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Fractional Difference Equations with Initial Time Difference

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Abstract:

In this paper, we consider non-linear fractional difference equations at different initial times and establish the existence of solutions using monotone iterative technique.

Keywords: fractional order, monotone iterative technique

1. Introduction.

Fractional calculus gained importance during the past three decades due to its applicability in diverse fields of science and engineering. The notions of fractional calculus may be traced back to the works of Euler, but the idea of fractional difference is very recent. G.V.S.R. Deekshitulu and J. Jagan Mohan [7] modified the definition of fractional difference given by Nagai [9] and discussed some basic inequalities, comparison theorems and qualitative properties of the solutions of fractional difference equations [2,3,4,5,7].

In the study of initial and boundary value problems most of the times, it is assumed that the independent variable is unchanged and the dependent variable or space variable is perturbed [10,11,12]. But in real world problems, it is almost impossible to measure the initial time or initial value of space variable with out at any error.

It is difficult to compare any two solutions if the initial times are different. This has attracted many mathematicians to study the corresponding problems. Though some literature is available on fractional differential equations with initial time difference, not much of work has been yet donein discrete case.

In this paper, using lower solutions and upper solutions starting at different initial times, comparison and existence results for fractional difference equations are established.

2. Fractional difference equations with initialtime difference

In this section, we consider the following Initial value problem (IVP)

$$\nabla^{\mu}\rho(n+1) = f\big(n, \rho(n)\big), \;\; \rho(n_0) = \rho_0, \qquad (2.1)$$

For n $\in N_{n_0}^+$, $n_0 \geq 0$.

Theorem 2.1. [6] Let $\xi, \omega : N_0^+ \rightarrow R$ be L.S and U.S of (2.1). Further assume that f (n, r), for r$\in$ R is strictly non decreasing in r for each n. Then $\xi(n) \leq \omega(n)$ provided $\xi(0) \leq \omega(0)$.

Theorem 2.2.[6] Suppose that $\nabla^{\mu} m(n+1) \leq -Mm(n)$ for $M>0$ and $m(0) \leq m(N+1)$, then $m(n) \leq 0$.

Theorem 2.3. Assume that

1.$\xi(n)$ and $\omega(n)$ be such that

$$\nabla^{\mu}\xi(n+1) \leq f\big(n, \xi(n)\big), \qquad \xi(n_0) \leq \rho_0, \quad n \in N_{n_0}^+$$

$$\nabla^{\mu}\omega(n+1) \geq f\big(n, \omega(n)\big), \qquad \omega(s_0) \geq \rho_0, \qquad n \in N_{s_0}^+$$

with $\xi(n_0) \leq \omega(s_0)$

2. For $x \geq y$, $M>0$, $f(n, x) - f(n, y) \leq -M(x-y)$.

3. $s_0 > n_0$ and f (n, ρ) is nondecreasing in n for each ρ.

Then

a. $\xi(n) \leq \omega(n+\eta)$, $n \geq n_0$,

b. $\xi(n-\eta) \leq \omega(n)$, $n \geq s_0$, and $\eta = s_0 - n_0$.

Proof. (a) Let $\tilde{\omega}(n) = \omega(n+\eta)$, $n \geq N_{n_0}^+$,

$\tilde{\omega}(n_0) = \omega(n_0+\eta) = \omega(s_0) \geq \rho_0$.

Since f is nondecreasing in n for each ρ.

Also $\nabla^{\mu}\tilde{\omega}(n+1) = \nabla^{\mu}\tilde{\omega}(n+\eta+1) = f(n+\eta, \omega(n+\eta)) \geq f(n, \tilde{\omega}(n))$.

Consolidating the above,

$\nabla^{\mu}\tilde{\omega}(n+1) \geq f(n, \tilde{\omega}(n))$, $\tilde{\omega}(n_0) \geq \rho_0$.

$\nabla^{\mu}\xi(n+1) \leq f(n, \xi(n))$, $\xi(n_0) \leq \rho_0$.

Using Theorem 2.1, clearly

$\xi(n) \leq \tilde{\omega}(n) = \omega(n+\eta)$.

(b) Let $\tilde{\xi}(n) = \xi(n-\eta)$, $n \geq N_{n_0}^+$

For n = s_0, $\tilde{\xi}(s_0) = \xi(s_0-\eta) = \xi(n_0) \leq \rho_0$.

Now, since f is non decreasing in n,

$\nabla^{\mu}\tilde{\xi}(n+1) = f(n-\eta, \xi(n-\eta)) \leq f(n, \tilde{\xi}(n))$.

Consolidating the above,

$\nabla^{\mu}\tilde{\xi}(n+1) \leq f(n, \tilde{\xi}(n))$, $\tilde{\xi}(s_0) \leq \rho_0$,

$\nabla^{\mu}\omega(n+1) \geq f(n, \omega(n)), \omega(n_0) \geq \rho_0$.

By using Theorem 2.1, $\xi(n-\eta) = \tilde{\xi}(n) \leq \omega(n)$, for $n \geq s_0$.

Here we shall discuss the monotone iterative method to obtain extremal solutions for Fractional difference equations with initialtime difference.

Theorem 2.4. Assume that

1. $\nabla^{\mu}\xi(n+1) \leq f(n, \xi(n)), \xi(n_0) = \rho_0, n \in N_{n_0}^{+}$, $n_0 \geq 0$,

$\nabla^{\mu}\omega(n+1) \geq f(n, \omega(n)), \omega(s_0) = \rho_0, n \in N_{s_0}^{+}$,

such that $\xi_0 \leq \omega_0$,

2. $f(n, x) - f(n, y) \geq -M(x-y)$, where $M > 0$ and $y \leq x$,

3. $s_0 > n_0 > 0$ and $f(n, \rho(n))$ is nondecreasing n for each $\rho(n)$ and $\xi(n) \leq \omega(n+\eta)$, $\eta = s_0 - n_0$.

Then there exit monotone sequences $\tilde{\xi}_n$, $\tilde{\omega}_n$ which converge uniformly and monotonically for $n \in N_{n_0}^{+}$ such that $\tilde{\xi}_m \rightarrow \tilde{\xi}$ and $\tilde{\omega}_m \rightarrow \tilde{\omega}$ for $m \rightarrow \infty$. Moreover $\tilde{\xi}$ and $\tilde{\omega}$ are minimal and maximal solutions of (2.1).

Proof: Let $\tilde{\omega}_0(n) = \omega(n+\eta)$ and $\tilde{\xi}_0(n) = \xi(n)$ for $n \in N_{n_0}^{+}$ where $\eta = s_0 - n_0$.

Let $\Theta: N_{n_0}^{+} \rightarrow R$ be such that $\tilde{\xi}_0(n) \leq \Theta \leq \tilde{\xi}_0(n)$, and the following linear

fractional difference equation for $0 < \mu < 1$,

$$\nabla^{\mu}\rho(n+1) = f(n, \Theta(n)) - M[\rho(n) - \Theta(n)], \rho(n_0) = \rho_0. \quad (2.2)$$

It is clear that, if $\Theta(n) = \rho(n)$, $\rho(n)$ is the unique solution of (3.1) on $n \in N_0^{+}$.

If $\Theta(n) \neq \rho(n)$, the non-homogeneous fractional difference equation (2.2) has unique solution ρ_n [1]. In order to construct monotone sequences $\tilde{\xi}(n)$, $\tilde{\omega}(n)$, a mapping A: R → R is defined such that $A\Theta(n) = \rho(n)$.

Now the following properties of A are proved below.

(a). $\tilde{\xi}_0(n) \leq A\tilde{\xi}_0(n)$, $\tilde{\omega}_0(n) \geq A\tilde{\omega}_0(n)$.

(b). A is monotone operator on $[\tilde{\xi}_0, \tilde{\omega}_0] = \{\rho(n)/ \tilde{\xi}_0 \leq \rho(n) \leq \tilde{\omega}_0\}$.

To prove (a), set $A\tilde{\xi}_0(n) = \tilde{\xi}_1(n)$, where $\tilde{\xi}_1(n)$ is the unique solution of (2.2).

with $\Theta(n) = \tilde{\xi}_0(n)$.

Set $p(n) = \tilde{\xi}_0(n) - \tilde{\xi}_1(n)$.

Consider
$$\begin{aligned}\nabla^{\mu}p(n+1) &= \nabla^{\mu}[\tilde{\xi}_0(n+1) - \tilde{\xi}_1(n+1)] \\ &\leq f(n, \tilde{\xi}_0(n)) - f(n, \tilde{\xi}_0(n)) + M[\tilde{\xi}_1(n) - \tilde{\xi}_0(n)] \\ &= -M[\tilde{\xi}_0(n) - \tilde{\xi}_1(n)]\end{aligned}$$

or $\nabla^{\mu}p(n+1) \leq -M\,p(n)$.

Also, $\rho(n_0) = \tilde{\xi}_0(n_0) - \tilde{\xi}_1(n_0) \le \rho_0 - \rho_0 \le 0$.

By using Theorem 2.2, $p(n) \le 0$ or $\tilde{\xi}_0(n) \le \tilde{\xi}_1(n) = A\tilde{\xi}_0(n)$.

Similarly, it can be proved that $\tilde{\omega}_0(n) \ge A\{\tilde{\omega}_0(n)\}$.

To prove (b), Let $k \in N_0^+$ and Θ_k and $\Theta_{k+1} \in [\tilde{\xi}_0(n), \tilde{\omega}_0(n)]$ such that $\Theta_k \le \Theta_{k+1}$.

Suppose $A\{\Theta_k(n)\} = \tilde{\xi}_k(n)$ and $A\{\Theta_{k+1}(n)\} = \tilde{\xi}_{k+1}(n)$.

Take $q(n) = \tilde{\xi}_k - \tilde{\xi}_{k+1}$. Consider

$$\nabla^\mu q(n+1) = \nabla^\mu [\tilde{\xi}_k(n+1) - \tilde{\xi}_{k+1}(n+1)]$$

$$= f(n, \Theta_k(n)) - M[\tilde{\xi}_k(n) - \Theta_k(n)] - f(n, \Theta_{k+1}(n)) + M[\tilde{\xi}_{k+1}(n) - \Theta_{k+1}(n)]$$

$$= M[\Theta_k(n) - \Theta_{k+1}(n)] + M[\tilde{\xi}_{k+1}(n) - \tilde{\xi}_k(n)] + f(n, \Theta_k) - f(n, \Theta_{k+1})$$

$$\le -M\, q(n)$$

Also, $q(n_0) = \tilde{\xi}_k(n_0) - \tilde{\xi}_{k+1}(n_0) \le \rho_0 - \rho_0 \le 0$.

By using Theorem 2.2, hence $q(n) \le 0$ or $\tilde{\xi}_k(n) \le \tilde{\xi}_{k+1}(n) = A\tilde{\xi}_k(n)$.

Similarly, it can be proved that $\omega_k(n) \ge \omega_{k+1}(n)$.

that means, a sequence can be constructed as

$$\tilde{\xi}_0(n) \le \tilde{\xi}_1(n) \le \tilde{\xi}_2(n) \le \dots\dots\dots\dots \le \tilde{\xi}_m(n) \le \tilde{\omega}_m(n) \le \dots\dots\dots\dots \le \tilde{\omega}_2(n) \le \tilde{\omega}_1(n) \le \tilde{\omega}_0(n)$$

for $n \in N_{n_0}^+$.

By Dini's theorem as $m \to \infty$, $\tilde{\xi}_m \to \tilde{\xi}$ and $\tilde{\omega}_m \to \tilde{\omega}$ where $\tilde{\xi}$ and $\tilde{\omega}$ are any two functions defined for $n \in N_{n_0}^+$.

Also $\tilde{\xi}_m(n)$ and $\tilde{\omega}_m(n)$ satisfy

$$\nabla^\mu \tilde{\xi}_m(n+1) = f(n, \tilde{\xi}_{m-1}(n) - M[\tilde{\xi}_m(n) - \tilde{\xi}_{m-1}(n)], \ \tilde{\xi}_m(n_0) = \rho_0, \quad (2.3)$$

$$\nabla^\mu \tilde{\omega}_m(n+1) = f(n, \tilde{\omega}_{m-1}(n)) - M[\tilde{\omega}_m(n) - \tilde{\omega}_{m-1}(n)], \ \tilde{\omega}_m(n_0) = \rho_0 \quad (2.4)$$

As $m \to \infty$, $\tilde{\xi}(n) \to \tilde{\xi}$ and $\tilde{\omega}(n) \to \tilde{\omega}$,

Hence

$$\nabla^\mu \tilde{\xi} = f(n, \tilde{\xi}), \ \tilde{\xi}(n_0) = \rho_0, \quad (2.5)$$

$$\nabla^\mu \tilde{\omega} = f(n, \tilde{\omega}), \ \tilde{\omega}(n_0) = \rho_0. \quad (2.6)$$

It implies that the functions $\tilde{\xi}$ and $\tilde{\omega}$ defined on $n \in N_{n_0}^+$ are solutions of (2.1). Now to prove that $\tilde{\xi}$ and $\tilde{\omega}$ are minimal and maximal solutions of (2.1) respectively,

it is sufficient to prove that $\tilde{\xi} \le \rho(n) \le \tilde{\omega}$ (2.7)

If $\rho(n)$ is any solution of (2.1) such that

$$\tilde{\xi}_0 \le \rho \le \tilde{\omega}_0. \quad (2.8)$$

Let the statement

$\tilde{\xi}_k \leq \rho(n) \leq \tilde{\omega}_k$. (2.9)

be true. Let $\tilde{\xi}_k$ satisfy (2.2) and consider $r(n) = \tilde{\xi}_{k+1}(n) - \rho(n)$.

Consider

$\nabla^\mu r(n+1) = \nabla^\mu [\tilde{\xi}_{k+1}(n+1) - \rho(n+1)]$

$= f(n, \tilde{\xi}_k(n)) - f(n, \rho(n) - M[\tilde{\xi}_{k+1}(n) - \tilde{\xi}_k(n)]$

$\leq -M[\tilde{\xi}_k(n) - \rho(n)] - M[\tilde{\xi}_{k+1}(n) - \tilde{\xi}_k(n)]$

$= -Mr(n)$.

Also, $r(n_0) = \tilde{\xi}_{k+1}(n_0) - \rho(n_0) \leq \rho_0 - \rho_0 \leq 0$.

By using Theorem 2.2, we have $\tilde{\xi}_{k+1}(n) \leq \rho(n)$. Similarly, it can be proved

that $\rho(n) \leq \omega_{k+1}(n)$.

Therefore $\tilde{\xi}_{k+1}(n) \leq \rho(n) \leq \omega_{k+1}(n)$. Hence the statement (2.9) is true for m = k + 1.

By the principle of mathematical induction, the statement (2.9) is true for every $m \in N_0^+$

Thus $\tilde{\xi}_m(n) \leq \rho(n) \leq \tilde{\omega}_m(n)$. Taking the limit as $m \to \infty$, we get $\tilde{\xi} \leq \rho(n) \leq \tilde{\omega}$ such that $\tilde{\xi}_0(n) \leq \rho(n) \leq \tilde{\omega}_0$. Hence the functions $\tilde{\xi}$ and $\tilde{\omega}$ defined for $n \in N_{n_0}^+$ are minimal and maximal solutions of (2.1) respectively.

Theorem 2.5. In addition to the hypothesis of Theorem (2.4), if we assume that

$f(n, x) - f(n, y) \geq -M(x - y)$, (2.10)

for $\xi \leq y \leq x \leq \omega$ and $M \geq 0$. Then $\xi = \omega = \rho$ is the unique solution of (2.1).

Proof: Since $\xi(n) \leq \omega(n)$, it is enough to prove that $\xi(n) \geq \omega(n)$. Take

$s(n) = \xi(n) - \omega(n)$.

Consider

$\nabla^\mu s(n+1) = \nabla^\mu [\xi(n+1) - \omega(n+1)]$

$= -(f(n, \omega(n)) - f(n, \xi(n)))$

$\leq M[\omega(n) - \xi(n)]$

$= -Ms(n)$.

Also $r(n_0) = \xi(n_0) - \rho(n_0) \leq \rho_0 - \rho_0 \leq 0$.

By using Theorem2.2, hence $s(n) \leq 0$ for $n \in N_{n_0}^+$.

Thus $\omega(n) \leq \xi(n)$. Hence $\xi(n) = \omega(n) = \rho(n)$ is the unique solution of (2.1).

Conclusion

In this chapter, using monotone iterative technique on solutions of fractional difference equations at different initial times the convergence of monotone sequences to maximum and minimum solutions is established. Using lower and upper solutions, existence and uniqueness of solution to fractional difference equations at different initial times is also obtained.

Future work: This work can be extended onto the systems of Fractional Difference equations This work may make better some processes concerned with signal processing and image processing, system trajectories and various other fields.

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