z}),\mathcal{M}) \geq \mathcal{A}_1{}'(\mathfrak{B}^{nF}\eth(\mathfrak{Z}),\mathcal{M})\\[1em]
\mathcal{A}_2{}'(\eth(11^{nF}\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_2{}'(\mathfrak{B}^{nF}\eth(\mathfrak{Z}),\mathcal{M})\\[1em]
\mathcal{A}_3{}'(\eth(11^{nF}\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_3{}'(\mathfrak{B}^{nF}\eth(\mathfrak{Z}),\mathcal{M})
\end{array}\right\} \tag{27}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$ and

$$\left.\begin{array}{l}
\lim_{n\to\infty}\mathcal{A}_1{}'(\eth(11^{Fn}\mathfrak{Z}),11^{12Fn}\mathcal{M}) = 1\\[1em]
\lim_{n\to\infty}\mathcal{A}_2{}'(\eth(11^{Fn}\mathfrak{Z}),11^{12Fn}\mathcal{M}) = 0\\[1em]
\lim_{n\to\infty}\mathcal{A}_3{}'(\eth(11^{Fn}\mathfrak{Z}),11^{12Fn}\mathcal{M}) = 0
\end{array}\right\} \tag{28}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Let an even function $\mathfrak{F}:\mathcal{X}_M \longrightarrow \mathcal{Y}_M$ satisfying

$$\left.\begin{array}{l}
\mathcal{A}_1(\mathfrak{F}(11\mathfrak{Z})-1,88,30,57,02,60,326\mathfrak{F}(\mathfrak{Z})+1,56,92,14,18,83,605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \geq \mathcal{A}_1{}'(\eth(\mathfrak{Z}),\mathcal{M})\\[1em]
\mathcal{A}_2(\mathfrak{F}(11\mathfrak{Z})-1,88,30,57,02,60,326\mathfrak{F}(\mathfrak{Z})+1,56,92,14,18,83,605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_2{}'(\eth(\mathfrak{Z}),\mathcal{M})\\[1em]
\mathcal{A}_3(\mathfrak{F}(11\mathfrak{Z})-1,88,30,57,02,60,326\mathfrak{F}(\mathfrak{Z})+1,56,92,14,18,83,605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_3{}'(\eth(\mathfrak{Z}),\mathcal{M})
\end{array}\right\} \tag{29}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Then there exists a unique duodecic mapping $\mathcal{D}:\mathcal{X}_M \longrightarrow \mathcal{Y}_M$ satisfying (1) and

$$\left.\begin{aligned}
&\mathcal{A}_1(\mathfrak{F}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M}) \geq \mathcal{A}_1{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{B}|\mathcal{M})\\
&\mathcal{A}_2(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_2{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{B}|\mathcal{M})\\
&\mathcal{A}_3(\mathfrak{F}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_3{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{B}|\mathcal{M})
\end{aligned}\right\} \tag{30}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$.

Proof. For the first case $F = 1$. By applying the evenness condition of $\mathfrak{F}$ in in (29), we arrive

$$\left.\begin{aligned}
&\mathcal{A}_1(\mathfrak{F}(11\mathfrak{Z})-1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(\mathfrak{Z}),\mathcal{M}) \geq \mathcal{A}_1{}'(\eth(\mathfrak{Z}),\mathcal{M})\\
&\mathcal{A}_2(\mathfrak{F}(11\mathfrak{Z})-1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_2{}'(\eth(\mathfrak{Z}),\mathcal{M})\\
&\mathcal{A}_3(\mathfrak{F}(11\mathfrak{Z})-1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_3{}'(\eth(\mathfrak{Z}),\mathcal{M})
\end{aligned}\right\} \tag{31}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. From (31) we have

$$\left.\begin{aligned}
&\mathcal{A}_1(\mathfrak{F}(11\mathfrak{Z})-11^{12}\mathfrak{F}(\mathfrak{Z}),\mathcal{M}) \geq \mathcal{A}_1{}'(\eth(\mathfrak{Z}),\mathcal{M})\\
&\mathcal{A}_2(\mathfrak{F}(11\mathfrak{Z})-11^{12}\mathfrak{F}(\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_2{}'(\eth(\mathfrak{Z}),\mathcal{M})\\
&\mathcal{A}_3(\mathfrak{F}(11\mathfrak{Z})-11^{12}\mathfrak{F}(\mathfrak{Z}),\mathcal{M}) \leq \mathcal{A}_3{}'(\eth(\mathfrak{Z}),\mathcal{M})
\end{aligned}\right\} \tag{32}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$.

Corollary 3.4 *Let $\mathcal{L}, r$ are constants, with $\mathcal{L} > 0$ and $r \neq 12$ and let an even mapping $\mathfrak{F}: \mathcal{X}_M \longrightarrow \mathcal{Y}_M$ satisfies*

$$\left.\begin{aligned}
&\mathcal{A}_1(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \geq \begin{cases}\mathcal{A}_1{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_1{}'(\mathcal{L}(||\mathfrak{Z}||^{r}),\mathcal{M}),\end{cases}\\
&\mathcal{A}_2(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \leq \begin{cases}\mathcal{A}_2{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_2{}'(\mathcal{L}(||\mathfrak{Z}||^{r}),\mathcal{M}),\end{cases}\\
&\mathcal{A}_3(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \leq \begin{cases}\mathcal{A}_3{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_3{}'(\mathcal{L}(||\mathfrak{Z}||^{r}),\mathcal{M}),\end{cases}
\end{aligned}\right\} \tag{33}$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$, Then there exists a unique duodecic function $\mathcal{D}: \mathcal{X}_M \longrightarrow \mathcal{Y}_M$ such that

$$\left.\begin{array}{l}
\mathcal{A}_1(\mathfrak{F}(\mathfrak{z})-\mathcal{D}(\mathfrak{z}),\mathcal{M}) \geq \begin{cases} \mathcal{A}_1{}'(\mathcal{L},|11^{12}-1|\mathcal{M}), \\ \mathcal{A}_1{}'(\mathcal{L}||\mathfrak{z}||^{r},|11^{12}-11^{r}|\mathcal{M}), \\ \end{cases} \\[3em]
\mathcal{A}_2(\mathfrak{F}(\mathfrak{z})-\mathcal{D}(\mathfrak{z}),\mathcal{M}) \leq \begin{cases} \mathcal{A}_2{}'(\mathcal{L},|11^{12}-1|\mathcal{M}), \\ \mathcal{A}_2{}'(\mathcal{L}||\mathfrak{z}||^{r},|11^{12}-11^{r}|\mathcal{M}), \\ \end{cases} \\[3em]
\mathcal{A}_3(\mathfrak{F}(\mathfrak{z})-\mathcal{D}(\mathfrak{z}),\mathcal{M}) \leq \begin{cases} \mathcal{A}_3{}'(\mathcal{L},|11^{12}-1|\mathcal{M}), \\ \mathcal{A}_3{}'(\mathcal{L}||\mathfrak{z}||^{r},|11^{12}-11^{r}|\mathcal{M}), \\ \end{cases}
\end{array}\right\} \qquad (34)$$

for all $\mathfrak{z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$.

Theorem 3.5 *Assume that* $\mathcal{X}_M$ *is a LS,* $(\mathcal{Z}_m, \mathcal{A}_1{}', \mathcal{A}_2{}', \mathcal{A}_3{}')$ *is a NNS and* $(\mathcal{Y}_M, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ *an NBS . Let* $\eth: \mathcal{X}_M \longrightarrow \mathcal{Z}_m$ *be a function such that for some* $0 < \left(\frac{\mathfrak{V}}{11^{12}}\right)^{F} < 1, 0 < \left(\frac{\mathfrak{V}}{11^{13}}\right)^{F} < 1$ *with* $F \in \{1,-1\}$. *Let* $\mathfrak{F}: \mathcal{X}_M \longrightarrow \mathcal{Y}_M$ *be a function satisfying the inequality with conditions (1), (27), (2) and (28). Then .*

$$\left.\begin{array}{l}
\mathcal{A}_1(\mathfrak{F}(11\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{z}),\mathcal{M}) \geq \mathcal{A}_1{}'(\eth(\mathfrak{z}),\mathcal{M}) \\[1em]
\mathcal{A}_2(\mathfrak{F}(11\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{z}),\mathcal{M}) \leq \mathcal{A}_2{}'(\eth(\mathfrak{z}),\mathcal{M}) \\[1em]
\mathcal{A}_3(\mathfrak{F}(11\mathfrak{z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{z}),\mathcal{M}) \leq \mathcal{A}_3{}'(\eth(\mathfrak{z}),\mathcal{M})
\end{array}\right\} \qquad (35)$$

for all $\mathfrak{z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Then there exists a unique tridecic function $\mathcal{T}: \mathcal{X}_M \longrightarrow \mathcal{Y}_M$ and a unique duodecic function $\mathcal{D}: \mathcal{X}_M \longrightarrow \mathcal{Y}_M$ satisfying (1) and

$$\left.\begin{array}{l}
\mathcal{A}_1(g(\mathfrak{z})-\mathcal{T}(\mathfrak{z})-\mathcal{D}(\mathfrak{z}),\mathcal{M}) \\
\geq \mathcal{A}_1{}'(\eth(\mathfrak{z}),|11^{13}-\mathfrak{V}|\mathcal{M}) * \mathcal{A}_1{}'(\eth(-\mathfrak{z}),|11^{13}-\mathfrak{V}|\mathcal{M}) \\
\quad * \mathcal{A}_1{}'(\eth(\mathfrak{z}),|11^{12}-\mathfrak{V}|\mathcal{M}) * \mathcal{A}_1{}'(\eth(-\mathfrak{z}),|11^{12}-\mathfrak{V}|\mathcal{M}) \\[1em]
\mathcal{A}_2(g(\mathfrak{z})-\mathcal{T}(\mathfrak{z})-\mathcal{D}(\mathfrak{z}),\mathcal{M}) \\
\leq \mathcal{A}_2{}'(\eth(\mathfrak{z}),|11^{13}-\mathfrak{V}|\mathcal{M}) \diamond \mathcal{A}_2{}'(\eth(-\mathfrak{z}),|11^{13}-\mathfrak{V}|\mathcal{M}) \\
\quad \diamond \mathcal{A}_2{}'(\eth(\mathfrak{z}),|11^{12}-\mathfrak{V}|\mathcal{M}) \diamond \mathcal{A}_2{}'(\eth(-\mathfrak{z}),|11^{12}-\mathfrak{V}|\mathcal{M}) \\[1em]
\mathcal{A}_3(g(\mathfrak{z})-\mathcal{T}(\mathfrak{z})-\mathcal{D}(\mathfrak{z}),\mathcal{M}) \\
\leq \mathcal{A}_3{}'(\eth(\mathfrak{z}),|11^{13}-\mathfrak{V}|\mathcal{M}) \oslash \mathcal{A}_3{}'(\eth(-\mathfrak{z}),|11^{13}-\mathfrak{V}|\mathcal{M}) \\
\quad \oslash \mathcal{A}_3{}'(\eth(\mathfrak{z}),|11^{12}-\mathfrak{V}|\mathcal{M}) \oslash \mathcal{A}_3{}'(\eth(-\mathfrak{z}),|11^{12}-\mathfrak{V}|\mathcal{M})
\end{array}\right\} \qquad (36)$$

for all $\mathfrak{z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$.

Proof. Let $\mathfrak{F}_o(\mathfrak{z}) = \frac{\mathfrak{F}(\mathfrak{z})-\mathfrak{F}(-\mathfrak{z})}{2}$ for all $\mathfrak{z} \in \mathcal{X}_M$. Then $\mathfrak{F}_o(0) = 0$ and $\mathfrak{F}_o(-\mathfrak{z}) = -\mathfrak{F}_o(\mathfrak{z})$ for all $\mathfrak{z} \in \mathcal{X}_M$. Hence by Theorem 3.1, we have

$$\left.\begin{aligned}&\mathcal{A}_1(\mathfrak{F}_o(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M})\ge \mathcal{A}_1{}'(\eth(\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})*\mathcal{A}_1{}'(\eth(-\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\\&\mathcal{A}_2(\mathfrak{F}_o(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M})\le \mathcal{A}_2{}'(\eth(\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\diamond \mathcal{A}_2{}'(\eth(-\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\\&\mathcal{A}_3(\mathfrak{F}_o(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M})\le \mathcal{A}_3{}'(\eth(\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\oslash \mathcal{A}_3{}'(\eth(-\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\end{aligned}\right\}(37)$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$. Also, let $\mathfrak{F}_e(\mathfrak{Z})=\frac{\mathfrak{F}(\mathfrak{Z})+\mathfrak{F}(-\mathfrak{Z})}{2}$ for all $\mathfrak{Z}\in\mathcal{X}_M$. Then $\mathfrak{F}_e(0)=0$ and $\mathfrak{F}_e(-\mathfrak{Z})=\mathfrak{F}_e(\mathfrak{Z})$ for all $\mathfrak{Z}\in\mathcal{X}_M$. Hence by Theorem 3.3, we have

$$\left.\begin{aligned}&\mathcal{A}_1(\mathfrak{F}_e(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\ge \mathcal{A}_1{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})*\mathcal{A}_1{}'(\eth(-\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\\&\mathcal{A}_2(\mathfrak{F}_e(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\le \mathcal{A}_2{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\diamond \mathcal{A}_2{}'(\eth(-\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\\&\mathcal{A}_3(\mathfrak{F}_e(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\le \mathcal{A}_3{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\oslash \mathcal{A}_1{}'(\eth(-\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\end{aligned}\right\}(38)$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$. Define

$$g(\mathfrak{Z})=\mathfrak{F}_o(\mathfrak{Z})+\mathfrak{F}_e(\mathfrak{Z}) \qquad (39)$$

for all $\mathfrak{Z}\in\mathcal{X}_M$. From (37),(38) and (39), we arrive

$$\begin{aligned}\mathcal{A}_1(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),2\mathcal{M})&=\mathcal{A}_1(\mathfrak{F}_o(\mathfrak{Z})+\mathfrak{F}_e(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),2\mathcal{M})\\&\ge \mathcal{A}_1(\mathfrak{F}_o(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M})*\mathcal{A}_1(\mathfrak{F}_e(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\\&\ge \mathcal{A}_1{}'(\eth(\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})*\mathcal{A}_1{}'(\eth(-\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\\&\quad *\mathcal{A}_1{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})*\mathcal{A}_1{}'(\eth(-\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_2(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),2\mathcal{M})&=\mathcal{A}_2(\mathfrak{F}_o(\mathfrak{Z})+\mathfrak{F}_e(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),2\mathcal{M})\\&\le \mathcal{A}_2(\mathfrak{F}_o(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M})\diamond\mathcal{A}_2(\mathfrak{F}_e(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\\&\le \mathcal{A}_2{}'(\eth(\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\diamond\mathcal{A}_2{}'(\eth(-\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\\&\quad \diamond\mathcal{A}_2{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\diamond\mathcal{A}_2{}'(\eth(-\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_3(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),2\mathcal{M})&=\mathcal{A}_3(\mathfrak{F}_o(\mathfrak{Z})+\mathfrak{F}_e(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),2\mathcal{M})\\&\le \mathcal{A}_2(\mathfrak{F}_o(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z}),\mathcal{M})\oslash\mathcal{A}_3(\mathfrak{F}_e(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\\&\le \mathcal{A}_3{}'(\eth(\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\oslash\mathcal{A}_3{}'(\eth(-\mathfrak{Z}),|11^{13}-\mathfrak{V}|\mathcal{M})\\&\quad \oslash\mathcal{A}_3{}'(\eth(\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\oslash\mathcal{A}_3{}'(\eth(-\mathfrak{Z}),|11^{12}-\mathfrak{V}|\mathcal{M})\end{aligned}$$

for all $\mathfrak{Z}\in\mathcal{X}_M$ and all $\mathcal{M}>0$.

Corollary 3.6 *Let* $\mathcal{L},r$ *are constants with* $\mathcal{L}>0$ *and* $r\neq 13,12$ *and let the function* $\mathfrak{F}:\mathcal{X}_M\longrightarrow\mathcal{Y}_M$ *satisfies*

$$\left.\begin{array}{l}\mathcal{A}_1(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \geq \begin{cases}\mathcal{A}_1{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_1{}'(\mathcal{L}(||\mathfrak{Z}||^r),\mathcal{M}),\end{cases}\\ \mathcal{A}_2(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \leq \begin{cases}\mathcal{A}_2{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_2{}'(\mathcal{L}(||\mathfrak{Z}||^r),\mathcal{M}),\\ \end{cases}\\ \mathcal{A}_3(\mathfrak{F}(11\mathfrak{Z})-1{,}88{,}30{,}57{,}02{,}60{,}326\mathfrak{F}(\mathfrak{Z})+1{,}56{,}92{,}14{,}18{,}83{,}605\mathfrak{F}(-\mathfrak{Z}),\mathcal{M}) \leq \begin{cases}\mathcal{A}_3{}'(\mathcal{L},\mathcal{M}),\\ \mathcal{A}_3{}'(\mathcal{L}(||\mathfrak{Z}||^r),\mathcal{M}),\\ \end{cases}\end{array}\right\} \quad (40)$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$. Then there exists a unique tridecic function $\mathcal{T}: \mathcal{X}_M \longrightarrow \mathcal{Y}_M$ and a unique duodecic function $\mathcal{D}: \mathcal{X}_M \longrightarrow \mathcal{Y}_M$ such that

$$\left.\begin{array}{l}\mathcal{A}_1(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\\ \geq \begin{cases}\mathcal{A}_1{}'(\mathcal{L},|11^{13}-1|\mathcal{M}) * \mathcal{A}_1{}'(\mathcal{L},|11^{12}-1|\mathcal{M}),\\ \mathcal{A}_1{}'(\mathcal{L}||\mathfrak{Z}||^r,|11^{13}-11^r|\mathcal{M}) * \mathcal{A}_1{}'(\mathcal{L}||\mathfrak{Z}||^r,|11^{12}-11^r|\mathcal{M}),\\ \end{cases}\\ \\ \mathcal{A}_2(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\\ \leq \begin{cases}\mathcal{A}_2{}'(\mathcal{L},|11^{13}-1|\mathcal{M}) \diamond \mathcal{A}_2{}'(\mathcal{L},|11^{12}-1|\mathcal{M}),\\ \mathcal{A}_2{}'(\mathcal{L}||\mathfrak{Z}||^r,|11^{13}-11^r|\mathcal{M}) \diamond \mathcal{A}_2{}'(\mathcal{L}||\mathfrak{Z}||^r,|11^{12}-11^r|\mathcal{M}),\\ \end{cases}\\ \\ \mathcal{A}_3(\mathfrak{F}(\mathfrak{Z})-\mathcal{T}(\mathfrak{Z})-\mathcal{D}(\mathfrak{Z}),\mathcal{M})\\ \leq \begin{cases}\mathcal{A}_3{}'(\mathcal{L},|11^{13}-1|\mathcal{M}) \oslash \mathcal{A}_3{}'(\mathcal{L},|11^{12}-1|\mathcal{M}),\\ \mathcal{A}_3{}'(\mathcal{L}||\mathfrak{Z}||^r,|11^{13}-11^r|\mathcal{M}) \oslash \mathcal{A}_3{}'(\mathcal{L}||\mathfrak{Z}||^r,|11^{12}-11^r|\mathcal{M}),\\ \end{cases}\end{array}\right\} \quad (41)$$

for all $\mathfrak{Z} \in \mathcal{X}_M$ and all $\mathcal{M} > 0$.

4 Conclusion

The numerical stability analysis conducted in this study underscores the viability and robustness of mixed-type duodecic and tridecic functional equations in neutrosophic normed spaces. By employing the Hyers classical direct method, we provided a thorough examination of stability characteristics, ensuring that these functional equations can be confidently applied in various scientific and engineering disciplines. This methodology not only validated the stability but also enriched the theoretical landscape, paving the way for future research in this domain.

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