Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol. 32 No. 8s (2025)

Memory Effect of Deformable Mean and Deformable Standard Deviation

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Article History:
Received: 30-10-2024
Revised: 04-12-2024
Accepted: 27-12-2024

Abstract: Memory effects can be easily understood with the help of fractional derivatives. In this paper with the help of Deformable fractional derivative we try to understand the role of deformability on memory effects of a normal distribution using moment generating function. First, we develop a formula for Deformable mean and Deformable standard deviation with the help of Deformable fractional derivative. Secondly, we calculate the value of Deformable mean and Deformable standard deviation for varying fractional order. Lastly, we draw plot of normal distribution for calculated value of Deformable mean and Deformable deviation with varying fractional order and give conclusion.

Keywords: Moment generating function, Fractional order, Deformable derivative, Deformable mean and Deformable standard deviation.

1. Introduction

On the basis of Deformable fractional derivative, we will define Deformable mean and Deformable standard deviation.

Deformable Fractional Derivative: Let $f(x)$ be a real valued function defined on interval (a, b) for a given number $\alpha,\ 0 \le \alpha \le 1$

$$\lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)\, f(x+\epsilon\alpha) - f(x)}{\epsilon} \tag{1}$$

Where $\alpha + \beta = 1$

If this limit exists, we denote it by $D^{\alpha}[f(x)]$ [1].

$$D^{\alpha}[f(x)] = \alpha\, Df(x) + \beta f(x) \tag{2}$$

or

$$D^{\alpha}[f(x)] = \alpha\, Df(x) + (1-\alpha)\, f(x)$$

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Moment Generating Function: Let X be random variable (r.v.) such that for some $\epsilon > 0$, the expected value of e^{tx} exists for $-\epsilon < t < \epsilon$. Then moment generating function (m.g.f.) of X is defined to be the function $M_X(t) = E[e^{tx}]$, for $-\epsilon < t < \epsilon$ [3],[5]-[6].

$$M_X(t) = E[e^{tx}] = \int e^{tx} \ f_X(x)dx \tag{3}$$

Where X is a continuous r.v. and $f_X(x)$ is a probability density function s.t. [3], [5].

(i) $f_X(x) \geq 0$

(ii) $\int f_X(x)dx = 1$

or

$$M_X(t) = E[e^{tx}] = \sum_{x \in D} e^{tx} \, p_X(x)$$

Where X is a discrete r.v. with space Ω and $p_X(x)$ is a probability mass function s.t. [3], [5].

(i) $0 \leq p_X(x) \leq 1 \,, x \in \Omega$

(ii) $\sum_{x \in \Omega} p_X(x) = 1$

2. Deformable Mean

If $M_X(t)$ is a well-defined moment generating function then we can define Deformable mean (μ^α) with the help of moment generating function as

$$\mu^\alpha = {M_X}^\alpha(0)$$

Deformable derivative of equation (3) w.r.to t

$$M_X{}^\alpha(t) = \int D^\alpha[e^{tx}] \ f_X(x)dx \tag{4}$$

$$M_X{}^\alpha(t) = \int \{\alpha \, x \, e^{tx} + (1-\alpha) \, e^{tx}\} \, f_X(x)dx$$

$$M_X{}^\alpha(t) = \alpha \int x \, e^{tx} \, f_X(x) \, dx + (1-\alpha) \int e^{tx} \ f_X(x)dx \tag{5}$$

$$M_X{}^\alpha(0) = \alpha \int x \, e^0 \, f_X(x) \, dx + (1-\alpha) \int e^0 \ f_X(x)dx$$

$$M_X{}^\alpha(0) = \alpha \int x \ f_X(x) \, dx + (1-\alpha) \int f_X(x) \ dx$$

$$M_X{}^\alpha(0) = \alpha \, E(X) + (1-\alpha) \tag{6}$$

or

$$\mu^\alpha = \alpha \, \mu + (1-\alpha)$$

For μ^α (Deformable mean) following cases arises:

(i) $\alpha = 0$, and $\mu \in R$ then μ^α=1 (7)

(ii) $\alpha = \frac{1}{2}$, and $\mu \in R$ then $\mu^{\alpha} = \frac{1}{2}(\mu + 1)$ (8)

(iii) $\alpha = 1$, and $\mu \in R$ then $\mu^{\alpha} = \mu$ (9)

(iv) $0 < \alpha < 1$, and $\mu \in R$ then $\mu^{\alpha} = \alpha\mu + (1 - \alpha)$ (10)

3. Deformable Standard Deviation

If $M_X(t)$ is a well-defined moment generating function then we denote Deformable variance by $({\sigma_X}^{\alpha})^2$ and defined as

$$({\sigma_X}^{\alpha})^2 = {M_X}^{\alpha.\alpha}(0) - ({M_X}^{\alpha}(0))^2 \tag{11}$$

Deformable derivative of equation (5) w.r.to t

$${M_X}^{\alpha.\alpha}(t) = \alpha \int x\, D^{\alpha}\{ e^{tx}\} f_X(x)\, dx + (1-\alpha) \int D^{\alpha}\{e^{tx}\}\ f_X(x)dx$$

$${M_X}^{\alpha.\alpha}(t) = \alpha \int x\ \{\alpha\, x\, e^{tx} + (1-\alpha)\, e^{tx}\} f_X(x)\, dx + (1-\alpha) \int \{\alpha\, x\, e^{tx} + (1-\alpha)\, e^{tx}\}\ f_X(x)dx$$

$${M_X}^{\alpha.\alpha}(t) = \alpha^2 \int x^2\, e^{tx}\, f_X(x)\, dx + 2\,\alpha\,(1-\alpha) \int x\, e^{tx}\ \ f_X(x)dx + (1-\alpha)^2 \int e^{tx}\, f_X(x)dx$$

$${M_X}^{\alpha.\alpha}(0) = \alpha^2 \int x^2\, e^{0}\, f_X(x)\, dx + 2\,\alpha\,(1-\alpha) \int x\, e^{0}\ \ f_X(x)dx + (1-\alpha)^2 \int e^{0}\, f_X(x)dx$$

$${M_X}^{\alpha.\alpha}(0) = \alpha^2\, E(X^2) + 2\,\alpha\,(1-\alpha)E(X) + (1-\alpha)^2 \tag{12}$$

$$({\sigma_X}^{\alpha})^2 = \alpha^2\, E(X^2) + 2\,\alpha\,(1-\alpha)E(X) + (1-\alpha)^2 - \alpha^2\left(E(X)\right)^2 - (1-\alpha)^2 - 2\,\alpha\,(1-\alpha)\,E(X)$$

$$({\sigma_X}^{\alpha})^2 = \alpha^2\, {\sigma_X}^2$$

$${\sigma_X}^{\alpha} = \alpha\, \sigma_X \tag{13}$$

We get a linear relation between Deformable standard deviation and an ordinary standard deviation. Deformable standard deviation is α-times the ordinary standard deviation. For $\alpha = 0$, Deformable standard deviation is zero. As fractional order increases Deformable standard deviation also increase.

$${\sigma_X}^{\alpha}\ \le\ \sigma_X \tag{14}$$

Example 1: For a continuous random variable X, Probability density function of normal distribution is defined as

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\,\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_X}\right)^2} \tag{15}$$

Where $-\infty < x < \infty$ and, $-\infty < \mu < \infty$, $\sigma_x > 0$ [3], [5].

On the basis of equations (6) and (13) calculated value of Deformable mean and Deformable standard deviation given in the following table, for mean $\mu = 50$ and standard deviation $\sigma_X = 25.3$ and fractional order $\alpha = 0.3, 0.5, 0.7, 0.9, 1.$

Fractional Order (α)	**Deformable Mean (μ^α)**	**Deformable S.D. (${\sigma_X}^\alpha$)**
0.3	15.7	7.59
0.5	25.5	12.65
0.7	35.3	17.71
0.9	45.5	22.77
1	50	25.3

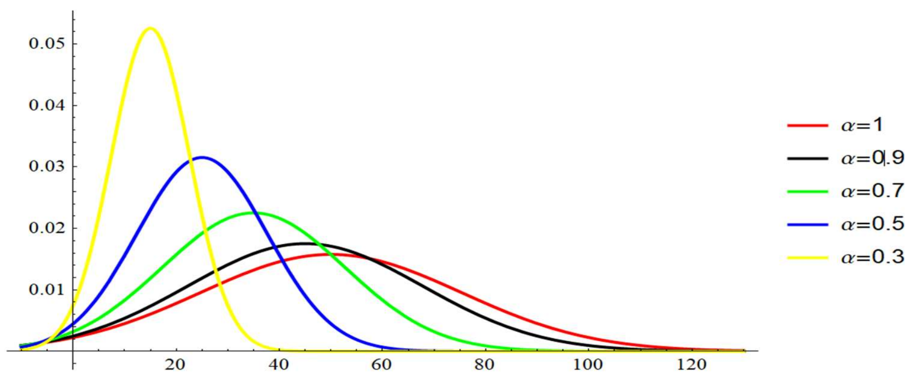

Figure: Normal distribution plot of fractional order $\alpha = 0.3, 0.5, 0.7, 0.9, 1.$

4. Conclusion

Deformable and ordinary mean have a linear relationship and Deformable standard deviation is less than and equal to ordinary standard deviation. Therefore, Deformable standard deviation is more reliable as compare to ordinary standard deviation. Deformable mean along with Deformable standard deviation gives more insight information of a normal distribution. We can see that Deformable derivative base mean and standard deviation affects the sharpness and flatness of a given normal distribution, which shows the memory of a normal distribution over varying fractional order. Therefore, we can say that moment generating base Deformable means and Deformable standard deviation is extension or ordinary mean and standard deviation. Further we can also extend this result to other measure of dispersion.

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