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Ternary algebraic structure convey to new type of intuitionistic Q1 anti fuzzy ideals of an regular ordered ternary semigroups

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Abstract

We examine some of the properties of these ordered ternary semigroups, such as intuitionistic Q1 anti fuzzy left ideal ,Q1 anti fuzzy right ideal, Q1 anti fuzzy lateral ideal, Q1 anti fuzzy ideal and Q1 anti fuzzy bi-ideal. We introduce the idea of TSS. The Q1 anti anti fuzzy ideal is extended in a new way over ternary semigroups $\mathfrak{B}$.

Keywords: ternary semigroups, anti fuzzy ideals, anti fuzzy bi-ideals, Q1 anti fuzzy bi-ideals.

1 Introduction

The ternary algebraic systems known as triplex structures were initially conceived in 1932 by D. H. Lehmer.[1] Vandiver came up with the idea of the semiring in 1934. In 1962 Hestenes[2] used linear transformation and matrices as examples to establish the concept of ternary algebra. In 1971, after describing those additive subgroups of rings that are closed under the triple ring product, Lister talked about this algebraic system as a ternary ring. The fuzzy set (FS) theory, which was created by Zadeh,[3] works best when it comes to handling ambiguity and uncertainty. An element with a single value inside the interval is called a member of an FS. The NMG might not always be equal to one minus the MG in practice, though, because of potential pushback. A growing number of hybrid fuzzy models are being created as FS theory develops quickly. As a result of the uncertainties, several theories of uncertainty have been created, such as Pythagorean FS (PFS),[5] intuitionistic FS (IFS),[4] and FS.[3] An FS is composed of MG sets, or sets with grades ranging from 0 to 1. Although Atanassov[4] asserts that non-membership grades (NMG) can only be worth 1, IFS is classified as MG. There are times when the sum of MGs and NMGs throughout a decision-making process can approach 1. The generalized MG and NMG logic, which is based on the square of the MGs and NMGs and has a value of no more than 1, was developed by Yager[5] using PFS logic. Since the neutral state is neither positive nor negative, it cannot be described by these theories. Palanikumar and colleagues have introduced an intuitionistic fuzzy normal subbisemiring of bisemiring.[6] The idea of bisemiring was created by Palanikumar et al.[7] employing bipolar-valued neutrosophic normal sets. Bi-ideals on ordered semigroups were examined by Hila and associates.[8] Dutta T.K. and associates presented novel concepts based on ternary semiring prime ideals and prime radicals.[9] Several prime and semiprime bi-ideals of the rings were discussed by Palanikumar and associates[11] and others. The several ideals of semigroups, semirings, and ternary semirings were examined by Palanikumar et al.[12,22–25]

2 $(\propto_1, \propto_2)$ intuitionistic Q1 anti fuzzy ideals

Here $\mathfrak{B}$ denotes the ordered ternary semigroup and $(\propto_1, \propto_2) \in [0,1]$ be such that $0 \leqslant \propto_1 < \propto_2 \leqslant 1$ both $(\propto_1, \propto_2)$ are arbitrary fixed.

Definition 2.1. A IFS $A = [\pitchfork_A, \top_A]$ of $\mathfrak{B}$ and $Q1$ is a any non-empty set of A, the pair $(A, Q1)$ Q1IFS is called a $(\propto_1, \propto_2)$ IQ1TFSS of $\mathfrak{B}$ if

1. If $\sharp \leqslant \emptyset$, then $\pitchfork(\sharp) \leqslant \pitchfork(\emptyset)$ and $\top(\sharp) \geqslant \top(\emptyset)$,
2. $\min\{\pitchfork(\sharp\partial\emptyset, \iota), \propto_1\} \leqslant \max\{\pitchfork(\sharp, \iota), \pitchfork(\partial, \iota), \pitchfork(\emptyset, \iota), \propto_2\}$
3. $\max\{\top(\sharp\partial\emptyset, \iota), \propto_1\} \geqslant \min\{\top(\sharp, \iota), \top(\partial, \iota), \top(\emptyset, \iota), \propto_2\}$ for all $\sharp, \partial, \emptyset \in \mathfrak{B}$ and $q \in Q1$.

Example 2.2. Let $\mathfrak{B} = \{\emptyset_l, \emptyset_m, \emptyset_n, \emptyset_o\}$ with the Cayley table:

$\cdot$	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
$\emptyset_l$	t	t	t	t
$\emptyset_m$	t	o	p	c
$\emptyset_n$	t	p	p	p
$\emptyset_o$	t	p	p	p

$\cdot$	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
t	$\emptyset_l$	$\emptyset_l$	$\emptyset_l$	$\emptyset_l$
o	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
p	$\emptyset_l$	$\emptyset_n$	$\emptyset_n$	$\emptyset_n$
c	$\emptyset_l$	$\emptyset_n$	$\emptyset_n$	$\emptyset_n$

$\leqslant := \{(\emptyset_l, \emptyset_l), (\emptyset_l, \emptyset_m), (\emptyset_l, \emptyset_n), (\emptyset_l, \emptyset_o), (\emptyset_m, \emptyset_m), (\emptyset_m, \emptyset_n), (\emptyset_m, \emptyset_o), (\emptyset_n, \emptyset_n), (\emptyset_o, \emptyset_n), (\emptyset_o, \emptyset_o)\}$.
Define $A = [\pitchfork_A, \top_A] : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0,1]$.

$$\pitchfork(\emptyset, \iota) = \begin{cases} 0.26 & if\ \emptyset = \emptyset_l \\ 0.33 & if\ \emptyset = \emptyset_m \\ 0.43 & if\ \emptyset = \emptyset_n \\ 0.38 & if\ \emptyset = \emptyset_o \end{cases} \quad \top(\emptyset, \iota) = \begin{cases} 0.58 & if\ \emptyset = \emptyset_l \\ 0.38 & if\ \emptyset = \emptyset_m \\ 0.08 & if\ \emptyset = \emptyset_n \\ 0.18 & if\ \emptyset = \emptyset_o \end{cases}$$

Then A is a $(0.48, 0.63)$ $IQ1TFSS$ of $\mathfrak{B}$.

Definition 2.3. A intuitionistic Q1 subset A of $\mathfrak{B}$ is called a $(\propto_1, \propto_2)$-IQ1AFBI of $\mathfrak{B}$ if

1. If $\sharp \leqslant \emptyset$, then $\pitchfork(\sharp) \leqslant \pitchfork(\emptyset)$ and $\top(\sharp) \geqslant \top(\emptyset)$,
2. $\min\{\pitchfork(\sharp\partial_1\emptyset, \iota), \propto_1\} \leqslant \max\{\pitchfork(\sharp, \iota), \pitchfork(\emptyset, \iota), \propto_2\}$,
   $\max\{\top(\sharp\partial_1\emptyset, \iota), \propto_1\} \geqslant \min\{\top(\sharp, \iota), \top(\emptyset, \iota), \propto_2\}$,
3. $\min\{\pitchfork(\sharp\partial_1\emptyset\partial_2\epsilon, \iota), \propto_1\} \leqslant \max\{\pitchfork(\sharp, \iota), \pitchfork(\epsilon, \iota), \propto_2\}$,
   $\max\{\top(\sharp\partial_1\emptyset\partial_2\epsilon, \iota), \propto_1\} \geqslant \min\{\top(\sharp, \iota), \top(\epsilon, \iota), \propto_2\}$, for $\sharp, \emptyset, \epsilon, \partial_1, \partial_2 \in \mathfrak{B}$ and $q \in Q1$.

Example 2.4. Let $\mathfrak{B} = \{\emptyset_l, \emptyset_m, \emptyset_n, \emptyset_o\}$ with Cayley table:

$\cdot$	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
$\emptyset_l$	t	t	t	t
$\emptyset_m$	t	o	p	c
$\emptyset_n$	t	p	p	p
$\emptyset_o$	t	p	p	p

$\cdot$	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
t	$\emptyset_l$	$\emptyset_l$	$\emptyset_l$	$\emptyset_l$
o	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
p	$\emptyset_l$	$\emptyset_n$	$\emptyset_n$	$\emptyset_n$
c	$\emptyset_l$	$\emptyset_n$	$\emptyset_n$	$\emptyset_n$

$\cdot$	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
t	t	t	t	t
o	t	o	p	d
p	t	p	p	p
c	t	o	p	c

$\cdot$	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
t	$\emptyset_l$	$\emptyset_l$	$\emptyset_l$	$\emptyset_l$
o	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$
p	$\emptyset_l$	$\emptyset_n$	$\emptyset_n$	$\emptyset_n$
c	$\emptyset_l$	$\emptyset_m$	$\emptyset_n$	$\emptyset_o$

$\leqslant:= \{(\varnothing_l,\varnothing_l),(\varnothing_l,\varnothing_m),(\varnothing_l,\varnothing_n),(\varnothing_l,\varnothing_o),(\varnothing_m,\varnothing_m),(\varnothing_m,\varnothing_n),(\varnothing_m,\varnothing_o),(\varnothing_n,\varnothing_n),(\varnothing_o,\varnothing_n),(\varnothing_o,\varnothing_o)\}$. Define $\sqsupset=[\pitchfork,\top]:\mathfrak{B}\times\mathfrak{B}\times\mathfrak{B}\rightarrow[0,1]$

$$\pitchfork(\varnothing,\iota)=\begin{cases}0.32 \;\; if\; \varnothing=\varnothing_l\\ 0.37\;\; if\; \varnothing=\varnothing_m\\ 0.47\;\; if\; \varnothing=\varnothing_n\\ 0.42\;\; if\; \varnothing=\varnothing_o\end{cases}\quad \top(\varnothing,\iota)=\begin{cases}0.49\;\; if\; \varnothing=\varnothing_l\\ 0.30\;\; if\; \varnothing=\varnothing_m\\ 0.02\;\; if\; \varnothing=\varnothing_n\\ 0.11\;\; if\; \varnothing=\varnothing_o\end{cases}$$

Then $\sqsupset$ is a $(0.35,0.50)$ IQ1AFBI of $\mathfrak{B}$.

Theorem 2.5. *A non-empty subset $\sqsupset_{\propto_1}$ is a $\pitchfork_{\propto_1}$ is a $(\propto_1,\propto_2)$-IQ1TFSS (IQ1AFLI,IQ1AFLATI, IQ1AFRI, IQ1AFBI) of $\mathfrak{B}$. Then the lower level set $\pitchfork_{\propto_1}$ is an TSS (TLI,TLATI, TRI, TBI) of $\mathfrak{B}$, where $\pitchfork_{\propto_1}=\{\sharp\in\mathfrak{B}|\pitchfork(\sharp,\iota)\prec\propto_1\}$ and $\top_{\propto_1}=\{\sharp\in\mathfrak{B}|\top(\sharp,\iota)\succ\propto_1\}$.*

Proof. Suppose that $\sqsupset_{\propto_1}$ is a $(\propto_1,\propto_2)$-IQ1TFSS of $\mathfrak{B}$. Let $\sharp,\partial,\varnothing\in\mathfrak{B}$ such that $\sharp,\partial,\varnothing\in\pitchfork_{\propto_1}$. Then $\pitchfork(\sharp,\iota)\prec\propto_1,\pitchfork(\partial,\iota)\prec\propto_1,\pitchfork(\varnothing,\iota)\prec\propto_1$.
Therefore $\min\{\pitchfork(\sharp\partial\varnothing,\iota),\propto_1\}\leqslant\max\{\pitchfork(\sharp,\iota),\pitchfork(\partial,\iota),\pitchfork(\varnothing,\iota),\propto_2\}\prec\max\{\propto_1,\propto_1,\propto_1,\propto_2\}=\propto_2$. Hence $\pitchfork(\sharp\partial\varnothing,\iota)\prec\propto_1$. It shows that $\sharp\partial\varnothing\in\pitchfork_{\propto_1}$. Therefore $\pitchfork_{\propto_1}$ is a TSS of $\mathfrak{B}$. Let $\sharp,\partial,\varnothing\in\mathfrak{B}$ such that $\sharp,\partial,\varnothing\in\top_{\propto_1}$. Then $\top(\sharp,\iota)\succ\propto_1,\top(\partial,\iota)\succ\propto_1\;\top(\varnothing,\iota)\succ\propto_1$. Therefore $\max\{\top(\sharp\partial\varnothing,\iota),\propto_1\}\geqslant\min\{\top(\sharp,\iota),\top(\partial,\iota),\top(\varnothing,\iota),\propto_2\}\succ\min\{\propto_1,\propto_1,\propto_1,\propto_2\}=\propto_1$. Hence $\top(\sharp\partial\varnothing,\iota)\succ\propto_1$. It shows that $\sharp\partial\varnothing\in\top_{\propto_1}$. Therefore $\top_{\propto_1}$ is a TSS of $\mathfrak{B}$. Therefore $\sqsupset_{\propto_1}$ is a TSS of $\mathfrak{B}$.

Theorem 2.6. *A non-empty subset $\dashv$ of $\mathfrak{B}$ is a SS $[TLI,TLATI,TRI,TBI]$ of $\mathfrak{B}$ if and only if the IQ1FS $\sqsupset=[\pitchfork,\top]$ of $\mathfrak{B}$ is defined as*

$$\pitchfork(\sharp,\iota)=\begin{cases}\leqslant\propto_2\;\; for\,all\;\;\sharp\in(\dashv]\\ \propto_1\;\; for\,all\;\sharp\notin(\dashv]\end{cases}\quad \top(\sharp,\iota)=\begin{cases}\geqslant\propto_2\;\; for\,all\;\;\sharp\in(\dashv]\\ \propto_1\;\; for\,all\;\sharp\notin(\dashv]\end{cases}$$

is a $(\propto_1,\propto_2)IQ1TFSS[IQ1AFLI,IQ1AFLATI,IQ1AFRI,IQ1AFBI]$ of $\mathfrak{B}$.

Proof. Suppose that $\dashv$ is an TSS of $\mathfrak{B}$. Let $\sharp,\partial,\varnothing\in\mathfrak{B}$ be such that $\sharp,\partial,\varnothing\in(\dashv]$ then $\sharp\partial\varnothing\in(\dashv]$. Hence $\pitchfork(\sharp\partial\varnothing,\iota)\leqslant\propto_2$ and $\top(\sharp\partial\varnothing,\iota)\geqslant\propto_2$. Thus, $\min\{\pitchfork(\sharp\partial\varnothing,\iota),\propto_1\}\leqslant\propto_2=\max\{\pitchfork(\sharp,\iota),\pitchfork(\partial,\iota),\pitchfork(\varnothing,\iota),\propto_2\}$ and $\max\{\top(\sharp\partial\varnothing,\iota),\propto_1\}\geqslant\propto_2=\min\{\top(\sharp,\iota),\top(\partial,\iota),\top(\varnothing,\iota),\propto_2\}$. If $\sharp\notin(\dashv]$ or $\partial\notin(\dashv]$ or $\varnothing\notin(\dashv]$,
then $\max\{\pitchfork(\sharp,\iota),\pitchfork(\partial,\iota),\pitchfork(\varnothing,\iota),\propto_2\}=\propto_1$ and $\min\{\top(\sharp,\iota),\top(\partial,\iota),\top(\varnothing,\iota),\propto_2\}=\propto_2$.
That is $\min\{\pitchfork(\sharp\partial\varnothing,\iota),\propto_1\}\leqslant\max\{\pitchfork(\sharp,\iota),\pitchfork(\partial,\iota),\pitchfork(\varnothing,\iota),\propto_2\}$ and
$\max\{\top(\sharp\partial\varnothing,\iota),\propto_1\}\geqslant\min\{\top(\sharp,\iota),\top(\partial,\iota),\top(\varnothing,\iota),\propto_2\}$. Therefore $\sqsupset$ is a $(\propto_1,\propto_2)$ IQ1TFSS of $\mathfrak{B}$.
Conversely assume that $\sqsupset=[\pitchfork,\top]$ is a $(\propto_1,\propto_2)$-IQ1TFSS of $\mathfrak{B}$. Let $\sharp\partial\varnothing\in(\dashv]$. Then $\pitchfork(\sharp,\iota)\leqslant\propto_2,\pitchfork(\partial,\iota)\leqslant\propto_2,\pitchfork(\varnothing,\iota)\leqslant\propto_2$ and $\top(\sharp,\iota)\geqslant\propto_2,\top(\partial,\iota)\geqslant\propto_2,\top(\varnothing,\iota)\geqslant\propto_2$. Now $\sqsupset=[\pitchfork,\top]$ is a $(\propto_1,\propto_2)$-IQ1TFSS of $\mathfrak{B}$. Therefore $\min\{\pitchfork(\sharp\partial\varnothing,\iota),\propto_1\}\leqslant\max\{\pitchfork(\sharp,\iota),\pitchfork(\partial,\iota),\pitchfork(\varnothing,\iota),\propto_2\}\leqslant\max\{\propto_2,\propto_2,\propto_2,\propto_2\}=\propto_2$ and $\max\{\top(\sharp\partial\varnothing,\iota),\propto_1\}\geqslant\min\{\top(\sharp,\iota),\top(\partial,\iota),\top(\varnothing,\iota),\propto_2\}\geqslant\min\{\propto_2,\propto_2,\propto_2,\propto_2\}=\propto_2$. It follows that $\sharp\partial\varnothing\in(\dashv]$. Therefore $\dashv$ is a TSS of $\mathfrak{B}$.

Theorem 2.7. *A subset $\sqsupset=[\pitchfork,\top]$ is a $(\propto_1,\propto_2)-IQ1TFSS[IQ1AFLI,IQ1AFLATI,IQ1AFRI,IQ1AFBI]$ of $\mathfrak{B}$ if and only if each non-empty level subset $\sqsupset_t$ is a TSS [TLI,TLATI,TRI,TBI] of $\mathfrak{B}$ for all $t\in(\propto_1,\propto_2]$.*

Proof. Assume that $\sqsupset_t$ is a TSS of $\mathfrak{B}$ for each $t\in[0,1]$.
Let $t=\max\{\pitchfork(\sharp_a,\iota),\pitchfork(\sharp_b,\iota),\pitchfork(\sharp_c,\iota)\}$. Then $\sharp_a,\sharp_b,\sharp_c\in\pitchfork_t$ for each $\sharp_a,\sharp_b,\sharp_c\in\mathfrak{B}$.
Thus $\min\{\pitchfork(\sharp\partial\varnothing,\iota),\propto_1\}\leqslant t=\max\{\pitchfork(\sharp_a,\iota),\pitchfork(\sharp_b,\iota),\pitchfork(\sharp_c,\iota),\propto_2\}$.
Let $t=\min\{\top(\sharp_a,\iota),\top(\sharp_b,\iota),\top(\sharp_c,\iota)\}$.
Then $\sharp_a,\sharp_b,\sharp_c\in\top_t$ for each $\sharp_a,\sharp_b,\sharp_c\in\mathfrak{B}$.
Thus $\max\{\top(\sharp\partial\varnothing,\iota),\propto_1\}\geqslant t=\min\{\top(\sharp_a,\iota),\top(\sharp_b,\iota),\top(\sharp_c,\iota),\propto_2\}$.
This shows that $\sqsupset^t$ is IQ1TFSS of $\mathfrak{B}$.
Conversely, assume that $\sqsupset^t$ is a IQ1TFSS of $\mathfrak{B}$. For each $t\in[0,1]$ and $\sharp_a,\sharp_b,\sharp_c\in\pitchfork_t$. We have $\pitchfork(\sharp_a,\iota)\leqslant t,\pitchfork(\sharp_b,\iota)\leqslant t,\pitchfork(\sharp_c,\iota)\leqslant t$. Since $\pitchfork$ is a TSS of $\mathfrak{B}$,
$\min\{\pitchfork(\sharp_a\sharp_b\sharp_c,\iota),\propto_1\}\leqslant\max\{\pitchfork(\sharp_a,\iota),\pitchfork(\sharp_b,\iota),\pitchfork(\sharp_c,\iota),\propto_2\}\leqslant t$. This implies that $\sharp_a\sharp_b\sharp_c\in\pitchfork_t$. We have $\top(\sharp_a,\iota)\geqslant t,\top(\sharp_b,\iota)\geqslant t,\top(\sharp_c,\iota)\geqslant t$. Since $\top$ is a TSS of $\mathfrak{B}$,$\max\{\top(\sharp_a\sharp_b\sharp_c,\iota),\propto_1\}\geqslant\min\{\top(\sharp_a,\iota),\top(\sharp_b,\iota),\top(\sharp_c,\iota),\propto_2\}\geqslant t$. This implies that $\sharp_a\sharp_b\sharp_c\in\top_t$. Therefore $\sqsupset_t$ is a TSS of $\mathfrak{B}$ for each $t\in(\propto_1,\propto_2]$.

Example 2.8. Every IQ1TFSS $\beth$ of $\mathfrak{B}$ is a $(\propto_1, \propto_2)$-IQ1TFSS of $\mathfrak{B}$, but converse is not true.

For the Example 2.2, we define subset $\beth$ by

$$\pitchfork(\varnothing, \iota) = \begin{cases} 0.19 & if\ \varnothing = \varnothing_l \\ 0.24 & if\ \varnothing = \varnothing_m \\ 0.34 & if\ \varnothing = \varnothing_n \\ 0.29 & if\ \varnothing = \varnothing_o \end{cases} \quad \top(\varnothing, \iota) = \begin{cases} 0.34 & if\ \varnothing = \varnothing_l \\ 0.27 & if\ \varnothing = \varnothing_m \\ 0.17 & if\ \varnothing = \varnothing_n \\ 0.22 & if\ \varnothing = \varnothing_o \end{cases}$$

Then $\beth$ is a $(0.25, 0.39)$-IQ1TFSS of $\mathfrak{B}$, but not a IQ1TFSS.
Since $\pitchfork(\varnothing_o\partial\varnothing_o, \iota) = 0.34 \nleq \max\{\pitchfork(\varnothing_o, q), \pitchfork(\varnothing_o, q)\} = 0.29$ and
$\top(\varnothing_o\partial\varnothing_o, \iota) = 0.17 \ngeq \min\{\top(\varnothing_o, q), \top(\varnothing_o, q)\} = 0.22$.

Example 2.9. Every IQ1AFBI $\beth = [\pitchfork, \top]$ of $\mathfrak{B}$ is a $(\propto_1, \propto_2)$-IQ1AFBI of $\mathfrak{B}$, but converse need not be true.

For the Example 2.4, we define subset $\beth$ by,

$$\pitchfork(\varnothing, \iota) = \begin{cases} 0.08 & if\ \varnothing = \varnothing_l \\ 0.23 & if\ \varnothing = \varnothing_m \\ 0.33 & if\ \varnothing = \varnothing_n \\ 0.28 & if\ \varnothing = \varnothing_o \end{cases} \quad \top(\varnothing, \iota) = \begin{cases} 0.42 & if\ \varnothing = \varnothing_l \\ 0.23 & if\ \varnothing = \varnothing_m \\ 0.01 & if\ \varnothing = \varnothing_n \\ 0.04 & if\ \varnothing = \varnothing_o \end{cases}$$

Then $\beth$ is a $(0.18, 0.43)$IQ1AFBI, but not a IQ1AFBI.
Since $\pitchfork(\varnothing_o\partial_1\varnothing_o\partial_2\varnothing_o, \iota) = \pitchfork(\varnothing_n, \iota) = 0.33 \nleq \max\{\pitchfork(\varnothing_o, \iota), \pitchfork(\varnothing_o, \iota)\} = 0.28$ and
$\top(\varnothing_o\partial_1\varnothing_o\partial_2\varnothing_o, \iota) = \top(\varnothing_n, \iota) = 0.01 \ngeq \min\{\pitchfork(\varnothing_o, \iota), \pitchfork(\varnothing_o, \iota)\} = 0.04$.

Definition 2.10. If $\jmath_{\dashv}$ is the characteristic function of $\dashv$, then $(\jmath_{\dashv})^{\propto_2}_{\propto_1}$ is defined as

$$(\jmath^{\mathbb{T}}_{\dashv})^{\propto_2}_{\propto_1}(\sharp, \iota) = \begin{cases} \propto_2 & if\ \sharp \in (\dashv] \\ \propto_1 & if\ \sharp \notin (\dashv] \end{cases} \quad (\jmath^{\mathbb{F}}_{\dashv})^{\propto_2}_{\propto_1}(\sharp, \iota) = \begin{cases} \propto_1 & if\ \sharp \in (\dashv] \\ \propto_2 & if\ \sharp \notin (\dashv] \end{cases}$$

Theorem 2.11. *A non empty subset $\dashv$ of $\mathfrak{B}$ is a TSS $[TLI, TLATI, TRI, TBI]$ of $\mathfrak{B}$ if and only if subset $\jmath_{(\dashv]}$ is a $(\propto_1, \propto_2)$-$IQ1TFSS[IQ1AFLI, IQ1AFLATI, IQ1AFRI, IQ1AFBI]$ of $\mathfrak{B}$.*

Proof. Assume that $\dashv$ is a TSS of $\mathfrak{B}$. Then $\jmath_{(\dashv]}$ is a IQ1TFSS of $\mathfrak{B}$ and hence $\jmath_{(\dashv]}$ is an $(\propto_1, \propto_2)$-IQ1TFSS of $\mathfrak{B}$.
Conversely, Let $\jmath_{(\dashv]}$ is an $(\propto_1, \propto_2)$-IQ1TFSS of $\mathfrak{B}$. Let $\sharp, \partial, \varnothing \in \mathfrak{B}$ be such that $\sharp, \partial, \varnothing \in (\dashv]$. Then $\jmath^{\mathbb{T}}_{(\dashv]}(\sharp, \iota) = \propto_2, \jmath^{\mathbb{T}}_{(\dashv]}(\partial, \iota) = \propto_2, \jmath^{\mathbb{T}}_{(\dashv]}(\varnothing, \iota) = \propto_2$. Since $\jmath^{\mathbb{T}}_{(\dashv]}$ is a $(\propto_1, \propto_2)$IQ1TFSS. Consider

$$\begin{aligned} \min\{\jmath^{\mathbb{T}}_{(\dashv]}(\sharp\partial\varnothing, \iota), \propto_1\} &\leqslant \max\{\jmath^{\mathbb{T}}_{(\dashv]}(\sharp, \iota), \jmath^{\mathbb{T}}_{(\dashv]}(\partial, \iota), \jmath^{\mathbb{T}}_{(\dashv]}(\varnothing, \iota), \propto_2\} \\ &= \max\{\propto_2, \propto_2, \propto_2, \propto_2\} \\ &= \propto_2 \end{aligned}$$

as $\propto_1 \prec \propto_2$, this implies that $\jmath^{\mathbb{T}}_{(\dashv]}(\sharp\partial\varnothing, \iota) \leqslant \propto_2$. Thus $\sharp\partial\varnothing \in (\dashv]$. Thus $\sharp\partial\varnothing \in (\dashv]$.

Let $\sharp, \partial, \varnothing \in \mathfrak{B}$ be such that $\sharp, \partial, \varnothing \in (\dashv]$. Then $\jmath^{\mathbb{F}}_{(\dashv]}(\sharp, \iota) = \propto_1, \jmath^{\mathbb{F}}_{(\dashv]}(\partial, \iota) = \propto_1, \jmath^{\mathbb{F}}_{(\dashv]}(\varnothing, \iota) = \propto_1$. Since $\jmath^{\mathbb{F}}_{(\dashv]}$ is a $(\propto_1, \propto_2)$IQ1TFSS. Consider

$$\begin{aligned} \max\{\jmath^{\mathbb{F}}_{(\dashv]}(\sharp\partial\varnothing, \iota), \propto_1\} &\geqslant \min\{\jmath^{\mathbb{F}}_{(\dashv]}(\sharp, \iota), \jmath^{\mathbb{F}}_{(\dashv]}(\partial, \iota), \jmath^{\mathbb{F}}_{(\dashv]}(\varnothing, \iota), \propto_2\} \\ &= \min\{\propto_1, \propto_1, \propto_1, \propto_2\} \\ &= \propto_1 \end{aligned}$$

as $\propto_1 \prec \propto_2$, this implies that $\jmath^{\mathbb{F}}_{(\dashv]}(\sharp\partial\varnothing, \iota) \geqslant \propto_1$. Thus $\sharp\partial\varnothing \in (\dashv]$. Thus $\sharp\partial\varnothing \in (\dashv]$.
Therefore $\dashv$ is a TSS of $\mathfrak{B}$.

Let $\sharp, \partial, \varnothing \in \mathfrak{B}$ be such that $\sharp, \partial, \varnothing \notin (\daleth]$. Then $\jmath^{\mathbb{T}}_{(\daleth]}(\sharp, \iota) = \propto_1, \jmath^{\mathbb{T}}_{(\daleth]}(\partial, \iota) = \propto_1, \jmath^{\mathbb{T}}_{(\daleth]}(\varnothing, \iota) = \propto_1$.
Since $\jmath^{\mathbb{T}}_{(\daleth]}$ is a $(\propto_1, \propto_2)$IQ1TFSS.

$$\begin{aligned}\min\{\jmath^{\mathbb{T}}_{(\daleth]}(\sharp\partial\varnothing, \iota), \propto_1\} &\leqslant \max\{\jmath^{\mathbb{T}}_{(\daleth]}(\sharp, \iota), \jmath^{\mathbb{T}}_{(\daleth]}(\partial, \iota), \jmath^{\mathbb{T}}_{(\daleth]}(\varnothing, \iota), \propto_2\}\\ &= \max\{\propto_1, \propto_1, \propto_1, \propto_2\}\\ &=\propto_2\end{aligned}$$

as $\propto_1 \prec \propto_2$, this implies that $\jmath^{\mathbb{T}}_{(\daleth]}(\sharp\partial\varnothing, \iota) \leqslant \propto_1$. Thus $\sharp\partial\varnothing \notin (\daleth]$.

Let $\sharp, \partial, \varnothing \in \mathfrak{B}$ be such that $\sharp, \partial, \varnothing \notin (\daleth]$. Then $\jmath^{\mathbb{F}}_{(\daleth]}(\sharp, \iota) = \propto_2, \jmath^{\mathbb{F}}_{(\daleth]}(\partial, \iota) = \propto_2, \jmath^{\mathbb{F}}_{(\daleth]}(\varnothing, \iota) = \propto_2$.
Since $\jmath^{\mathbb{F}}_{(\daleth]}$ is a $(\propto_1, \propto_2)$IQ1TFSS.

$$\begin{aligned}\max\{\jmath^{\mathbb{F}}_{(\daleth]}(\sharp\partial\varnothing, \iota), \propto_1\} &\geqslant \min\{\jmath^{\mathbb{F}}_{(\daleth]}(\sharp, \iota), \jmath^{\mathbb{F}}_{(\daleth]}(\partial, \iota), \jmath^{\mathbb{F}}_{(\daleth]}(\varnothing, \iota), \propto_2\}\\ &= \min\{\propto_2, \propto_2, \propto_2, \propto_2\}\\ &=\propto_2\end{aligned}$$

as $\propto_1 \prec \propto_2$, this implies that $\jmath^{\mathbb{F}}_{(\daleth]}(\sharp\partial\varnothing, \iota) \geqslant \propto_2$. Thus $\sharp\partial\varnothing \notin (\daleth]$.
Therefore $\daleth$ is a TSS of $\mathfrak{B}$.

Definition 2.12. For three IQ1AFSs $\beth$, ∂ and $\eth$ of $\mathfrak{B}$. Their product $\beth \cdot \partial \cdot \eth$ is defined as

$$(\beth^{\mathbb{T}} \cdot \partial^{\mathbb{T}} \cdot \eth^{\mathbb{T}})(\sharp, \iota) = \begin{cases} \inf\limits_{(r,s,t)\in\daleth_\sharp} \{\beth^{\mathbb{T}}(r, \iota) \bigtriangledown \partial^{\mathbb{T}}(s, \iota) \bigtriangledown \eth^{\mathbb{T}}(t, \iota)\}\ if\ \daleth_\sharp \neq 0 \\ 1\ otherwise \end{cases}$$

$$(\beth^{\mathbb{F}} \cdot \partial^{\mathbb{F}} \cdot \eth^{\mathbb{F}})(\sharp, \iota) = \begin{cases} \sup\limits_{(r,s,t)\in\daleth_\sharp} \{\beth^{\mathbb{F}}(r, \iota) \bigtriangleup \partial^{\mathbb{F}}(s, \iota) \bigtriangleup \eth^{\mathbb{F}}(t, \iota)\}\ if\ \daleth_\sharp \neq 0 \\ 0\ otherwise \end{cases}$$

Definition 2.13. Let $\beth$ be subset of $\mathfrak{B}$, we define the subset $(\pitchfork)^{\propto_2}_{\propto_1}(\sharp, \iota) = \{\pitchfork(\sharp, \iota) \bigtriangledown \propto_2\} \bigtriangleup \propto_1$ and $(\top)^{\propto_2}_{\propto_1}(\sharp, \iota) = \{\top(\sharp, \iota) \bigtriangleup \propto_2\} \bigtriangledown \propto_1$, for all $\sharp \in \mathfrak{B}$.

Lemma 2.14. *Let $\daleth$, $\daleth_1$ and $\daleth_2$ be non-empty subsets of $\mathfrak{B}$. Then*

1. $(\jmath_{(\daleth]} \bigtriangledown \jmath_{(\daleth_1]} \bigtriangledown \jmath_{(\daleth_2]})^{\propto_2}_{\propto_1} = (\jmath_{(\daleth\cup\daleth_1\cup\daleth_2]})^{\propto_2}_{\propto_1}$,
2. $(\jmath_{(\daleth]} \bigtriangleup \jmath_{(\daleth_1]} \bigtriangleup \jmath_{(\daleth_2]})^{\propto_2}_{\propto_1} = (\jmath_{(\daleth\cap\daleth_1\cap\daleth_2]})^{\propto_2}_{\propto_1}$,
3. $(\jmath_{(\daleth]} \cdot \jmath_{(\daleth_1]} \cdot \jmath_{(\daleth_2]})^{\propto_2}_{\propto_1} = (\jmath_{(\daleth\daleth_1\daleth_2]})^{\propto_2}_{\propto_1}$.

Proof. (iii) Let $\sharp \in \mathfrak{B}$. If $\sharp \in (\daleth\daleth_1\daleth_2]$, then $(\jmath_{(\daleth\daleth_1\daleth_2]})(\sharp, \iota) = \propto_2$.
Since $\sharp \leqslant \tau_1\tau_2\tau_3$ for some $\tau_1 \in (\daleth]$,$\tau_2 \in (\daleth_1]$ and $\tau_3 \in (\daleth_2]$. We have $(\tau_1, \tau_2, \tau_3) \in \daleth_\sharp$ and $\daleth_\sharp \neq 0$.

$$\begin{aligned}(\jmath^{\mathbb{T}}_{(\daleth]} \cdot \jmath^{\mathbb{T}}_{(\daleth_1]} \cdot \jmath^{\mathbb{T}}_{(\daleth_2]})(\sharp, \iota) &= \inf_{\sharp=\beta_1\beta_2\beta_3} \max\{\jmath^{\mathbb{T}}_{(\daleth]}(\beta_1, \iota), \jmath^{\mathbb{T}}_{(\daleth_1]}(\beta_2, \iota), \jmath^{\mathbb{T}}_{(\daleth_2]}(\beta_3, \iota)\}\\ &\leqslant \max\{\jmath^{\mathbb{T}}_{(\daleth]}(\tau_1, \iota), \jmath^{\mathbb{T}}_{(\daleth_1]}(\tau_2, \iota), \jmath^{\mathbb{T}}_{(\daleth_2]}(\tau_3, \iota)\}\\ &=\propto_2\end{aligned}$$

$$\begin{aligned}(\jmath^{\mathbb{F}}_{(\daleth]} \cdot \jmath^{\mathbb{F}}_{(\daleth_1]} \cdot \jmath^{\mathbb{F}}_{(\daleth_2]})(\sharp, \iota) &= \sup_{\sharp=\beta_1\beta_2\beta_3} \min\{\jmath^{\mathbb{F}}_{(\daleth]}(\beta_1, \iota), \jmath^{\mathbb{F}}_{(\daleth_1]}(\beta_2, \iota), \jmath^{\mathbb{F}}_{(\daleth_2]}(\beta_3, \iota)\}\\ &\geqslant \min\{\jmath^{\mathbb{F}}_{(\daleth]}(\tau_1, \iota), \jmath^{\mathbb{F}}_{(\daleth_1]}(\tau_2, \iota), \jmath^{\mathbb{F}}_{(\daleth_2]}(\tau_3, \iota)\}\\ &=\propto_1\end{aligned}$$

Therefore $(\jmath_{(\dashv]}\cdot\jmath_{(\dashv_1]}\cdot\jmath_{(\dashv_2]})(\sharp,\iota)=(\jmath_{(\dashv\dashv_1\dashv_2]})(\sharp,\iota)$.

If $\sharp\notin(\dashv\dashv_1\dashv_2]$ then $(\jmath^{\mathbb{T}}_{(\dashv\dashv_1\dashv_2]})(\sharp,\iota)=\propto_1$ and $(\jmath^{\mathbb{F}}_{(\dashv\dashv_1\dashv_2]})(\sharp,\iota)=\propto_2$. Since $\sharp\leqslant\tau_1\tau_2\tau_3$ for some $\tau_1\notin(\dashv]$, $\tau_2\notin(\dashv_1]$ and $\tau_3\notin(\dashv_2]$. We have

$$\begin{aligned}(\jmath^{\mathbb{T}}_{(\dashv]}\cdot\jmath^{\mathbb{T}}_{(\dashv_1]}\cdot\jmath^{\mathbb{T}}_{(\dashv_2]})(\sharp,\iota)&=\inf_{\sharp=\beta_1\beta_2\beta_3}\max\{\jmath^{\mathbb{T}}_{(\dashv]}(\beta_1,\iota),\jmath^{\mathbb{T}}_{(\dashv_1]}(\beta_2,\iota),\jmath^{\mathbb{T}}_{(\dashv_2]}(\beta_3,\iota)\}\\&\leqslant\max\{\jmath^{\mathbb{T}}_{(\dashv]}(\tau_1,\iota),\jmath^{\mathbb{T}}_{(\dashv_1]}(\tau_2,\iota),\jmath^{\mathbb{T}}_{(\dashv_2]}(\tau_3,\iota)\}\\&=\propto_1\end{aligned}$$

$$\begin{aligned}(\jmath^{\mathbb{F}}_{(\dashv]}\cdot\jmath^{\mathbb{F}}_{(\dashv_1]}\cdot\jmath^{\mathbb{F}}_{(\dashv_2]})(\sharp,\iota)&=\sup_{\sharp=\beta_1\beta_2\beta_3}\min\{\jmath^{\mathbb{F}}_{(\dashv]}(\beta_1,\iota),\jmath^{\mathbb{F}}_{(\dashv_1]}(\beta_2,\iota),\jmath^{\mathbb{F}}_{(\dashv_2]}(\beta_3,\iota)\}\\&\geqslant\min\{\jmath^{\mathbb{F}}_{(\dashv]}(\tau_1,\iota),\jmath^{\mathbb{F}}_{(\dashv_1]}(\tau_2,\iota),\jmath^{\mathbb{F}}_{(\dashv_2]}(\tau_3,\iota)\}\\&=\propto_2\end{aligned}$$

Hence $(\jmath_{(\dashv]}\cdot\jmath_{(\dashv_1]}\cdot\jmath_{(\dashv_2]})(\sharp,\iota)=(\jmath_{(\dashv\dashv_1\dashv_2]})(\sharp,\iota)$.

Theorem 2.15. *For $\dashv,\dashv_2\subseteq\mathfrak{B}$ and $\{\dashv_j|j\in J\}$ be a family of subsets of $\mathfrak{B}$. Then*
(i) $(\dashv]\subseteq(\dashv_1]$ if and only if $(\jmath_{(\dashv]})^{\propto_2}_{\propto_1}\leqslant(\jmath_{(\dashv_1]})^{\propto_2}_{\propto_1}$.
(ii) $(\bigcap_{j\in J}\jmath_{(\dashv_j]})^{\propto_2}_{\propto_1}=(\jmath_{\bigcap_{j\in J}(\dashv_j]})^{\propto_2}_{\propto_1}$.
(iii) $(\bigcup_{j\in J}\jmath_{(\dashv_j]})^{\propto_2}_{\propto_1}=(\jmath_{\bigcup_{j\in J}(\dashv_j]})^{\propto_2}_{\propto_1}$.

Theorem 2.16. *Let $\dashv$ be an $(\propto_1,\propto_2)$IQ1AFRI, $\dashv_1$ be an $(\propto_1,\propto_2)$IQ1AFLATI and $\dashv_2$ be an $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$ then $((\dashv\cdot\dashv_1\cdot\dashv_2])^{\propto_2}_{\propto_1}\subseteq(\dashv\cap\dashv_1\cap\dashv_2]^{\propto_2}_{\propto_1}$.*

Proof. Let $\dashv=[\pitchfork_{\dashv},\top_{\dashv}]$ be an $(\propto_1,\propto_2)$IQ1AFRI, $\dashv_1=[\pitchfork_{\dashv_1},\top_{\dashv_1}]$ be an $(\propto_1,\propto_2)$IQ1AFLATI and $\dashv_2=[\pitchfork_{\dashv_2},\top_{\dashv_2}]$ be an $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$. Let $(\sharp,\partial,\varnothing)\in X_\epsilon$. If $X_\epsilon\neq\emptyset$, then $\epsilon\leqslant\sharp\partial\varnothing$. Thus $\pitchfork_{\dashv}(\epsilon,\iota)\leqslant\pitchfork_{\dashv}(\sharp\partial\varnothing,\iota)\leqslant\pitchfork_{\dashv}(\sharp,\iota)$ and $\top_{\dashv}(\epsilon,\iota)\geqslant\top_{\dashv}(\sharp\partial\varnothing,\iota)\geqslant\top_{\dashv}(\sharp,\iota)$.
Similarly $\pitchfork_{\dashv_1}(\epsilon,\iota)\leqslant\pitchfork_{\dashv_1}(\sharp\partial\varnothing,\iota)\leqslant\pitchfork_{\dashv_1}(\partial,\iota)$ and $\top_{\dashv_1}(\epsilon,\iota)\geqslant\top_{\dashv_1}(\sharp\partial\varnothing,\iota)\geqslant\top_{\dashv_1}(\partial,\iota)$.
Similarly, $\pitchfork_{\dashv_2}(\epsilon,\iota)\leqslant\pitchfork_{\dashv_2}(\sharp\partial\varnothing,\iota)\leqslant\pitchfork_{\dashv_2}(\varnothing,\iota)$ and $\top_{\dashv_2}(\epsilon,\iota)\geqslant\top_{\dashv_2}(\sharp\partial\varnothing,\iota)\geqslant\top_{\dashv_2}(\varnothing,\iota)$.
We have

$$\begin{aligned}&(\pitchfork_{(\dashv\cdot\dashv_1\cdot\dashv_2]})^{\propto_2}_{\propto_1}(\epsilon,\iota)\\&=(\pitchfork_{(\dashv\cdot\dashv_1\cdot\dashv_2]}(\epsilon,\iota)\bigtriangledown\propto_2)\bigtriangleup\propto_1\\&=\Big[\big[\inf_{\epsilon\leqslant\sharp\partial\varnothing}\{\pitchfork_{\dashv}(\sharp,\iota)\bigtriangledown\pitchfork_{\dashv_1}(\partial,\iota)\bigtriangledown\pitchfork_{\dashv_2}(\varnothing,\iota)\}\bigtriangledown\propto_2\big]\Big]\bigtriangleup\propto_1\\&=\Big[\inf_{\epsilon\leqslant\sharp\partial\varnothing}\{\pitchfork_{\dashv}(\sharp,\iota)\bigtriangledown\pitchfork_{\dashv_1}(\partial,\iota)\bigtriangledown\pitchfork_{\dashv_2}(\varnothing,\iota)\}\bigtriangledown\propto_2\bigtriangledown\propto_2\bigtriangledown\propto_2\bigtriangledown\propto_2\Big]\bigtriangleup\propto_1\\&=\Big[\inf_{\epsilon\leqslant\sharp\partial\varnothing}\{(\pitchfork_{\dashv}(\sharp,\iota)\bigtriangledown\propto_2)\bigtriangledown(\pitchfork_{\dashv_1}(\partial,\iota)\bigtriangledown\propto_2)\bigtriangledown(\pitchfork_{\dashv_2}(\varnothing,\iota)\bigtriangledown\propto_2)\}\bigtriangledown\propto_2\Big]\bigtriangleup\propto_1\\&\geqslant(\{(\pitchfork_{\dashv}(\epsilon,\iota)\bigtriangleup\propto_1)\bigtriangledown(\pitchfork_{\dashv_1}(\epsilon,\iota)\bigtriangleup\propto_1)\bigtriangledown(\pitchfork_{\dashv_2}(\epsilon,\iota)\bigtriangleup\propto_1)\}\bigtriangledown\propto_2)\bigtriangleup\propto_1\\&=\{((\pitchfork_{\dashv}(\epsilon,\iota)\bigtriangledown\pitchfork_{\dashv_1}(\epsilon,\iota)\bigtriangledown\pitchfork_{\dashv_2}(\epsilon,\iota))\bigtriangleup\propto_1)\bigtriangledown\propto_2\}\bigtriangleup\propto_1\\&=\{((\pitchfork_{\dashv}\bigtriangledown\pitchfork_{\dashv_1}\bigtriangledown\pitchfork_{\dashv_2})(\epsilon,\iota)\bigtriangledown\propto_2\}\bigtriangleup\propto_1\\&=(\pitchfork_{\dashv\cap\dashv_1\cap\dashv_2})^{\propto_2}_{\propto_1}(\epsilon,\iota)\end{aligned}$$

$$(\top_{(\dashv\cdot\dashv_1\cdot\dashv_2]})^{\propto_2}_{\propto_1}(\epsilon,\iota)$$
$$= (\top_{(\dashv\cdot\dashv_1\cdot\dashv_2]}(\epsilon,\iota)\triangle \propto_2)\triangledown \propto_1$$
$$= \Big[\big[\sup_{\epsilon\leqslant\sharp\partial\o}\{\top_{\dashv}(\sharp,\iota)\triangle \top_{\dashv_1}(\partial,\iota)\triangle \top_{\dashv_2}(\o,\iota)\}\triangle \propto_2\big]\Big]\triangledown \propto_1$$
$$= \Big[\sup_{\epsilon\leqslant\sharp\partial\o}\{\top_{\dashv}(\sharp,\iota)\triangle \top_{\dashv_1}(\partial,\iota)\triangle \top_{\dashv_2}(\o,\iota)\}\triangle \propto_2 \triangle \propto_2 \triangle \propto_2 \triangle \propto_2\Big]\triangledown \propto_1$$
$$= \Big[\sup_{\epsilon\leqslant\sharp\partial\o}\{(\top_{\dashv}(\sharp,\iota)\triangle \propto_2)\triangle (\top_{\dashv_1}(\partial,\iota)\triangle \propto_2)\triangle (\top_{\dashv_2}(\o,\iota)\triangle \propto_2)\}\triangle \propto_2\Big]\triangledown \propto_1$$
$$\leqslant (\{(\top_{\dashv}(\epsilon,\iota)\triangledown \propto_1)\triangle (\top_{\dashv_1}(\epsilon,\iota)\triangledown \propto_1)\triangle (\top_{\dashv_2}(\epsilon,\iota)\triangledown \propto_1)\}\triangle \propto_2)\triangledown \propto_1$$
$$= \{((\top_{\dashv}(\epsilon,\iota)\triangle \top_{\dashv_1}(\epsilon,\iota)\triangle \top_{\dashv_2}(\epsilon,\iota))\triangledown \propto_1)\triangle \propto_2\}\triangledown \propto_1$$
$$= \{((\top_{\dashv}\triangle \top_{\dashv_1}\triangle \top_{\dashv_2})(\epsilon,\iota)\triangle \propto_2\}\triangledown \propto_1$$
$$= (\top_{\dashv\cup\dashv_1\cup\dashv_2})^{\propto_2}_{\propto_1}(\epsilon,\iota)$$

Let $\sharp,\partial,\o\notin X_\epsilon$. If $X_\epsilon=\emptyset$, then $(\pitchfork_{\dashv}\cdot\dashv_1\cdot\pitchfork_{\dashv_2})(\epsilon,\iota)=1$ and $(\top_{\dashv}\cdot\dashv_1\cdot\top_{\dashv_2})(\epsilon,\iota)=0$ such that $\epsilon\leqslant\sharp\partial\o$.

$$\begin{aligned}(\pitchfork_{(\dashv\cdot\dashv_1\cdot\dashv_2]})^{\propto_2}_{\propto_1}(\epsilon,\iota) &= (\pitchfork_{(\dashv\cdot\dashv_1\cdot\dashv_2]}(\epsilon,\iota)\triangledown \propto_2)\triangle \propto_1\\ &= 1\triangle \propto_1\\ &\geqslant (\pitchfork_{\dashv\cap\dashv_1\cap\dashv_2}(\epsilon,\iota)\triangledown \propto_2)\triangle \propto_1\\ &= (\pitchfork_{\dashv\cap\dashv_1\cap\dashv_2}(\epsilon,\iota)\triangledown \propto_2)\end{aligned}$$

$$\begin{aligned}(\top_{(\dashv\cdot\dashv_1\cdot\dashv_2]})^{\propto_2}_{\propto_1}(\epsilon,\iota) &= (\top_{(\dashv\cdot\dashv_1\cdot\dashv_2]}(\epsilon,\iota)\triangle \propto_2)\triangledown \propto_1\\ &= 0\triangledown \propto_1\\ &=\propto_1\\ &\leqslant (\top_{\dashv\cup\dashv_1\cup\dashv_2}(\epsilon,\iota)\triangle \propto_2)\triangledown \propto_1\\ &= (\top_{\dashv\cup\dashv_1\cup\dashv_2}(\epsilon,\iota)\triangle \propto_2)\end{aligned}$$

Therefore $((\dashv\cdot\dashv_1\cdot\dashv_2])^{\propto_2}_{\propto_1}\subseteq((\dashv\cap\dashv_1\cap\dashv_2])^{\propto_2}_{\propto_1}$.

Theorem 2.17. *An ordered -semigroup $\mathfrak{B}$ is regular, $\dashv$ be an $(\propto_1,\propto_2)$IQ1AFRI, $\dashv_1$ be an $(\propto_1,\propto_2)$IQ1AFLATI and $\dashv_2$ be an $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$ if and only if $((\dashv\cdot\dashv_1\cdot\dashv_2])^{\propto_2}_{\propto_1}=((\dashv\cap\dashv_1\cap\dashv_2])^{\propto_2}_{\propto_1}$.*

Proof. Let $\mathfrak{B}$ be an ordered -regular ternary semigroup and $\dashv$ be an $(\propto_1,\propto_2)$IQ1AFRI, $\dashv_1$ be an $(\propto_1,\propto_2)$IQ1AFLATI and $\dashv_2$ be an $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$. Let $(\sharp,\o)\in X_\epsilon$. If $X_\epsilon\neq\emptyset$, then $\epsilon\leqslant\sharp\partial\o$. Thus $\pitchfork_{\dashv}(\epsilon,\iota)\leqslant\pitchfork_{\dashv}(\sharp\partial\o,\iota)\leqslant\pitchfork_{\dashv}(\sharp,\iota)$ and $\top_{\dashv}(\epsilon,\iota)\geqslant\top_{\dashv}(\sharp\partial\o,\iota)\geqslant\top_{\dashv}(\sharp,\iota)$.
Similarly $\pitchfork_{\dashv_1}(\epsilon,\iota)\leqslant\pitchfork_{\dashv_1}(\sharp\partial\o,\iota)\leqslant\pitchfork_{\dashv_1}(\partial,\iota)$ and $\top_{\dashv_1}(\epsilon,\iota)\geqslant\top_{\dashv_1}(\sharp\partial\o,\iota)\geqslant\top_{\dashv_1}(\partial,\iota)$.
Similarly, $\pitchfork_{\dashv_2}(\epsilon,\iota)\leqslant\pitchfork_{\dashv_2}(\sharp\partial\o,\iota)\leqslant\pitchfork_{\dashv_2}(\o,\iota)$ and $\top_{\dashv_2}(\epsilon,\iota)\geqslant\top_{\dashv_2}(\sharp\partial\o,\iota)\geqslant\top_{\dashv_2}(\o,\iota)$.
For $\epsilon\in\mathfrak{B}$, there exists $x\in\mathfrak{B}$ such that $\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon$. Then $\epsilon,(\zeta_1\epsilon\zeta_2\epsilon\zeta_3),\epsilon\in X_\epsilon$. We have

$$(\pitchfork_{(\dashv\cdot\dashv_1\cdot\dashv_2]})^{\propto_2}_{\propto_1}(\epsilon,\iota)$$
$$= (\pitchfork_{(\dashv\cdot\dashv_1\cdot\dashv_2]}(\epsilon,\iota)\triangledown \propto_2)\triangle \propto_1$$
$$= \Big[\big[\inf_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon}\{\pitchfork_{\dashv}(\sharp,\iota)\triangledown \pitchfork_{\dashv_1}(\partial,\iota)\triangledown \pitchfork_{\dashv_2}(\o,\iota)\}\triangledown \propto_2\big]\Big]\triangle \propto_1$$
$$= \Big[\inf_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon}\{\pitchfork_{\dashv}(\sharp,\iota)\triangledown \pitchfork_{\dashv_1}(\partial,\iota)\triangledown \pitchfork_{\dashv_2}(\o,\iota)\}\triangledown \propto_2\triangledown \propto_2\triangledown \propto_2\triangledown \propto_2\Big]\triangle \propto_1$$
$$= \Big[\inf_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon}\{(\pitchfork_{\dashv}(\sharp,\iota)\triangledown \propto_2)\triangledown (\pitchfork_{\dashv_1}(\partial,\iota)\triangledown \propto_2)\triangledown (\pitchfork_{\dashv_2}(\o,\iota)\triangledown \propto_2)\}\triangledown \propto_2\Big]\triangle \propto_1$$
$$\leqslant (\{(\pitchfork_{\dashv}(\epsilon,\iota)\triangle \propto_1)\triangledown (\pitchfork_{\dashv_1}(\zeta_1\epsilon\zeta_2\epsilon\zeta_3)\triangle \propto_1)\triangledown (\pitchfork_{\dashv_2}(\epsilon,\iota)\triangle \propto_1)\}\triangledown \propto_2)\triangle \propto_1$$
$$\leqslant (\{(\pitchfork_{\dashv}(\epsilon,\iota)\triangle \propto_1)\triangledown (\pitchfork_{\dashv_1}(\epsilon,\iota)\triangle \propto_1)\triangledown (\pitchfork_{\dashv_2}(\epsilon,\iota)\triangle \propto_1)\}\triangledown \propto_2)\triangle \propto_1$$
$$= \{((\pitchfork_{\dashv}(\epsilon,\iota)\triangledown \pitchfork_{\dashv_1}(\epsilon,\iota)\triangledown \pitchfork_{\dashv_2}(\epsilon,\iota))\triangle \propto_1)\triangledown \propto_2\}\triangle \propto_1$$
$$= \{((\pitchfork_{\dashv}\triangledown \pitchfork_{\dashv_1}\triangledown \pitchfork_{\dashv_2})(\epsilon,\iota)\triangledown \propto_2\}\triangle \propto_1$$
$$= (\pitchfork_{\dashv\cap\dashv_1\cap\dashv_2})^{\propto_2}_{\propto_1}(\epsilon,\iota)$$

$$
\begin{aligned}
&(\top_{(\daleth\cdot\daleth_1\cdot\daleth_2]})_{\propto_1}^{\propto_2}(\epsilon,\iota)\\
&=(\top_{(\daleth\cdot\daleth_1\cdot\daleth_2]}(\epsilon,\iota)\triangle\propto_2)\bigtriangledown\propto_1\\
&=\Big[\big[\sup_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon}\{\top_{\daleth}(\sharp,\iota)\triangle\top_{\daleth_1}(\partial,\iota)\triangle\top_{\daleth_2}(\emptyset,\iota)\}\triangle\propto_2\big]\Big]\bigtriangledown\propto_1\\
&=\Big[\sup_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon}\{\top_{\daleth}(\sharp,\iota)\triangle\top_{\daleth_1}(\partial,\iota)\triangle\top_{\daleth_2}(\emptyset,\iota)\}\triangle\propto_2\triangle\propto_2\triangle\propto_2\triangle\propto_2\Big]\bigtriangledown\propto_1\\
&=\Big[\sup_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon}\{(\top_{\daleth}(\sharp,\iota)\triangle\propto_2)\triangle(\top_{\daleth_1}(\partial,\iota)\triangle\propto_2)\triangle(\top_{\daleth_2}(\emptyset,\iota)\triangle\propto_2)\}\triangle\propto_2\Big]\bigtriangledown\propto_1\\
&\geqslant(\{(\top_{\daleth}(\epsilon,\iota)\bigtriangledown\propto_1)\triangle(\top_{\daleth_1}(\zeta_1\epsilon\zeta_2\epsilon\zeta_3)\bigtriangledown\propto_1)\triangle(\top_{\daleth_2}(\epsilon,\iota)\bigtriangledown\propto_1)\}\triangle\propto_2)\bigtriangledown\propto_1\\
&\geqslant(\{(\top_{\daleth}(\epsilon,\iota)\bigtriangledown\propto_1)\triangle(\top_{\daleth_1}(\epsilon,\iota)\bigtriangledown\propto_1)\triangle(\top_{\daleth_2}(\epsilon,\iota)\bigtriangledown\propto_1)\}\triangle\propto_2)\bigtriangledown\propto_1\\
&=\{((\top_{\daleth}(\epsilon,\iota)\triangle\top_{\daleth_1}(\epsilon,\iota)\triangle\top_{\daleth_2}(\epsilon,\iota))\bigtriangledown\propto_1)\triangle\propto_2\}\bigtriangledown\propto_1\\
&=\{((\top_{\daleth}\triangle\top_{\daleth_1}\triangle\top_{\daleth_2})(\epsilon,\iota)\triangle\propto_2\}\bigtriangledown\propto_1\\
&=(\top_{\daleth\cup\daleth_1\cup\daleth_2})_{\propto_1}^{\propto_2}(\epsilon,\iota)
\end{aligned}
$$

Thus $((\daleth\cdot\daleth_1\cdot\daleth_2])_{\propto_1}^{\propto_2}\supseteq((\daleth\cap\daleth_1\cap\daleth_2])_{\propto_1}^{\propto_2}$ and by Theorem 2.16.
Hence $((\daleth\cdot\daleth_1\cdot\daleth_2])_{\propto_1}^{\propto_2}=((\daleth\cap\daleth_1\cap\daleth_2])_{\propto_1}^{\propto_2}$.
Conversely assume that $((\daleth\cdot\daleth_1\cdot\daleth_2])_{\propto_1}^{\propto_2}=((\daleth\cap\daleth_1\cap\daleth_2])_{\propto_1}^{\propto_2}$. Let $\daleth=(\pitchfork_{\daleth},\top_{\daleth})$ be an $(\propto_1,\propto_2)$IQ1AFRI, $\daleth_1=(\pitchfork_{\daleth_1},\top_{\daleth_1})$ be an $(\propto_1,\propto_2)$IQ1AFLATI and $\daleth_2=(\pitchfork_{\daleth_2},\Xi_{\daleth_2},\top_{\daleth_2})$ be an $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$. Then by Theorem 2.11, $\jmath_{\daleth}$ is a $(\propto_1,\propto_2)$IQ1AFRI, $\jmath_{\daleth_1}$ is a $(\propto_1,\propto_2)$IQ1AFLATI and $\jmath_{\daleth_2}$ be a $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$. By Lemma 2.14 and Theorem 2.15, $(\jmath_{(\daleth\cap\daleth_1\cap\daleth_2]})_{\propto_1}^{\propto_2}=(\jmath_{\daleth}\cap\jmath_{\daleth_1}\cap\jmath_{\daleth_2})_{\propto_1}^{\propto_2}=(\jmath_{\daleth}\cdot\jmath_{\daleth_1}\cdot\jmath_{\daleth_2})_{\propto_1}^{\propto_2}=(\jmath_{(\daleth\cdot\daleth_1\cdot\daleth_2]})_{\propto_1}^{\propto_2}$. This implies $(\daleth\cap\daleth_1\cap\daleth_2]_{\propto_1}^{\propto_2}=((\daleth\cdot\daleth_1\cdot\daleth_2])_{\propto_1}^{\propto_2}$. Hence by Corollary **??**, $\mathfrak{B}$ is regular.

Theorem 2.18. *An TSS $\mathfrak{B}$ is regular, $\daleth$ be an $(\propto_1,\propto_2)$IQ1AFBI, $\daleth_1$ be an $(\propto_1,\propto_2)$IQ1AFLATI and $\daleth_2$ be an $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$ if and only if $((\daleth\cdot\daleth_1\cdot\daleth_2])_{\propto_1}^{\propto_2}=((\daleth\cap\daleth_1\cap\daleth_2])_{\propto_1}^{\propto_2}$.*

Proof. Let $\mathfrak{B}$ be an TSS and $\daleth$ be an $(\propto_1,\propto_2)$IQ1AFBI and $\daleth_2$ be an $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$. Let $(\sharp,\emptyset)\in X_\epsilon$. If $X_\epsilon\neq\emptyset$, then $\epsilon\leqslant\sharp\partial\emptyset$. Thus $\pitchfork_{\daleth}(\epsilon,\iota)\leqslant\pitchfork_{\daleth}(\sharp\partial\emptyset,\iota)\leqslant\pitchfork_{\daleth}(\sharp,\iota)$ and $\top_{\daleth}(\epsilon,\iota)\geqslant\top_{\daleth}(\sharp\partial\emptyset,\iota)\geqslant\top_{\daleth}(\sharp,\iota)$.
Similarly $\pitchfork_{\daleth_1}(\epsilon,\iota)\leqslant\pitchfork_{\daleth_1}(\sharp\partial\emptyset,\iota)\leqslant\pitchfork_{\daleth_1}(\partial,\iota)$ and $\top_{\daleth_1}(\epsilon,\iota)\geqslant\top_{\daleth_1}(\sharp\partial\emptyset,\iota)\geqslant\top_{\daleth_1}(\partial,\iota)$.
Similarly, $\pitchfork_{\daleth_2}(\epsilon,\iota)\leqslant\pitchfork_{\daleth_2}(\sharp\partial\emptyset,\iota)\leqslant\pitchfork_{\daleth_2}(\emptyset,\iota)$ and $\top_{\daleth_2}(\epsilon,\iota)\geqslant\top_{\daleth_2}(\sharp\partial\emptyset,\iota)\geqslant\top_{\daleth_2}(\emptyset,\iota)$.
For $\epsilon\in\mathfrak{B}$, there exists $x\in\mathfrak{B}$ such that $\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon\zeta_4\epsilon\zeta_5\epsilon$.
Then $\epsilon\leqslant(\epsilon\zeta_1\epsilon\zeta_2\epsilon),(\zeta_3\epsilon\zeta_4\epsilon\zeta_5),\epsilon\in X_\epsilon$. We have

$$
\begin{aligned}
&(\pitchfork_{(\daleth\cdot\daleth_1\cdot\daleth_2]})_{\propto_1}^{\propto_2}(\epsilon,\iota)\\
&=(\pitchfork_{(\daleth\cdot\daleth_1\cdot\daleth_2]}(\epsilon,\iota)\bigtriangledown\propto_2)\triangle\propto_1\\
&=\Big[\big[\inf_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon\zeta_4\epsilon\zeta_5\epsilon}\{\pitchfork_{\daleth}(\sharp,\iota)\bigtriangledown\pitchfork_{\daleth_1}(\partial,\iota)\bigtriangledown\pitchfork_{\daleth_2}(\emptyset,\iota)\}\bigtriangledown\propto_2\big]\Big]\triangle\propto_1\\
&=\Big[\inf_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon\zeta_4\epsilon\zeta_5\epsilon}\{\pitchfork_{\daleth}(\sharp,\iota)\bigtriangledown\pitchfork_{\daleth_1}(\partial,\iota)\bigtriangledown\pitchfork_{\daleth_2}(\emptyset,\iota)\}\bigtriangledown\propto_2\bigtriangledown\propto_2\bigtriangledown\propto_2\bigtriangledown\propto_2\Big]\triangle\propto_1\\
&=\Big[\inf_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon\zeta_4\epsilon\zeta_5\epsilon}\{(\pitchfork_{\daleth}(\sharp,\iota)\bigtriangledown\propto_2)\bigtriangledown(\pitchfork_{\daleth_1}(\partial,\iota)\bigtriangledown\propto_2)\bigtriangledown(\pitchfork_{\daleth_2}(\emptyset,\iota)\bigtriangledown\propto_2)\}\bigtriangledown\propto_2\Big]\triangle\propto_1\\
&\leqslant(\{(\pitchfork_{\daleth}(\epsilon\zeta_1\epsilon\zeta_2\epsilon,\iota)\triangle\propto_1)\bigtriangledown(\pitchfork_{\daleth_1}(\zeta_3\epsilon\zeta_4\epsilon\zeta_5)\triangle\propto_1)\bigtriangledown(\pitchfork_{\daleth_2}(\epsilon,\iota)\triangle\propto_1)\}\bigtriangledown\propto_2)\triangle\propto_1\\
&\leqslant(\{(\pitchfork_{\daleth}(\epsilon,\iota)\triangle\propto_1)\bigtriangledown(\pitchfork_{\daleth_1}(\epsilon,\iota)\triangle\propto_1)\bigtriangledown(\pitchfork_{\daleth_2}(\epsilon,\iota)\triangle\propto_1)\}\bigtriangledown\propto_2)\triangle\propto_1\\
&=\{((\pitchfork_{\daleth}(\epsilon,\iota)\bigtriangledown\pitchfork_{\daleth_1}(\epsilon,\iota)\bigtriangledown\pitchfork_{\daleth_2}(\epsilon,\iota))\triangle\propto_1)\bigtriangledown\propto_2\}\triangle\propto_1\\
&=\{((\pitchfork_{\daleth}\bigtriangledown\pitchfork_{\daleth_1}\bigtriangledown\pitchfork_{\daleth_2})(\epsilon,\iota)\bigtriangledown\propto_2\}\triangle\propto_1\\
&=(\pitchfork_{\daleth\cap\daleth_1\cap\daleth_2})_{\propto_1}^{\propto_2}(\epsilon,\iota)
\end{aligned}
$$

$$
\begin{aligned}
&(\top_{(\dashv\cdot\dashv_1\cdot\dashv_2]})^{\propto_2}_{\propto_1}(\epsilon,\iota)\\
&=(\top_{(\dashv\cdot\dashv_1\cdot\dashv_2]}(\epsilon,\iota)\triangle\propto_2)\bigtriangledown\propto_1\\
&=\Big[[\sup_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon\zeta_4\epsilon\zeta_5\epsilon}\{\top_\dashv(\sharp,\iota)\triangle\top_{\dashv_1}(\partial,\iota)\triangle\top_{\dashv_2}(\emptyset,\iota)\}\triangle\propto_2]\Big]\bigtriangledown\propto_1\\
&=\Big[\sup_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon\zeta_4\epsilon\zeta_5\epsilon}\{\top_\dashv(\sharp,\iota)\triangle\top_{\dashv_1}(\partial,\iota)\triangle\top_{\dashv_2}(\emptyset,\iota)\}\triangle\propto_2\triangle\propto_2\triangle\propto_2\triangle\propto_2\Big]\bigtriangledown\propto_1\\
&=\Big[\sup_{\epsilon\leqslant\epsilon\zeta_1\epsilon\zeta_2\epsilon\zeta_3\epsilon\zeta_4\epsilon\zeta_5\epsilon}\{(\top_\dashv(\sharp,\iota)\triangle\propto_2)\triangle(\top_{\dashv_1}(\partial,\iota)\triangle\propto_2)\triangle(\top_{\dashv_2}(\emptyset,\iota)\triangle\propto_2)\}\triangle\propto_2\Big]\bigtriangledown\propto_1\\
&\geqslant(\{(\top_\dashv(\epsilon\zeta_1\epsilon\zeta_2\epsilon,\iota)\bigtriangledown\propto_1)\triangle(\top_{\dashv_1}(\zeta_3\epsilon\zeta_4\epsilon\zeta_5)\bigtriangledown\propto_1)\triangle(\top_{\dashv_2}(\epsilon,\iota)\bigtriangledown\propto_1)\}\triangle\propto_2)\bigtriangledown\propto_1\\
&\geqslant(\{(\top_\dashv(\epsilon,\iota)\bigtriangledown\propto_1)\triangle(\top_{\dashv_1}(\epsilon,\iota)\bigtriangledown\propto_1)\triangle(\top_{\dashv_2}(\epsilon,\iota)\bigtriangledown\propto_1)\}\triangle\propto_2)\bigtriangledown\propto_1\\
&=\{((\top_\dashv(\epsilon,\iota)\triangle\top_{\dashv_1}(\epsilon,\iota)\triangle\top_{\dashv_2}(\epsilon,\iota))\bigtriangledown\propto_1)\triangle\propto_2\}\bigtriangledown\propto_1\\
&=\{((\top_\dashv\triangle\top_{\dashv_1}\triangle\top_{\dashv_2})(\epsilon,\iota)\triangle\propto_2\}\bigtriangledown\propto_1\\
&=(\top_{\dashv\cup\dashv_1\cup\dashv_2})^{\propto_2}_{\propto_1}(\epsilon,\iota)
\end{aligned}
$$

Thus, $((\dashv\cdot\dashv_1\cdot\dashv_2])^{\propto_2}_{\propto_1}\supseteq((\dashv\cap\dashv_1\cap\dashv_2])^{\propto_2}_{\propto_1}$ and by Theorem 2.16 and hence $((\dashv\cdot\dashv_1\cdot\dashv_2])^{\propto_2}_{\propto_1}=((\dashv\cap\dashv_1\cap\dashv_2])^{\propto_2}_{\propto_1}$. Conversely assume that $((\dashv\cdot\dashv_1\cdot\dashv_2])^{\propto_2}_{\propto_1}=((\dashv\cap\dashv_1\cap\dashv_2])^{\propto_2}_{\propto_1}$. Let $\dashv=(\pitchfork_\dashv,\top_\dashv)$ be an $(\propto_1,\propto_2)$IQ1AFBI, $\dashv_1=(\pitchfork_{\dashv_1},\Xi_{\dashv_1},\top_{\dashv_1})$ be an $(\propto_1,\propto_2)$IQ1AFLATI and $\dashv_2=(\pitchfork_{\dashv_2},\top_{\dashv_2})$ be an $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$. Then by Theorem 2.11, $\jmath_\dashv$ is a $(\propto_1,\propto_2)$IQ1AFBI, $\jmath_{\dashv_1}$ is a $(\propto_1,\propto_2)$IQ1AFLATI and $\jmath_{\dashv_2}$ be a $(\propto_1,\propto_2)$IQ1AFLI of $\mathfrak{B}$. By Lemma 2.14 and Theorem 2.15, $(\jmath_{(\dashv\cap\dashv_1\cap\dashv_2]})^{\propto_2}_{\propto_1}=(\jmath_\dashv\cap\jmath_{\dashv_1}\cap\jmath_{\dashv_2})^{\propto_2}_{\propto_1}=(\jmath_\dashv\cdot\jmath_{\dashv_1}\cdot\jmath_{\dashv_2})^{\propto_2}_{\propto_1}=(\jmath_{(\dashv\cdot\dashv_1\cdot\dashv_2]})^{\propto_2}_{\propto_1}$. This implies $(\dashv\cap\dashv_1\cap\dashv_2]^{\propto_2}_{\propto_1}=((\dashv\cdot\dashv_1\cdot\dashv_2])^{\propto_2}_{\propto_1}$. Hence by Corollary **??**, $\mathfrak{B}$ is regular.

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