

# Effect of Edge Deletion on the Weak Roman Domination Number of a Graph

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## Abstract:

Consider a graph  $\Gamma$  with vertex set  $V$  and edge set  $E$ . For a function  $g$  defined on the vertex set having values in the set  $\{0, 1, 2\}$ , the *weight of a vertex*  $x$  is  $g(x)$ . The *weight of a subset*  $X$  of  $V$  is denoted by  $g(X)$  and is defined as the sum of the weights of all the vertices in  $X$ . An undefended vertex under  $g$  is a vertex  $a$  in  $\Gamma$  with a neighbourhood with weight 0.

A weak Roman Dominating Function  $g$  (*WRDF*) is a function such that for any vertex  $x$  having weight 0, there is a vertex  $y$  in the open neighbourhood of  $x$  having positive weight such that, under the function  $h$  defined on  $V$  having values in the set  $\{0, 1, 2\}$  defined by  $h(x) = 1$ ,  $h(y) = g(y) - 1$ ,  $h(z) = g(z)$ , if  $z \in V - \{x, y\}$ , there are no undefended vertices in  $\Gamma$ . The number  $\gamma_r(\Gamma)$ , the minimum of the weights of all the *WRDFs* defined on  $\Gamma$  is called the *weak Roman Domination Number of*  $\Gamma$ . The *WRDF* whose weight is  $\gamma_r(\Gamma)$  is called a  $\gamma_r$ -function and  $\gamma_r(\Gamma)$  is called the  $\gamma_r$ -value of  $\Gamma$ .

With respect to a graph  $\Gamma$ , we say that an edge  $x \in E^-$  if and only if its removal will reduce the  $\gamma_r$ -value of  $\Gamma$ . Similarly, we say that an edge  $x \in E^+$  if and only if its removal will increase the  $\gamma_r$ -value of  $\Gamma$  and an edge  $x \in E^0$  if and only if its removal will leave unaltered the  $\gamma_r$ -value of  $\Gamma$ .

In this paper, we categorize edges of a graph as belonging to  $E^-$ ,  $E^+$  or  $E^0$ .

**Keywords:** Weak Roman dominating number, Changing and Unchanging edges, edge deletion.

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## 1. Introduction

For a graph  $\Gamma$  having vertex set  $V$  and edge set  $E$ , the *open neighbourhood of the vertex*  $x$  (denoted by  $N(x)$ ) is the set of all those vertices that are adjacent to  $x$  and the closed neighbourhood of the vertex  $x$  (denoted by  $N[x]$ ) is the union of the open neighbourhood of  $x$  and  $\{x\}$ .

For a function  $g$  defined on the vertex set  $V$  and having values in the set  $\{0, 1, 2\}$ , the *weight of a vertex*  $x$  is  $g(x)$ . The *weight of a subset*  $X$  of  $V$  is denoted by  $g(X)$  and is defined as the sum of the weights of all the vertices in  $X$ .

For  $l \in \{0, 1, 2\}$ , we say that a vertex  $x \in V_l$  if and only if the weight of  $x$  under  $g$  is  $l$ .

In view of the one-to-one correspondence between the functions  $g$  defined on  $V$  and having values in the set  $\{0, 1, 2\}$  and the sets  $(\{V_l\}: l = 0, 1, 2)$ , we can write  $g = (V_0, V_1, V_2)$ .

The function  $g$  is called a Roman Dominating Function (*RDF*) [2], any vertex  $a$  of  $\Gamma$  having weight 0 has at least one vertex with weight 2 in its open neighbourhood. The number  $\gamma_R(\Gamma)$  is the minimum of the weights of all *RDFs* defined on  $\Gamma$ . The *RDF*  $g$  with weight  $\gamma_R(\Gamma)$  is called a  $\gamma_R$  – function on  $\Gamma$  and the number  $\gamma_R(\Gamma)$  itself is called the  $\gamma_R$  – value of  $\Gamma$ . Many researchers have investigated the Roman Dominating Number of graphs [1, 4, 5, 7, 8, 9].

A weak Roman Dominating Function  $g$  (*WRDF*) [3] is a function such that for any vertex  $x$  having positive weight, there is a vertex  $y$  in the open neighbourhood of  $x$  having positive weight and under the function  $h$  defined on  $V$  having values in the set  $\{0,1,2\}$  defined by  $h(x) = 1$ ,  $h(y) = g(y) - 1$ ,  $h(z) = g(z)$ ,  $z \in V - \{x, y\}$ , there are no undefended vertices in  $\Gamma$ . The number  $\gamma_r(\Gamma)$ , the minimum of the weights of all the *WRDFs* defined on  $\Gamma$  is called the *weak Roman Domination Number* of  $\Gamma$ . The *WRDF* whose weight is  $\gamma_r(\Gamma)$  is called a  $\gamma_r$  – function defined on  $\Gamma$  and  $\gamma_r(\Gamma)$  is called the  $\gamma_r$  – value of  $\Gamma$ . *WRDFs* are marginally compromising but more accommodative versions of *RDFs*. Researchers have extensively worked on the parameter weak Roman Domination Number of graphs [6, 10, 11, 12].

With respect to a graph  $\Gamma$ , we say that an edge  $x \in E^-$  if and only if its removal will reduce the  $\gamma_r$  – value of  $\Gamma$ . Similarly, we say that an edge  $x \in E^+$  if and only if its removal will increase the  $\gamma_r$  – value of  $\Gamma$  and an edge  $x \in E^0$  if and only if its removal will leave unaltered the  $\gamma_r$  – value of  $\Gamma$ . The behaviour of an edge to belong to  $E^-, E^+$  or  $E^0$  is called the changing and unchanging behaviour of the edge. This paper is dedicated to study the changing and unchanging behaviour of an edge with respect to a *WRDF*.

**Definition 1.1.** [12] Under a *WRDF*, a vertex  $a \in V$  with weight 0 is said to be dependent on another vertex  $b \in V$  of positive weight, if the increase in the weight of  $a$  by 1 and the simultaneous decrease of the weight of  $b$  by 1, will not create an undefended vertex in  $V$ . We then write,  $a \in D_G(b)$ .

**Notation 1.1.** [12]  $H_x = (V_1 \cup V_2) - \{x\}$ ,  $\bar{N}(u) = (N(u) \cap V_0) - \cup_{x \in H} D_\Gamma(x)$ .

**Definition 1.2.** [12] Let  $u \in V$ . A  $(u: 1)$  – set in  $\Gamma$  is a set  $S \subseteq N(u)$  such that for any  $x \in S$ ,  $S - N(H_u) \subseteq N[x]$ .

Definition 1.3. [12] Let  $u \in V$ . A  $(u: 1)$  – set in  $\Gamma$  is a set  $S \subseteq N(u)$  that is not a  $(u: 1)$  – set.

## 2. Categorizing an edge for membership in $E^-, E^+$ and $E^0$

We start by stating two trivial observations.

**Observation 2.1.**  $E^- = \emptyset$ .

**Observation 2.2.** If  $xy$  is an edge of  $\Gamma$  and if  $\{x, y\} \subseteq V_0$  or if  $\{x, y\} \subseteq V_1 \cup V_2$ , then  $xy \in E^0$ .

**Theorem 2.1.** Let  $xy$  be an edge of the graph  $\Gamma$  such that  $x$  has positive weight and  $y$  has weight 0 under a *WRDF*  $g$ . Then, the following conditions imply and are implied by the membership of  $x$  in  $E^0$ .

1.  $y \in \bar{N}(x)$  and
  - a.  $x \in V_2$  and  $\bar{N}(x) - \{y\}$  is a  $(x: 1)$  – set in  $\Gamma$  or

- b.  $x \in V_1$ ,  $x \in D_\Gamma(\omega)$ , for some  $\omega \in V_1 \cup V_2$  and for no vertex  $z \in N(x) \cap V_0$  having  $N(z) \cap V_2 = \emptyset$  and for every  $t \in (N(z) - \{x\}) \cap V_1$ , we have  $\bar{N}(t) \cup \{y\}$  is a  $(t:2)$  – set in  $\Gamma$ .
2.  $y \notin \bar{N}(x)$  and
- c.  $y \in D_\Gamma(r)$  for some  $r \in V_2$  or if  $N(y) \cap V_2 = \emptyset$ ,  $y \in D_\Gamma(r)$ , for some  $r \in V_1$  where  $|N(y) \cap V_1| > 2$  if  $|\bar{N}(r) \cup \{y\}| > 1$  and  $\bar{N}(r) \cup \{y\}$  does not induce a clique and  $\bar{N}(x) - \{y\}$  is a  $(x:i)$  – set in  $\Gamma$  where  $g(x) = i$  or
- d.  $r \in V_1$ ,  $\bar{N}(r) \cup \{y\}$  is a  $(r:2)$  – set and we can find  $m \in V_1 \cup V_2$  such that  $x \in D_\Gamma(m)$ .

Proof. Assume that  $y \in \bar{N}(x)$ ,  $x \in V_2$ . Let  $\bar{N}(x) - \{y\}$  be a  $(x:1)$  – set in  $\Gamma$ . Consider the function  $g'$  defined on  $V$  and having values in  $\{0,1,2\}$  by  $g'(y) = 1$ ,  $g'(x) = 1$  and  $g'(z) = g(z)$  for all  $z \in V - \{x, y\}$ . The function  $g'$  is a *WRDF* on  $\Gamma - e$  having weight  $\gamma_r(\Gamma)$ . Since  $e \notin E^-$ ,  $g'$  is a  $\gamma_r$  – function defined on  $\Gamma - e$ . Therefore,  $\gamma_r(\Gamma - e) = \gamma_r(\Gamma)$ . So,  $e \in E^0$ .

Assume now that,  $x \in V_1$ . If  $x \in D_G(w)$  for some  $w \in V_1 \cup V_2$  then there exists no  $z \in V_0 \cap N(x)$  with  $N(z) \cap V_2 = \emptyset$  and for every  $t \in N(z) \cap V_1$ , if we have  $\bar{N}(t) \cup \{y\}$  is a  $(t:2)$  – set in  $\Gamma$ , then the function  $g'$  defined on  $V$  having values in  $\{0,1,2\}$  defined by  $g'(y) = 1$ ,  $g'(x) = 0$ ,  $g'(s) = g(s)$  for all  $s \in V - \{x, y\}$  is a *WRDF* defined on  $\Gamma - e$  having weight  $\gamma_r(\Gamma)$ . Since  $e \notin E^-$  we have  $g'$  is a  $\gamma_r$  – function defined on  $\Gamma - e$ . Therefore,  $\gamma_r(\Gamma - e) = \gamma_r(\Gamma)$ . So,  $e \in E^0$ .

Assume now that  $y \notin \bar{N}(x)$ . Assume that there is some  $r \in V_2$ . Then,  $e \in E^0$ . Let  $N(y) \cap V_2 = \emptyset$ ,  $y \in \bar{N}(r)$  for some  $r \in V_1$  where  $|N(y) \cap V_1| > 2$  if  $|\bar{N}(r) \cup \{y\}| > 1$ ,  $\bar{N}(r) \cup \{y\}$  does not induce a clique in such a manner that  $\bar{N}(x) - \{y\}$  happens to be a  $(u:i)$  – set, where  $g(x) = i$ ,  $i = 1, 2$ . In this case,  $g$  is a *WRDF* defined on  $\Gamma - e$ . Again since  $e \notin E^-$ , we have that  $g'$  is a  $\gamma_r$  – function defined on  $\Gamma - e$ . Therefore,  $\gamma_r(\Gamma - e) = \gamma_r(\Gamma)$ . Consequently,  $e \in E^0$ .

If  $r \in V_1$ , and  $\bar{N}(r) \cup \{y\}$  is a  $(r:2)$  – set in  $\Gamma$  and if we can find  $m \in V_1 \cup V_2$  in such a manner that  $x \in D_G(m)$ , then the function  $g'$  defined on  $V$  having values in  $\{0,1,2\}$  so defined that  $g'(r) = 2$ ,  $g'(x) = 0$ ,  $g'(z) = g(z)$  when  $z \in V - \{r, x\}$  will be a *WRDF* defined on  $\Gamma - e$  having  $\gamma_r(\Gamma)$  as its weight. Arguing as before, we have  $e \in E^0$ .

Conversely, let  $e \in E^0$ . If  $y \in \bar{N}(x)$ , then  $e \in E^+$  if none of the conditions a or b is satisfied. Similarly, if  $y \in \bar{N}(x)$ , then  $e \in E^+$  if none of the conditions c or d is satisfied.

This proves the theorem completely. □

Theorem 2.2. For any graph  $\Gamma$  having an edge  $xy (= e)$ ,  $e \in E^+$  if and only if  $e \notin E^0$ .

Proof. The result is a consequence of the facts,  $E = E^- \cup E^0 \cup E^+$ ,  $E^- = \emptyset$  and  $E^0 \cap E^+ = \emptyset$ .

□

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