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Extension of primary and semiprimary bi-ideals of semirings

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Abstract

The 1PBID, 2PBID, and 3PBID of semirings are introduced. Additionally, we communicate with a number of the various SPBIs' attributes. 2PBID and 3PBID are generalizations of 1PBID and 2PBID, respectively, which we explain. We go over the m1, m2 and m3-systems and BI generator. It is possible to generalize the m1-system to the m2-system and the m2-system to the m3-system.

Keywords: 1PBID; 2PBID; 3PBID; m1-system, m2-system and m3-system.

1 Introduction

Numerous studies have looked at different kinds of ideals in mathematical structures, such as rings and semirings,[3,4] respectively. Associative rings made up the notion of ideals that Dedekind introduced into the theory of algebraic numbers. In this approach the notion was expanded to include algebraic numbers. Moreover, it is a particular case of Lajos (m, n)-ideal. In order to analyze regular and intra-regular semigroups, Lajos used generalized BIDs and quasi-ideals. To reference[5] while discussing different types of semigroups is a BID. The associative rings are somewhat arbitrary, however they are described in terms of BIDs. It would be almost perfect to expand LIs and RIs, which are specific types of BIDs. Semigroups and rings now referred to as quasi ideals were introduced by Otto Steinfeld. Quote[3] states that semirings provide a range of methods for elucidating prime ideals. Commutative ring theory has been extensively influenced by prime ideal theory. It has been less frequently used for non-commutative rings than for commutative rings. Palanikumar et al.[6] investigated different prime partial BIs in non-commutative partial rings.

Walt[9] studied the prime and semiprimate BIs of associative rings with unity. Associative rings without unity were extended to prime and semiprime BIs by Roux.[10] Some descriptions of BIs in basic semirings were given by Flaska et al.[11] Atani[12] also provided some results for the ideal theory of commutative semirings with non-zero identities. McCoy gives some details on prime ideals in general rings.[4] Information on the PID for rings and semirings was given in.[3,13,14] Van der Walt coined the words prime BID and semiprime BID.[9] The subsets X_1 and X_2 of $\S$ and the product $X_1 \cdot X_2$ may be understood as follows: the subring of $\S$ is produced by the set of all products $x_1 \cdot x_2$, where $x_1 \in X_1, x_2 \in X_2$. A BID $\eth_1$ of a ring $\S$ is defined as a subring $\eth_1$ of $\S$ that satisfies $\eth_1 \S \eth_1 \subseteq \eth_1$. For ideals $\eth$ and $\eth_1$ of $\S$, then $\eth \subseteq \aleph$ or $\eth_1 \subseteq \aleph$ means that an ID $\aleph$ of a ring $\S$ is PID if and only if whenever $\eth\eth_1 \subseteq \aleph$.[4] Each of the five components that make up this document is arranged differently. In section 2, we address the various kinds of main BIDs and their expansions. Section 3 offers a discussion of the semiprimary BIDs.

List of Abbreviations

RID	right ideal
LID	left ideal
ID	ideal
primary BID	primary bi-ideal
primary ID	primary ideal
TID	two sided ideal
semi primary BID	semi primary bi-ideal
semi primary ID	semi primary ideal

2 Characterization of PBIDs

Definition 2.1. A BID $\aleph$ of $\S$ is said to be
(1) 1PBID if $\Game_1\Game_2 \subseteq \aleph$ implies $\Game_1 \subseteq \aleph$ or $\Game_2 \subseteq \sqrt{\aleph}$ for any BIDs $\Game_1$ and $\Game_2$ of $\S$.
(ii) 2PBID if $\propto \S\wp \subseteq \aleph$ implies $\propto \in \aleph$ or $\wp \in \sqrt{\aleph}$.
(iii) 3PBID if $\pounds_1\pounds_2 \subseteq \aleph$ implies $\pounds_1 \subseteq \aleph$ or $\pounds_2 \subseteq \sqrt{\aleph}$ for any IDs $\pounds_1$ and $\pounds_2$ of $\S$.

Theorem 2.2. *Every 1PBID is a 2PBID.*

Proof. Let $\aleph$ be an 1PBID of $\S$. Let $\propto, \wp \in \S$ and $\propto \S\wp \subseteq \aleph$. Now, $(\propto \S) \cdot (\S\wp) \subseteq \propto \S\wp \subseteq \aleph$, since $\propto \S$ and $\S\wp$ are BIDs. Hence $\propto \S \subseteq \aleph$ or $\S\wp \subseteq \sqrt{\aleph}$. Suppose that $\propto \S \subseteq \aleph$. Consider $<\propto>_b \cdot <\propto>_b \subseteq \propto \S \subseteq \aleph$. Then $\propto \in \aleph$. Similarly if $\S\wp \subseteq \sqrt{\aleph}$ then $\wp \in \sqrt{\aleph}$. Thus $\aleph$ is a 2PBID of $\S$. $\square$

Converse is does not hold.

Example 2.3. Consider the semiring $\S = \mathfrak{D}_2(\mathbb{Z}_2)$ and $\aleph = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is a 2PBID, but not 1PBID.

Theorem 2.4. *Every 2PBID is a 3PBID.*

Proof. Let $\aleph$ be an 2PBID of $\S$. For the IDs $\pounds_1$ and $\pounds_2$ of $\S$ such that $\pounds_1 \cdot \pounds_2 \subseteq \aleph$. If $\pounds_1 \not\subseteq \aleph$, let $\propto \in \pounds_1 \setminus \aleph$. For any $\wp \in \pounds_2$, $\propto \S\wp \subseteq <\propto> \cdot < \wp > \subseteq \pounds_1 \cdot \pounds_2 \subseteq \aleph$. Hence $\wp \in \sqrt{\aleph}$. Then $\pounds_2 \subseteq \sqrt{\aleph}$. Thus $\aleph$ is a 3PBID of $\S$. $\square$

Definition 2.5. A subset $\mathfrak{D}$ of $\S$ is said to be
(i) m_{p_1}-sys if for any $\propto, \wp \in \mathfrak{D}$, $\exists \propto_1 \in <\propto>_b$ and $\wp_1 < \wp >_b$ such that $\propto_1 \wp_1 \in \mathfrak{D}$.
(ii) m_{p_2}-sys if for any $\propto, \wp \in \mathfrak{D}$, $\exists \propto_1 \in <\propto>_r$ and $\wp_1 \in < \wp >_l$ such that $\propto_1 \wp_1 \in \mathfrak{D}$.
(iii) m_{p_3}-sys if for any $\propto, \wp \in \mathfrak{D}$, $\exists \propto_1 \in <\propto>$ and $\wp_1 \in < \wp >$ such that $\propto_1 \wp_1 \in \mathfrak{D}$.

Theorem 2.6. *If $\aleph$ is a BID of $\S$, then $\aleph$ is a 1PBID (2PBID,3PBID) if and only if $\S \setminus \aleph$ is an m_{p_1}-sys (m_{p_2}-sys, m_{p_3}-sys) of $\S$.*

Proof. Let $\varsigma, \epsilon \in \S \setminus \aleph$. Hence $\varsigma, \epsilon \in \S$ but $\varsigma, \epsilon \notin \aleph$. So $< \varsigma >_b \cdot < \epsilon >_b \not\subseteq \aleph$. There exists $\varsigma^{'} \in < \varsigma >_b$ and $\epsilon^{'} \in < \epsilon >_b$ such that $\varsigma^{'} \cdot \epsilon^{'} \notin \aleph$. Hence $\varsigma^{'} \cdot \epsilon^{'} \in \S \setminus \aleph$. So we have proved that for $\varsigma, \epsilon \in \S \setminus \aleph \ \exists \varsigma^{'} \in < \varsigma >_b$ and $\epsilon^{'} \in < \epsilon >_b$ such that $\varsigma^{'} \cdot \epsilon^{'} \in \S \setminus \aleph$. So $\S \setminus \aleph$ is an m_{p_1}-sys.

Conversely, let $\S \setminus \aleph$ is an m_{p_1}-sys. Let us shows that $\Game_1 \subseteq \aleph$ or $\Game_2 \subseteq \sqrt{\aleph}$. Let us arrive at a contradiction. If $\Game_1 \not\subseteq \aleph$ and $\Game_2 \not\subseteq \sqrt{\aleph}$, let $\wp_1 \in \Game_1 \setminus \aleph$ and let $\wp_2 \in \Game_2 \setminus \sqrt{\aleph}$. Since $\wp_2 \notin \sqrt{\aleph}$, so $\exists$ an m_{p_1}-sys $\S \setminus \aleph$ in $\S$ such that $\wp_2 \in \S \setminus \aleph$ and $(\S \setminus \aleph) \cap \aleph = \phi$. Thus $\wp_1, \wp_2 \in \S \setminus \aleph$ implies $< \wp_1 >_b \cdot < \wp_2 >_b \not\subseteq \aleph$. Thus $\Game_1 \subseteq \aleph$ or $\Game_2 \subseteq \sqrt{\aleph}$. Hence $\aleph$ is a 1PBID of $\S$. $\square$

Corollary 2.7. *Every m_{p_1}-sys is an m_{p_2}-sys.*

Proof. Given that $\mathfrak{D}$ be an m_{p_1}-sys of $\S$. For any $\propto, \wp \in \mathfrak{D}$, $\exists \propto_1 \in <\propto>_b$ and $\wp_1 \in < \wp >_b$ such that $\propto_1 \cdot \wp_1 \in \mathfrak{D}$. Let us shows that $\mathfrak{D}$ is an m_{p_2}-sys. For $\propto, \wp \in \mathfrak{D}$, $\exists \propto_1 \in <\propto>_r$ and $\wp_1 \in < \wp >_l$. Since right and LIDs are BIDs also, we have $\propto_1 \cdot \wp_1 \in \mathfrak{D}$. Hence $\mathfrak{D}$ is an m_{p_2}-sys of $\S$. ☐

Corollary 2.8. *Every m_{p_2}-sys is an m_{p_3}-sys.*

Proof. Given that $\mathfrak{D}$ be an m_{p_2}-sys of $\S$. For any $\propto, \wp \in \mathfrak{D}$, $\exists \propto_1 \in <\propto>_r$ and $\wp_1 \in < \wp >_l$ such that $\propto_1 \wp_1 \in \mathfrak{D}$. Let us shows that $\mathfrak{D}$ is an m_{p_3}-sys. For $\propto, \wp \in \mathfrak{D}$, $\exists \propto_1 \in <\propto>$ and $\wp_1 \in < \wp >$. Since IDs are RID and LIDs also, we have $\propto_1 \wp_1 \in \mathfrak{D}$. Hence $\mathfrak{D}$ is an m_{p_3}-sys of $\S$. ☐

Remark 2.9. Let $\sqrt{\eth}$ be any BID of a ring $\S$. Then $\sqrt{ß^{\eth}} = \{\varsigma \in \sqrt{\eth} \mid \S\varsigma \subseteq \sqrt{\eth}\}$ and $\sqrt{\Finv^{\eth}} = \{\epsilon \in \sqrt{ß^{\eth}} \mid \epsilon\S \subseteq \sqrt{ß^{\eth}}\}$.

Corollary 2.10. *Let $\sqrt{\eth}$ be a BID of $\S$. Then $\sqrt{ß^{\eth}}$ is a LID of $\S$ such that $\sqrt{ß^{\eth}} \subseteq \sqrt{\eth}$.*

Proof. Let $\varsigma, \epsilon \in \sqrt{ß^{\eth}}$. Then $\varsigma, \epsilon \in \sqrt{\eth}$ and $\S\varsigma \subseteq \sqrt{\eth}$ and $\S\epsilon \subseteq \sqrt{\eth}$. Since $\sqrt{\eth}$ is a BID of $\S$, $\varsigma + \epsilon \in \sqrt{\eth}$ and $\varsigma\epsilon \in \sqrt{\eth}$. Now, $\S(\varsigma + \epsilon) \subseteq \S\varsigma + \S\epsilon \subseteq \sqrt{\eth}$. Thus, $\varsigma + \epsilon \in \sqrt{ß^{\eth}}$. Now, $\S(\varsigma\epsilon) \subseteq (\S\varsigma)(\S\epsilon) \subseteq \sqrt{\eth}$. Thus, $\varsigma\epsilon \in \sqrt{ß^{\eth}}$. Hence $\sqrt{ß^{\eth}}$ is a SSR of $\S$. Let $\varsigma \in \sqrt{ß^{\eth}}$ and $\hbar \in \S$. Since $\hbar\varsigma \in \S\varsigma \subseteq \sqrt{\eth}$, we have $\hbar\varsigma \in \sqrt{\eth}$ and $\S\hbar\varsigma \subseteq \S\S\varsigma \subseteq \S\varsigma \subseteq \sqrt{\eth}$. Thus, $\hbar\varsigma \in \sqrt{ß^{\eth}}$. Hence $\sqrt{ß^{\eth}}$ is a LID of $\S$ and $\sqrt{ß^{\eth}} \subseteq \sqrt{\eth}$. ☐

Corollary 2.11. *Let $\sqrt{\eth}$ be a BID of $\S$. Then $\sqrt{\Finv^{\eth}}$ is a SSR of $\S$.*

Proof. Let $\varsigma, \epsilon \in \sqrt{\Finv^{\eth}}$. Then $\varsigma, \epsilon \in \sqrt{ß^{\eth}}$ and $\varsigma\S \subseteq \sqrt{ß^{\eth}}$ and $\epsilon\S \subseteq \sqrt{ß^{\eth}}$. Since $\varsigma \in \sqrt{ß^{\eth}}$, $\varsigma \in \sqrt{\eth}$ and $\S\varsigma \subseteq \sqrt{\eth}$. Since $\epsilon \in \sqrt{ß^{\eth}}$, $\epsilon \in \sqrt{\eth}$ and $\S\epsilon \subseteq \sqrt{\eth}$. Since $\varsigma, \epsilon \in \sqrt{\eth}$ and $\sqrt{\eth}$ is a SSR of $\S$. We have $\varsigma + \epsilon \in \sqrt{\eth}$ and $\varsigma\epsilon \in \sqrt{\eth}$. Now, $\S(\varsigma + \epsilon) \subseteq \S\varsigma + \S\epsilon \subseteq \sqrt{\eth}$ implies $\varsigma + \epsilon \in \sqrt{ß^{\eth}}$. Now, $(\varsigma + \epsilon)\S \subseteq \varsigma\S + \epsilon\S \subseteq \sqrt{ß^{\eth}}$. Hence $\varsigma + \epsilon \in \sqrt{\Finv^{\eth}}$. Now, $\S(\varsigma\epsilon) \subseteq (\S\varsigma)(\S\epsilon) \subseteq \sqrt{\eth}$ implies $\varsigma\epsilon \in \sqrt{ß^{\eth}}$ and $(\varsigma\epsilon)\S \subseteq (\varsigma\S)(\epsilon\S) \subseteq \sqrt{ß^{\eth}}$. That is $\varsigma\epsilon \in \sqrt{\Finv^{\eth}}$. Hence $\sqrt{\Finv^{\eth}}$ is a SSR of $\S$. ☐

Corollary 2.12. *Let $\sqrt{\eth}$ be a LID of $\S$. Then $\sqrt{ß^{\eth}} = \sqrt{\eth}$.*

Proof. Clearly, $\sqrt{ß^{\eth}} \subseteq \sqrt{\eth}$. Let $\varsigma \in \sqrt{\eth}$, since $\sqrt{\eth}$ is a LID of $\S$. We have $\S\varsigma \subseteq \sqrt{\eth}$ implies $\varsigma \in \sqrt{ß^{\eth}}$. Thus, $\sqrt{\eth} \subseteq \sqrt{ß^{\eth}}$. Hence $\sqrt{ß^{\eth}} = \sqrt{\eth}$. ☐

Theorem 2.13. *Let $\sqrt{\eth}$ is a BID of $\S$. Then $\sqrt{\Finv^{\eth}}$ is the unique largest TID of $\S$ contained in $\sqrt{\eth}$.*

Proof. Let $\sqrt{\eth}$ is any BID of $\S$. To prove that $\sqrt{\Finv^{\eth}}$ is the TID of $\S$. Since $\sqrt{ß^{\eth}} \subseteq \sqrt{\eth}$ and $\sqrt{\Finv^{\eth}} \subseteq \sqrt{ß^{\eth}}$. Therefore $\sqrt{\Finv^{\eth}} \subseteq \sqrt{ß^{\eth}} \subseteq \sqrt{\eth}$. Let $\varsigma \in \sqrt{\Finv^{\eth}}$ and $\bowtie \in \S$. Then $\varsigma \in \sqrt{\Finv^{\eth}} \subseteq \sqrt{\eth} \implies \varsigma \in \sqrt{\eth}$. Since ς is an element of $\sqrt{ß^{\eth}}$. We have $\S\varsigma \subseteq \sqrt{\eth}$ and $\varsigma\S \subseteq \sqrt{ß^{\eth}}$. Then $\bowtie \varsigma \in \S\varsigma \subseteq \sqrt{\eth}$ implies $\bowtie \varsigma \in \sqrt{\eth}$ and $\S \bowtie \varsigma \subseteq \S\S\varsigma \subseteq \S\varsigma \subseteq \sqrt{\eth} \implies \bowtie \varsigma \in \sqrt{ß^{\eth}}$. Now, $\varsigma \bowtie \in \varsigma\S \subseteq \sqrt{ß^{\eth}}$. Hence $\varsigma \bowtie \in \sqrt{ß^{\eth}}$ and $\bowtie \varsigma \in \sqrt{ß^{\eth}}$. First to prove that $\varsigma \bowtie \in \sqrt{\Finv^{\eth}}$ and $\bowtie \varsigma \in \sqrt{\Finv^{\eth}}$. Now, $\varsigma \bowtie \S \subseteq \varsigma\S\S \subseteq \varsigma\S \subseteq \sqrt{ß^{\eth}}$. Hence $\varsigma \bowtie \S \subseteq \sqrt{ß^{\eth}}$ implies $\varsigma \bowtie \in \sqrt{\Finv^{\eth}}$. Now, $\bowtie \varsigma\S \subseteq \S\varsigma\S \subseteq \S\sqrt{ß^{\eth}} \subseteq \sqrt{ß^{\eth}}$. Since $\sqrt{ß^{\eth}}$ is a LID of $\S$, $\bowtie \varsigma \in \sqrt{\Finv^{\eth}}$. Hence $\sqrt{\Finv^{\eth}}$ is a TID of $\S$. It enough to prove $\sqrt{\Finv^{\eth}}$ is a largest two sided ID of $\S$. Let $\sqrt{\S}$ be any ID of $\S$ and $\sqrt{\S} \subseteq \sqrt{\eth}$. Let $\varrho \in \sqrt{\S}$, then $\varrho \in \sqrt{\eth}$ and $\S\varrho \subseteq \sqrt{\S} \subseteq \sqrt{\eth}$. Hence $\S\varrho \subseteq \sqrt{\eth} \implies \varrho \in \sqrt{ß^{\eth}}$. Hence $\sqrt{\S} \subseteq \sqrt{ß^{\eth}}$. Next, $\varrho \in \sqrt{ß^{\eth}}$ and $\varrho\S \subseteq \sqrt{\S} \subseteq \sqrt{ß^{\eth}}$. Therefore $\varrho\S \subseteq \sqrt{ß^{\eth}}$. Thus, $\varrho \in \sqrt{\Finv^{\eth}}$. Hence $\sqrt{\S} \subseteq \sqrt{\Finv^{\eth}}$. ☐

Theorem 2.14. *A BID $\eth$ of a semiring $\S$ is 2PBID if and only if $\pounds_1\pounds_2 \subseteq \eth$, with $\pounds_1$ is a RID of $\S$ and $\pounds_2$ is a LID of $\S$ implies $\pounds_1 \subseteq \eth$ or $\pounds_2 \subseteq \sqrt{\eth}$.*

Proof. Let $\eth$ be a 2PBID and $\pounds_1\pounds_2 \subseteq \eth$. Suppose $\pounds_1 \not\subseteq \eth$. For all $\wp \in \pounds_2$ and $\propto \in \pounds_1 \setminus \eth$, we have $\propto \S\wp \subseteq \pounds_1\pounds_2 \subseteq \eth$. Since $\eth$ is primary and $\propto \notin \eth$ and $\wp \in \sqrt{\eth}$ for all $\wp \in \pounds_2$. So $\pounds_2 \subseteq \sqrt{\eth}$.

Conversely, suppose that $\propto \S\wp \subseteq \eth$. Now, $(\propto \S)(\S\wp) \subseteq \propto \S\wp$ implies $\propto \S \subseteq \eth$ or $\S\wp \subseteq \sqrt{\eth}$. If $\propto \S \subseteq \eth$, then $<\propto>_r < \wp >_l = \{n \propto + \propto \S | n \in \mathbb{Z}^+\} \cdot \{m\wp + \S\wp | m \in \mathbb{Z}^+\} = n \propto m\wp + n \propto \S\wp + \propto \wp + \propto \S\S\wp \subseteq \propto \S \subseteq \eth$. Thus, $\propto \in \eth$ or $\wp \in \sqrt{\eth}$. Similarly, suppose that $\S\wp \subseteq \sqrt{\eth} \implies <\propto>_r < \wp >_l \subseteq \S\wp \subseteq \sqrt{\eth}$. Thus, $\propto \in \eth$ or $\wp \in \sqrt{\eth}$. ☐

Theorem 2.15. *A BID $\eth$ is a 3PBID of $\S$ if and only if $\Finv^{\eth}$ is a PID of $\S$.*

Proof. Let $\Game$ be an 3PBID of $\S$. To show that $\Finv^{\Game}$ is a PID of $\S$. Let $\pounds_1$ and $\pounds_2$ be the IDs of $\S$ such that $\pounds_1 \cdot \pounds_2 \subseteq \Finv^{\Game}$. By thm 2.13, 2.14 and Proposition 6,[10] $\Finv^{\Game}$ and $\sqrt{\Finv^{\Game}}$ are unique largest TID contained in $\Game$ and $\sqrt{\Game}$ respectively. Thus $\pounds_1 \subseteq \Finv^{\Game}$ or $\pounds_2 \subseteq \sqrt{\Finv^{\Game}}$.

Conversely, suppose that $\pounds_1$ and $\pounds_2$ are IDs of $\S$ such that $\pounds_1 \cdot \pounds_2 \subseteq \Game$. Then $\pounds_1 \cdot \pounds_2 \subseteq \Finv^{\Game}$, implies $\pounds_1 \subseteq \Finv^{\Game} \subseteq \Game$ or $\pounds_2 \subseteq \sqrt{\Finv^{\Game}} \subseteq \sqrt{\Game}$. Hence $\Game$ is a 3PBIDs of $\S$. □

Corollary 2.16. *If $\Game$ is a 1PBID of $\S$, then $\Finv^{\Game}$ is a PID of $\S$.*

Proof. Let $\Game$ be an 1PBID of $\S$. Let us show that $\Finv^{\Game}$ is a PID of $\S$. Let $\pounds_1$ and $\pounds_2$ be an IDs of $\S$ such that $\pounds_1\pounds_2 \subseteq \Finv^{\Game}$. To show that $\pounds_1 \subseteq \Finv^{\Game}$ or $\pounds_2 \subseteq \sqrt{\Finv^{\Game}}$. Since $\Finv^{\Game} \subseteq \Game$ and $\sqrt{\Finv^{\Game}} \subseteq \sqrt{\Game}$. Hence $\pounds_1\pounds_2 \subseteq \Game$. Since $\pounds_1$ and $\pounds_2$ are IDs of $\S$ is a BIDs also and $\Game$ is an 1PBID of $\S$. Hence $\pounds_1 \subseteq \Game$ or $\pounds_2 \subseteq \sqrt{\Game}$. By Proposition 6,[10] $\Finv^{\Game}$ is the largest ID of $\S$ such that $\Finv^{\Game} \subseteq \Game$ and by thm 2.13, $\sqrt{\Finv^{\Game}}$ is the largest ID of $\S$ such that $\sqrt{\Finv^{\Game}} \subseteq \sqrt{\Game}$. Thus $\pounds_1 \subseteq \Finv^{\Game}$ or $\pounds_2 \subseteq \sqrt{\Finv^{\Game}}$. Hence $\Finv^{\Game}$ is a PID of $\S$. □

Corollary 2.17. *If $\Game$ is a 2PBID of $\S$, then $\Finv^{\Game}$ is a PID of $\S$.*

Proof. Let $\Game$ be an 2PBID of $\S$. Let us show that $\Finv^{\Game}$ is a PID of $\S$. Let $\pounds_1$ and $\pounds_2$ be an IDs of $\S$ such that $\pounds_1\pounds_2 \subseteq \Finv^{\Game}$. To show that $\pounds_1 \subseteq \Finv^{\Game}$ or $\pounds_2 \subseteq \sqrt{\Finv^{\Game}}$. Since $\Finv^{\Game} \subseteq \Game$ and $\sqrt{\Finv^{\Game}} \subseteq \sqrt{\Game}$. Hence $\pounds_1\pounds_2 \subseteq \Game$. Since $\pounds_1$ is an ID of $\S$ is an RID also and $\pounds_2$ is an ID of the ring $\S$ is an LID also. Since $\Game$ is an 2PBID of $\S$. Hence $\pounds_1 \subseteq \Game$ or $\pounds_2 \subseteq \sqrt{\Game}$. By Proposition 6,[10] $\Finv^{\Game}$ is the largest ID of $\S$ such that $\Finv^{\Game} \subseteq \Game$ and by thm 2.13, $\sqrt{\Finv^{\Game}}$ is the largest ID of $\S$ such that $\sqrt{\Finv^{\Game}} \subseteq \sqrt{\Game}$. Thus $\pounds_1 \subseteq \Finv^{\Game}$ or $\pounds_2 \subseteq \sqrt{\Finv^{\Game}}$. Hence $\Finv^{\Game}$ is a PID of $\S$. □

Theorem 2.18. *Let $\mathfrak{D}$ be a m_{p_3}- sys and $\Game$ be a BID of $\S$ with $\Game \cap \mathfrak{D} = \phi$. Then $\exists$ a 3PBID $\aleph$ of $\S$ containing $\Game$ with $\aleph \cap \mathfrak{D} = \phi$.*

Proof. Let $\mathcal{X} = \left\{\pounds_2 | \pounds_2 \text{ is a BID with } \Game \subseteq \pounds_2 \text{ and } \pounds_2 \cap \mathfrak{D} = \phi\right\}$. Clearly $\mathcal{X}$ is non-empty. By Zorn's lem, $\exists$ an maximal element $\aleph$ in $\mathcal{X}$. We claim that $\aleph$ is a 3PBID of $\S$. It is enough if we show that $\Finv^{\aleph}$ is a PID in $\S$. Since $\Finv^{\aleph} \subseteq \aleph$ and $\aleph \cap \mathfrak{D} = \phi$, this $\implies$ $\Finv^{\aleph} \cap \mathfrak{D} = \phi$. Then $\Finv^{\aleph}$ is a largest ID in $\S$ such that $\Finv^{\aleph} \cap \mathfrak{D} = \phi$. We claim that $<\propto>< \wp >\subseteq \Finv^{\aleph}$. Then $<\propto>\subseteq \Finv(Q)$ or $< \wp >\subseteq \Finv(Q)$. If $<\propto>\nsubseteq \Finv^{\aleph}$ and $< \wp >\nsubseteq \sqrt{\Finv^{\aleph}}$, then $\varsigma \in<\propto> \setminus\Finv^{\aleph}$ and $\epsilon \in< \wp > \setminus\sqrt{\Finv^{\aleph}}$. Then $< \varsigma >\subseteq<\propto>$ and $< \epsilon >\subseteq< \wp >$. If $<\propto>< \wp >\subseteq \Finv^{\aleph}$ then $< \varsigma >< \epsilon >\subseteq<\propto>< \wp >\subseteq \Finv^{\aleph}$. Since $< \wp >\nsubseteq \sqrt{\Finv^{\aleph}}$ and hence $(< \wp >)^n \nsubseteq \Finv^{\aleph} \implies < \wp >\nsubseteq \Finv^{\aleph}$. Then $(\Finv^{\aleph}+ < \varsigma >) \cap \mathfrak{D} \neq \phi$ and $(\Finv^{\aleph}+ < \epsilon >) \cap \mathfrak{D} \neq \phi$. Thus $(\Finv^{\aleph}+ < \varsigma >)(\Finv^{\aleph}+ < \epsilon >) \subseteq \Finv^{\aleph}$. Then the BID $(\Finv(Q) + \varsigma)$ contains an element m_{p_1} of $\mathfrak{D}$. Then $\exists\ \varpi_1 \in (\Finv^{\aleph}+ < \varsigma >) \cap \mathfrak{D}$. Similarly the BID $(\Finv(Q) + \epsilon)$ contains an element m_{p_2} of $\mathfrak{D}$. Then $\exists\ \varpi_2 \in (\Finv^{\aleph}+ < \epsilon >) \cap \mathfrak{D}$. Since $\mathfrak{D}$ is m_{p_3}-sys of A, $\varpi_1{}' \in< \varpi_1 >$ and $\varpi_2{}' \in< \varpi_2 > \varpi_1{}^{'}\varpi_2{}^{'} \in \mathfrak{D}$ for some $\varpi_1{}^{'} \in< \varpi_1 >\subseteq (\Finv^{\aleph}+ < \varsigma >)$ and $\varpi_2{}^{'} \in< \varpi_2 >\subseteq (\Finv^{\aleph}+ < \epsilon >)$. Hence $\varpi_1{}^{'}\varpi_2{}^{'} \in (\Finv^{\aleph}+ < \varsigma >)(\Finv^{\aleph}+ < \epsilon >) \subseteq \Finv^{\aleph}$. Which is a contradiction. Thus $<\propto>< \wp >\nsubseteq \Finv^{\aleph}$. Hence $\Finv^{\aleph}$ is a primary ID of $\S$. By thm 2.15, then there is an maximal ID $\aleph'$ in $\S$ such that $\Finv^{\aleph} \subseteq \aleph'$ and $\aleph' \cap \mathfrak{D} = \phi$. Hence $\aleph'$ is the BID of $\S$. □

3 Characterization of SPBIDs

Definition 3.1. A BID $\aleph$ of $\S$ is said to be
(i) 1SPBID if $\Game^2 \subseteq \aleph$ implies $\Game \subseteq \aleph$ or $\Game \subseteq \sqrt{\aleph}$ for any BID $\Game$ of $\S$.
(ii) 2SPBID if $\propto \S \propto\subseteq \aleph$ implies $\propto\in \aleph$ or $\propto\in \sqrt{\aleph}$.
(iii) 3SPBID if $\pounds_1^2 \subseteq \aleph$ implies $\pounds_1 \subseteq \aleph$ or $\pounds_1 \subseteq \sqrt{\aleph}$ for any ID $\pounds_1$ of $\S$.

Theorem 3.2. *Every 1SPBID is a 2SPBID of $\S$.*

Proof. Let $\aleph$ is a 1SPBID of $\S$. Let $\propto\in \S$ and $\propto \S \propto\subseteq \aleph$. Now, $(\propto \S) \cdot (\S \propto) \subseteq\propto \S \propto\subseteq \aleph$, since $\propto \S$ and $\S \propto$ are BIDs. Hence $\propto \S \subseteq \aleph$ or $\S \propto\subseteq \sqrt{\aleph}$. Suppose that $\propto \S \subseteq \aleph$. Consider $<\propto>_b \cdot <\propto>_b\subseteq\propto \S \subseteq \aleph$. Then $\propto\in \aleph$. Similarly if $\S \propto\subseteq \sqrt{\aleph}$ then $\propto\in \sqrt{\aleph}$. Thus $\aleph$ is a 2SPBID of $\S$. □

Theorem 3.3. *Every 2SPBID(2PBID) is a 3SPBID of $\S$.*

Proof. Suppose that $\aleph$ is a 2SPBID and $\pounds^2 \subseteq \aleph$ for an ID $\pounds$ of $\S$. To show that $\pounds \subseteq \aleph$ or $\pounds \subseteq \sqrt{\aleph}$. If $\pounds \nsubseteq \aleph$ and $\pounds \nsubseteq \sqrt{\aleph}$. For $\propto\in \pounds$, but $\propto\notin \aleph$ and $\propto\notin \sqrt{\aleph}$. Now $\propto \S \propto\subseteq<\propto> \cdot <\propto>\subseteq \pounds^2 \subseteq \aleph$. Since $\aleph$ is a 2SPBID of $\S$, then $\propto\in \aleph$ or $\propto\in \sqrt{\aleph}$. Which is contradiction, hence $\pounds \subseteq \aleph$ or $\pounds \subseteq \sqrt{\aleph}$. Thus $\aleph$ is a 3SPBID of $\S$. ☐

Definition 3.4. A subset N of $\S$ is said to be
(i) n_{p_1}-sys if for any $\propto\in N$, $\exists \propto_1, \propto_2\in<\propto>_b$ such that $\propto_1\propto_2\in N$.
(ii) n_{p_2}-sys if for any $\propto\in N$, $\exists \propto_1, \propto_2\in<\propto>_r$ $(\propto_1, \propto_2\in<\propto>_l)$ such that $\propto_1\propto_2\in N$.
(iii) n_{p_3}-sys if for any $\propto\in N$, $\exists \propto_1, \propto_2\in<\propto>$ such that $\propto_1\propto_2\in N$.

Theorem 3.5. *If $\aleph$ is a BID of $\S$, then $\aleph$ is a 1SPBID (2SPBID, 3SPBID) if and only if $\S \setminus \aleph$ is an n_{p_1}-sys (n_{p_2}-sys, n_{p_3}-sys).*

Proof. Let $\aleph$ be a 1SPBID of $\S$. Let $\propto\in \S \setminus \aleph$. Hence $\propto\in \S$ but $\propto\notin \aleph$. So $<\propto>_b \cdot <\propto>_b\nsubseteq \aleph$. There exists $\propto^{'}, \propto^{''}\in<\propto>_b$ such that $\propto^{'} \cdot \propto^{''}\notin \aleph$. Hence $\propto^{'} \cdot \propto^{''}\in \S \setminus \aleph$. So we have proved that for $\propto\in \S \setminus \aleph$ $\exists$ $\propto^{'}, \propto^{''}\in<\propto>_b$ such that $\propto^{'} \cdot \propto^{''}\in \S \setminus \aleph$. So $\S \setminus \aleph$ is an n_{p_1}-sys.

Conversely, let $\S \setminus \aleph$ is an n_{p_1}-sys. Let $\Game^2 \subseteq \aleph$ for the BID $\Game$ of $\S$. Let us shows that $\Game \subseteq \aleph$ or $\Game \subseteq \sqrt{\aleph}$. If $\Game \nsubseteq \aleph$ and $\Game \nsubseteq \sqrt{\aleph}$, let $\wp_1 \in \Game \setminus \aleph$ and $\wp_1 \in \Game \setminus \sqrt{\aleph}$. Since $\wp_1 \notin \sqrt{\aleph}$, so $\exists$ an n_{p_1}-sys $\S \setminus \aleph$ in $\S$ such that $\wp_1 \in \S \setminus \aleph$ and $(\S \setminus \aleph) \cap \aleph = \phi$. Thus $\wp_1 \in \S \setminus \aleph$ implies $< \wp_1 >_b \cdot < \wp_1 >_b\nsubseteq \aleph$, which is a contradiction. Thus $\Game \subseteq \aleph$ or $\Game \subseteq \sqrt{\aleph}$. Hence $\aleph$ is a 1SPBID of $\S$. ☐

Corollary 3.6. *Every n_{p_1}-sys is an n_{p_2}-sys.*

Theorem 3.7. *Let $\Game$ be a 2SPBID of a ring $\S$. Then $\pounds^2 \subseteq \Game$ implies $\pounds \subseteq \Game$ or $\pounds \subseteq \sqrt{\Game}$ for any LID (RID)$\pounds$ of $\S$.*

Theorem 3.8. *A BID $\Game$ is a 3SPBID of $\S$ if and only if $\Finv^{\Game}$ is a SPID of $\S$.*

Corollary 3.9. *If $\Game$ is a 1SPBID (2SPBID) of $\S$, then $\Finv^{\Game}$ is a SPID of $\S$.*

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