

On (α, β) -Total Edge Irregularity Strength of Some Graphs

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Abstract:

Let $V(G)$, and $E(G)$ having cardinality l and m respectively, symbolize the vertex set, and edge set of a simple graph G together with a total \hbar - labeling $\ell: \mathcal{M} \rightarrow \{1, 2, 3, \dots, \hbar\}$ where $\mathcal{M} = V(G) \cup E(G)$. Let $\mathbb{C} = \{\alpha, \alpha + \beta, \alpha + 2\beta, \dots, \alpha + (m - 1)\beta\}$. If an invertible function exists, say $\mathcal{F}: E(G) \rightarrow \mathbb{C}$ specified by $\mathcal{F}(ij) = \ell(i) + \ell(j) + \ell(ij)$ for every edge ij belongs to $E(G)$ with $\alpha \geq 3, \beta \geq 2$, then ℓ is termed as (α, β) - total edge irregular labeling. Also, the value $\mathcal{F}(ij)$ is said to be the edge weight of ij . The least \hbar for which G admits (α, β) - edge irregular \hbar is indicated by (α, β) - $tes(G)$, known as (α, β) - Total Edge irregularity Strength of the graph G . The precise value of (α, β) - $tes(G)$ for standard graph families and Mycielskian path graphs is explored in this work. In addition, an open problem of $(3, 2)$ - $tes(P_n^k)$ and $(3, 2)$ - $tes(T)$ are solved partially.

Keywords: (α, β) - Irregular labeling, Edge irregular labeling, Irregular labeling, Irregularity strength, Total edge irregular labeling.

1. Introduction

This paper considers finite simple undirected graphs. Under certain conditions, graph labeling refers to assigning numerical values to graph components like vertices, edges, or both. These specifications described by means of some evaluating function values (weights). Chartrand et al. [1] initially proposed irregular graph labeling as follows: Let $G = (V, E)$ be a connected graph of order at least 3 having edge \hbar - labeling $\mathcal{g}: E \rightarrow \{1, 2, \dots, \hbar\}$ and $w(v) = \sum_{e \in E} \mathcal{g}(e)$, where $w(v)$ denotes vertex v weight. When the vertex weights for all vertices differ, this is termed as irregular labeling. The irregularity strength $s(G)$ is the least natural number \hbar for which G reveals irregular graph labeling. Baca et al. [2] reshaped irregular labeling as total vertex irregular labeling and total edge irregular labeling. The weight of an edge $e = st \in E$ is defined as $w(e) = \mathcal{f}(s) + \mathcal{f}(t) + \mathcal{f}(e)$ where $\mathcal{f}: \mathcal{D} \rightarrow \mathcal{Q}$ is a function with $\mathcal{D} = V \cup E$ and $\mathcal{Q} = \{\tau: 1 \leq \tau \leq m \text{ and } \tau \in \mathbb{N}\}$. If every one of the edge weights are distinct, \mathcal{f} is referred to as a Total Edge Irregular labeling. The Total Edge irregularity Strength $tes(G)$ is the least positive integer m for which total edge irregular labeling $\mathcal{f}: \mathcal{D} \rightarrow \mathcal{Q}$ exists. For a centralized uniform theta graphs, Riyan [3] computed the total edge irregularity

strength (TES). Susanti Y, et al. [4] predicted asymmetric graphs and symmetric graphs $tes(G)$. Rosyida I, et al. [5] determined the exact value of $tes(G)$ for n -uniform cactus chain graphs. Muthu Guru Packiam [6] introduced the concept of $(\alpha, \beta) - tes(G)$ as follows: Consider a simple graph $G = (V, E)$ on l vertices and m edges together with a total h -labeling ρ or $\rho: V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, h\}$. Then ρ is called (α, β) -total edge irregular labeling if there exists a one-to-one correspondence, say $\psi: E(G) \rightarrow \{\alpha, \alpha + \beta, \alpha + 2\beta, \dots, \alpha + (m - 1)\beta\}$ defined by $\psi(uv) = \rho(u) + \rho(v) + \rho(uv)$ for all $uv \in E(G)$, where $\alpha \geq 3, \beta \geq 2$. Also, the value $\psi(uv)$ is said to be the edge weight of uv . The least h for which G admits $(\alpha, \beta) -$ edge irregular h is indicated by $(\alpha, \beta) - tes(G)$, known as $(\alpha, \beta) -$ Total Edge irregularity Strength of the graph G . Additionally, they presented upper and lower limitations for the parameter and established the evaluations of $(\alpha, \beta) - tes(G)$ for some graph families. The purpose of this paper is to focus on the study of $(\alpha, \beta) - tes(G)$. Subdivision of a graph G that results from dividing each edge by a vertex [7]. In a graph G , attaching two pendant vertices to each all vertices of G is denoted by $G \circ S_2$. Middle graph of a graph $G = (V, E)$ denoted by $M(G)$ has vertex set $V \cup E$ and a pair of vertices are adjacent if and only if they are incident with one other or adjacent edges. Any two adjacent vertices of G are adjacent in the middle graph of G is named as Total graph $T(G)$ of a graph G . The vertices of path graph P_n are v_1, v_2, \dots, v_n . Graph P_n^3 can be constructed from P_n by connecting every v_i^{th} vertex to v_{i+2}^{th} vertex and every v_i^{th} vertex to the v_{i+3}^{th} vertex by an edge. If all unordered pairs of n vertices are adjacent in G is termed as complete graph [8] denoted by K_n . If a vertex set is separated like U and V with $|U| = a$ and $|V| = b$ such that every vertex of U is adjacent to every vertex of V is referred as $K_{a,b}$ - Complete bipartite graph. In case $a = 1$ or $b = 1$ in $K_{a,b}$ is named as star graph $K_{1,b}$ or $K_{a,1}$. The central vertex of two copies $K_{1,n}$ is joined by an edge is $B(n, n)$ - Bistar graph. Mycielskian [9] $\mu(G)$ of a graph $G = (V, E)$ has vertex set $V \cup V' \cup \{u\}$, where $V' = \{v_i' : v_i \in V\}$ and edge set $E \cup \{v_i v_j' : v_i v_j \in E\} \cup \{v_i' u : v_i' \in V'\}$.

Proposition 1 [6]: Let G be a graph with q edges and p vertices. Then $\left\lfloor \frac{\alpha+(q-1)\beta}{3} \right\rfloor \leq (\alpha, \beta) - tes(G) \leq \alpha - 2 + (q - 1)\beta$ for any two natural number $\alpha \geq 3$ and $\beta \geq 2$.

Proposition 2 [10]: $(3, 2) - tes(P_n) = \left\lfloor \frac{2n-1}{3} \right\rfloor$ for path P_n .

2. Results and Discussion

Theorem 1: $(3, 2) - tes(S(P_n \circ S_2)) = 4n - 1$.

Proof. Consider the index sets $I = \{1, 2, \dots, n\}$ and $J = \{1, 2, \dots, n - 1\}$. Let the vertex set of $S(P_n \circ S_2) = \cup_{\sigma=1}^6 V_\sigma$ where $V_1 = \{u_i\}, V_2 = \{v_i\}, V_3 = \{w_i\}, V_4 = \{x_j\}, V_5 = \{y_i\}$ and $V_6 = \{z_i\}$ and edge set of $(S(P_n \circ S_2)) = \cup_{\varphi=1}^6 E_\varphi$ where $E_1 = \{u_i x_i\}, E_2 = \{x_j u_{j+1}\}, E_3 = \{u_i y_i\}, E_4 = \{y_i v_i\}, E_5 = \{u_i z_i\}$ and $E_6 = \{z_i w_i\}$ for every $i \in I$ and $j \in J$.

Total labeling $\rho: \mathfrak{V} \rightarrow \mathfrak{N}$ where $\mathfrak{V} = V(S(P_n \circ S_2)) \cup E(S(P_n \circ S_2))$ and $\mathfrak{N} = \{1, 2, \dots, 4n - 1\}$ is given below:

$$\rho(u_1) = 3, \quad \rho(u_i) = 4i - 3, \quad \rho(v_i) = 4i - 1, \quad i \neq 1.$$

$$\rho(v_1) = 1,$$

$$\rho(w_i) = 4i - 1, \quad \rho(y_i) = 4i - 3, \quad \rho(z_i) = 4i - 1,$$

$$\begin{aligned} \rho(x_j) &= 4j + 1, \\ \rho(u_1x_1) &= 3, \\ \rho(u_ix_i) &= 1 + 4i, \quad i \neq 1. \\ \rho(x_iu_{i+1}) &= -1 + 4i, \\ \rho(u_iy_i) &= \rho(y_iv_i) = -3 + 4i, \\ \rho(u_1z_1) &= 1, \rho(u_iz_i) = 4i - 1, \quad \rho(z_iw_i) = -1 + 4i, \quad i \neq 1. \end{aligned}$$

Induced edge weight function $\psi: E(S(P_n \circ S_2)) \rightarrow \{3, 5, 7, \dots, 12n - 3\}$ is given by

$$\begin{aligned} \psi(u_ix_i) &= -1 + 12i, \quad \psi(x_iu_{i+1}) = 1 + 12i, \\ \psi(u_iy_i) &= 12i - 9, \quad \psi(y_iv_i) = 12i - 7, \quad i \neq 1 \\ \psi(u_1y_1) &= 5, \quad \psi(y_1v_1) = 3, \\ \psi(u_iz_i) &= 12i - 5, \quad \psi(z_iw_i) = 12i - 3, \end{aligned}$$

Therefore, the induced edge weights of $S(P_n \circ S_2)$ differs by 2 in the arithmetic progression. Thus, $(3, 2)\text{-tes}(S(P_n \circ S_2)) \leq -1 + 4n$.

Proposition 1 confirms that $(3, 2)\text{-tes}(S(P_n \circ S_2)) \geq -1 + 4n$, this concludes the proof.

Example 1: $(3, 2)$ total edge irregular labeling for $S(P_5 \circ S_2)$ is shown in Fig 1.

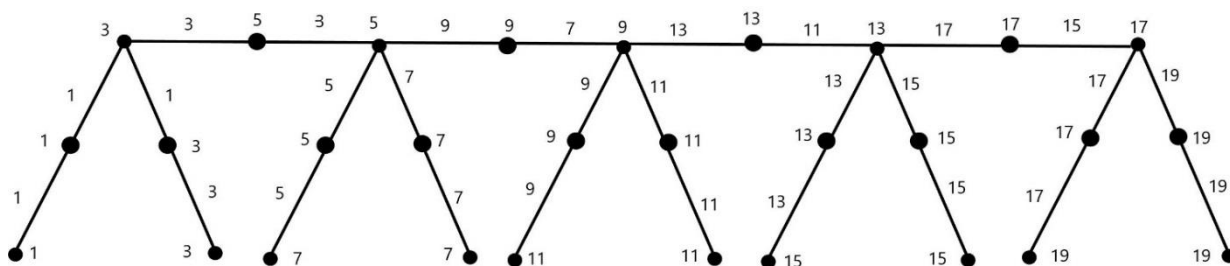


Figure 1. $(3, 2)\text{-tes}(S(P_5 \circ S_2)) = 19$.

Remark 1:

When $n = 1, P_1 \circ S_2 \cong P_5$ and hence it follows from the proposition 2. Requisition

Theorem 2: $(3, 2)\text{-tes}(M(P_n)) = 2n - 2$.

Proof. Consider the index sets $\mathbb{I} = \{1, 2, \dots, n\}$ and $\mathbb{J} = \{1, 2, \dots, n - 1\}$. Let $V(M(P_n)) = V_1 \cup V_2$ where $V_1 = \{u_i\}$ and $V_2 = \{v_j\}$ and let $E(M(P_n)) = \{v_\ell v_{\ell+1}\} \cup \{u_\ell v_\ell\} \cup \{v_\ell u_{\ell+1}\}$ denotes the vertex set and edge set of $M(P_n)$ for all $i \in \mathbb{I}, j, \ell \in \mathbb{J}$ and $\ell \in \mathbb{I} - \{n - 1, n\}$.

Total labeling $\rho: \mathcal{A} \rightarrow \mathcal{B}$ where $\mathcal{A} = V(M(P_n)) \cup E(M(P_n))$ and $\mathcal{B} = \mathbb{I} \cup \{n + 1, n + 2, \dots, 2n - 2\}$ is prescribed by: $\rho(v_1) = 1, \rho(v_2) = 3, \rho(v_i) = 2i, \quad i = 3, 4, 5, \dots, n$

$$\rho(u_1) = 1, \rho(u_2) = 3, \rho(u_i) = -2 + 2i, \quad 3 \leq i \leq n$$

$$\rho(v_1v_2) = 3, \quad \rho(v_2v_3) = 4, \rho(v_iv_{i+1}) = -1 + 2i, \quad i = 3, 4, 5, \dots, n - 2$$

$$\rho(u_jv_j) = -1 + 2j,$$

$$\rho(v_1u_2) = 1, \rho(v_2u_3) = 4, \quad \rho(v_iu_{i+1}) = -1 + 2i, \quad i = 3, 4, 5, \dots, n - 1$$

Induced edge weight function $\psi: E(M(P_n)) \rightarrow \{3, 5, 7, \dots, 6n - 7\}$ is given by

$$\psi(v_\ell v_{\ell+1}) = 1 + 6\ell, \quad \ell \in \mathbb{I} - \{n-1, n\}$$

$$\psi(u_j v_j) = -3 + 6j, \quad \psi(v_j u_{j+1}) = -1 + 6j$$

Therefore, the induced edge weights of $S(M(P_n))$ differs by 2 in the arithmetic progression.

Hence, $(3, 2)\text{-tes}(M(P_n)) \leq -2 + 2n$.

Proposition 1 reveals that $(3, 2)\text{-tes}(M(P_n)) \geq -2 + 2n$. Hence the result follows.

Example 2: $(3, 2)\text{-tes}(M(P_8))$ is shown in Fig 2.

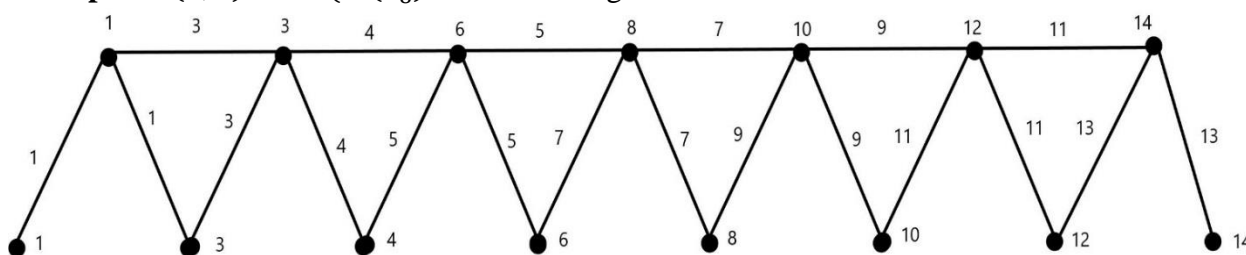


Figure 2. $(3, 2)\text{-tes}(M(P_8)) = 14$.

Theorem 3: $(3, 2)\text{-tes}(T(P_n)) = \left\lfloor \frac{8n}{3} \right\rfloor - 3$ for every positive integer n at least 3.

Proof. Consider the index sets $\mathbb{I} = \{1, 2, \dots, n\}$ and $\mathbb{J} = \{1, 2, \dots, n-1\}$. Let $V(T(P_n)) = \mathbb{A}_1 \cup \mathbb{V}_2$ where $\mathbb{A}_1 = \{a_i\}$ and $\mathbb{V}_2 = \{v_j\}$ and $E(T(P_n)) = \{a_j a_{j+1}\} \cup \{v_t v_{t+1}\} \cup \{a_j v_j\} \cup \{v_j a_{j+1}\}$ denotes the vertex set and edge set of $T(P_n)$ for all $i \in \mathbb{I}$, $j \in \mathbb{J}$ and $t \in \mathbb{J} - \{n-1\}$.

Total labeling $\rho: \mathcal{C} \rightarrow \mathcal{D}$ where $\mathcal{C} = V(T(P_n)) \cup E(T(P_n))$ and $\mathcal{D} = \{1, 2, \dots, \left\lfloor \frac{8n}{3} \right\rfloor - 3\}$ is represented by:

Case i Suppose $n \equiv 0 \pmod{3}$

$$\rho(v_1) = 1, \rho(v_2) = 5,$$

$$\rho(v_{3i}) = 8i - 1,$$

$$\rho(v_{3i+1}) = 3 + 8i, \quad \rho(v_{3i+2}) = 5 + 8i,$$

$$\rho(a_1) = 1, \rho(a_2) = 3,$$

$$\rho(a_{3i}) = -3 + 8i,$$

$$\rho(a_{3i+1}) = 8i + 1,$$

$$\rho(a_{3i+2}) = 3 + 8i,$$

$$\rho(v_1 v_2) = 3, \rho(v_{3i-1} v_{3i}) = 8i - 3,$$

$$\rho(v_{3i} v_{3i+1}) = 8i - 1,$$

$$\rho(v_{3i+1} v_{3i+2}) = 8i + 1,$$

$$\rho(a_1 a_2) = 1,$$

$$\mathcal{P}(a_{3i-1} a_{3i}) = -3 + 8i,$$

$$\mathcal{P}(a_{3i} a_{3i+1}) = -1 + 8i,$$

$$\mathcal{P}(a_{3i+1} a_{3i+2}) = 8i + 1,$$

$$\mathcal{P}(a_1 v_1) = 1, \mathcal{P}(a_{3i} v_{3i}) = \mathcal{P}(a_{3i+1} v_{3i+1}) = -1 + 8i,$$

$$\mathcal{P}(a_{3i-1} v_{3i-1}) = -5 + 8i,$$

$$\mathcal{P}(v_1 a_2) = 3, \mathcal{P}(v_{3i-1} a_{3i}) = -3 + 8i,$$

$$\mathcal{P}(v_{3i} a_{3i+1}) = 8i - 1,$$

$$\mathcal{P}(v_{3i+1} a_{3i+2}) = 8i + 1,$$

Case ii Let $n \equiv 1 \pmod{3}$

$$\mathcal{P}(v_1) = 1, \mathcal{P}(v_{3i-1}) = 8i - 3,$$

$$\mathcal{P}(v_{3i}) = 8i - 1,$$

$$\mathcal{P}(v_{3i+1}) = 3 + 8i,$$

$$\mathcal{P}(a_1) = 1, \mathcal{P}(a_2) = 3,$$

$$\mathcal{P}(a_n) = \frac{8}{3}(n - 1)$$

$$\mathcal{P}(a_{3i}) = -3 + 8i,$$

$$\mathcal{P}(a_{3i+1}) = 8i + 1,$$

$$\mathcal{P}(a_{3i+2}) = 3 + 8i,$$

$$\mathcal{P}(v_1 v_2) = 3, \mathcal{P}(v_{3i-1} v_{3i}) = -3 + 8i,$$

$$\mathcal{P}(v_{3i} v_{3i+1}) = -1 + 8i,$$

$$\mathcal{P}(v_{3i+1} v_{3i+2}) = 1 + 8i,$$

$$\mathcal{P}(a_1 a_2) = 1,$$

$$\mathcal{P}(a_{n-1} a_n) = \frac{8}{3}(n - 1)$$

$$\mathcal{P}(a_{3i-1} a_{3i}) = -3 + 8i,$$

$$\mathcal{P}(a_{3i} a_{3i+1}) = 8i - 1,$$

$$\mathcal{P}(a_{3i+1} a_{3i+2}) = 1 + 8i,$$

$$\mathcal{P}(v_1 a_2) = 3, \mathcal{P}(v_{3i-1} a_{3i}) = -3 + 8i,$$

$$\mathcal{P}(v_{n-1} a_n) = \frac{8}{3}(n - 1),$$

$$\mathcal{P}(v_{3i} a_{3i+1}) = 8i - 1,$$

$$p(v_{3i+1} a_{3i+2}) = 1 + 8i,$$

$$p(a_1 v_1) = 1$$

$$p(a_{3i-1} v_{3i-1}) = -5 + 8i,$$

$$p(a_{3i} v_{3i}) = -1 + 8i,$$

$$p(a_{3i+1} v_{3i+1}) = -1 + 8i,$$

Case iii Consider $n \equiv 2 \pmod{3}$

$$p(v_1) = 1, p(v_{3i-1}) = -3 + 8i,$$

$$p(v_{3i}) = 8i - 1,$$

$$p(v_{3i+1}) = 3 + 8i,$$

$$p(a_1) = 1$$

$$p(a_{3i-1}) = -5 + 8i,$$

$$p(a_{3i}) = -3 + 8i,$$

$$p(a_{3i+1}) = 8i + 1,$$

$$p(v_1 v_2) = 3, p(v_{3i-1} v_{3i}) = -3 + 8i,$$

$$p(v_{3i} v_{3i+1}) = 8i - 1,$$

$$p(v_{3i+1} v_{3i+2}) = 8i + 1,$$

$$p(a_1 a_2) = 1$$

$$p(a_{3i-1} a_{3i}) = -3 + 8i,$$

$$p(a_{3i} a_{3i+1}) = -1 + 8i,$$

$$p(a_{3i+1} a_{3i+2}) = 8i + 1,$$

$$p(a_1 v_1) = 1$$

$$p(a_{3i-1} v_{3i-1}) = -5 + 8i,$$

$$p(a_{3i} v_{3i}) = p(a_{3i+1} v_{3i+1}) = 8i - 1,$$

$$p(v_1 a_2) = 3, p(v_{3i-1} a_{3i}) = -3 + 8i,$$

$$p(v_{3i} a_{3i+1}) = 8i - 1,$$

$$p(v_{3i+1} a_{3i+2}) = 8i + 1.$$

Induced edge weight function $\psi: E(T(P_n)) \rightarrow \{3, 5, 7, \dots, 8n - 9\}$ is given by

$$\psi(a_j a_{j+1}) = -3 + 8j,$$

$$\psi(v_t v_{t+1}) = 1 + 8t,$$

$$\psi(a_j v_j) = -5 + 8j,$$

$$\psi(v_j a_{j+1}) = 8j - 1,$$

Therefore, the induced edge weights of $T(P_n)$ generates an arithmetic progression and it differs by 2 and hence $(3, 2)$ -tes($T(P_n)$) $\leq \left\lceil \frac{8n}{3} \right\rceil - 3$.

From Proposition 1, we get $(3, 2)$ -tes($T(P_n)$) $\geq \left\lceil \frac{8n}{3} \right\rceil - 3$ and hence the result follows.

Example 3: $(3, 2)$ -tes($T(P_9)$) is given in Fig 3.

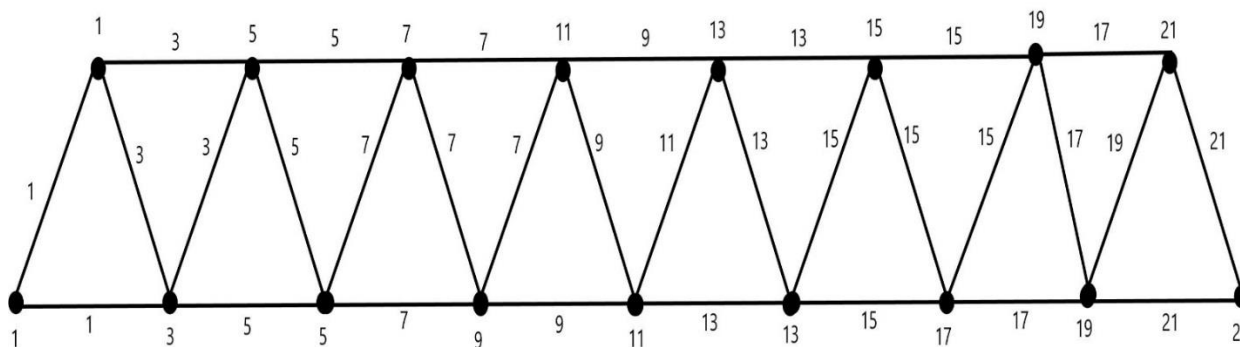


Figure 3. $(3, 2)$ -tes($T(P_9)$) = 21.

Remark 2:

When $n = 2$, $T(P_2) \cong C_3$ and hence $(3, 2)$ -tes($T(P_2)$) = 3.

$(3, 2)$ total edge irregular labeling for $T(P_9)$ is shown in Fig 4.

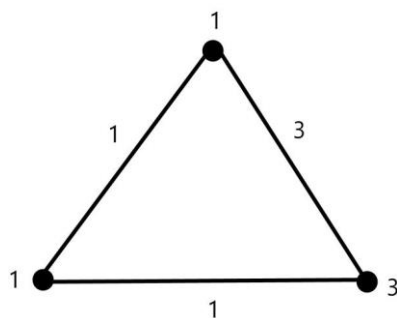


Figure 4. $(3, 2)$ -tes($T(P_2)$) = 3.

Theorem 4: $(3, 2)$ -tes($P_n \circ K_2$) = $\left\lceil \frac{8(n+1)}{3} \right\rceil - 3$.

Proof. Consider the index sets $\mathbb{I} = \{1, 2, \dots, n\}$ and $\mathbb{J} = \{1, 2, \dots, n - 1\}$. Let $V(P_n \circ K_2) = \cup_{n=1}^3 V_n$ where $V_1 = \{s_i\}$, $V_2 = \{v_i\}$ and $V_3 = \{t_i\}$ and let $E(P_n \circ K_2) = \{s_j s_{j+1}\} \cup \{s_i v_i\} \cup \{v_i t_i\} \cup \{s_i t_i\}$ denotes the vertex set and edge of $P_n \circ K_2$ respectively for all $i \in \mathbb{I}$, $j \in \mathbb{J}$.

Total labeling $\rho: \mathcal{L} \rightarrow \mathcal{M}$ where $\mathcal{L} = V(P_n \circ K_2) \cup E(P_n \circ K_2)$ and $\mathcal{M} = \left\{1, 2, \dots, \left\lceil \frac{8(n+1)}{3} \right\rceil - 3\right\}$ is described by:

Case i Consider $n \equiv 0 \pmod{3}$

$$\mathcal{P}(s_n) = \frac{8n}{3}, \mathcal{P}(v_n) = \frac{8n}{3} - 3, \mathcal{P}(t_n) = \frac{8n}{3}$$

$$\mathcal{P}(s_{3i}) = 1 + 8i,$$

$$\mathcal{P}(s_{3i-1}) = -3 + 8i,$$

$$\mathcal{P}(s_{3i-2}) = 8i - 5,$$

$$\mathcal{P}(v_{3i}) = -3 + 8i,$$

$$\mathcal{P}(v_{3i-1}) = 8i - 5,$$

$$\mathcal{P}(v_{3i-2}) = 8i - 7,$$

$$\mathcal{P}(t_1) = 1$$

$$\mathcal{P}(t_{3i}) = 8i + 1,$$

$$\mathcal{P}(t_{3i-1}) = -3 + 8i,$$

$$\mathcal{P}(t_{3i-2}) = 8i - 5, i \neq 1$$

$$\mathcal{P}(s_{3i-2} s_{3i-1}) = -7 + 8i,$$

$$\mathcal{P}(s_{3i-1} s_{3i}) = 8i - 5,$$

$$\mathcal{P}(s_{3i} s_{3i+1}) = 8i - 3,$$

$$\mathcal{P}(s_1 v_1) = 1$$

$$\mathcal{P}(s_{3i-1} v_{3i-1}) = 8i - 5,$$

$$\mathcal{P}(s_{3i} v_{3i}) = 8i - 3,$$

$$\mathcal{P}(s_{3i-2} v_{3i-2}) = 8i - 9, i \neq 1$$

$$\mathcal{P}(v_{3i-2} t_{3i-2}) = -7 + 8i,$$

$$\mathcal{P}(v_{3i-1} t_{3i-1}) = -3 + 8i,$$

$$\mathcal{P}(v_{3i} t_{3i}) = 8i - 1,$$

$$\mathcal{P}(s_1 t_1) = 3$$

$$\mathcal{P}(s_{3i-1} t_{3i-1}) = 8i - 3,$$

$$\mathcal{P}(s_{3i} t_{3i}) = 8i - 3,$$

$$\mathcal{P}(s_{3i-2} t_{3i-2}) = 8i - 7, i \neq 1$$

$$\mathcal{P}(s_{n-1} s_n) = \frac{8n}{3} - 4, \mathcal{P}(s_n v_n) = \frac{8n}{3} - 2, \mathcal{P}(s_n t_n) = \frac{8n}{3} - 1, \mathcal{P}(v_n t_n) = \frac{8n}{3}$$

Case ii If $n - 1 \equiv 0 \pmod{3}$

$$\mathcal{P}(v_{3i}) = -3 + 8i,$$

$$\mathcal{P}(v_{3i-1}) = 8i - 5,$$

$$\mathcal{P}(v_{3i-2}) = 8i - 7,$$

$$p(s_{3i}) = 1 + 8i,$$

$$p(s_{3i-1}) = -3 + 8i,$$

$$p(s_{3i-2}) = 8i - 5,$$

$$p(t_1) = 1$$

$$p(t_{3i}) = 8i + 1,$$

$$p(t_{3i-1}) = -3 + 8i,$$

$$p(t_{3i-2}) = -5 + 8i, \quad i \neq 1$$

$$p(s_{3i-1} s_{3i}) = 8i - 5,$$

$$p(s_{3i-2} s_{3i-1}) = -7 + 8i,$$

$$p(s_{3i} s_{3i+1}) = 8i - 3,$$

$$p(s_1 v_1) = 1$$

$$p(s_{3i-1} v_{3i-1}) = 8i - 5,$$

$$p(s_{3i} v_{3i}) = 8i - 3,$$

$$p(s_{3i-2} v_{3i-2}) = 8i - 9,$$

$$p(v_{3i-2} t_{3i-2}) = -7 + 8i,$$

$$p(v_{3i-1} t_{3i-1}) = -3 + 8i,$$

$$p(v_{3i} t_{3i}) = 8i - 1, \quad p(s_1 t_1) = 3$$

$$p(s_{3i-1} t_{3i-1}) = 8i - 3,$$

$$p(s_{3i} t_{3i}) = 8i - 3,$$

$$p(s_{3i-2} t_{3i-2}) = 8i - 7, \quad i \neq 1$$

Case iii If $n \equiv 2 \pmod{3}$

$$p(s_{3i-2}) = 8i - 5,$$

$$p(s_{3i-1}) = -3 + 8i,$$

$$p(s_{3i}) = 1 + 8i,$$

$$p(v_{3i-2}) = -7 + 8i,$$

$$p(v_{3i-1}) = 8i - 5,$$

$$p(v_{3i}) = -3 + 8i,$$

$$p(t_1) = 1$$

$$p(t_{3i}) = 8i + 1,$$

$$p(t_{3i-1}) = -3 + 8i,$$

$$p(t_{3i-2}) = -5 + 8i, \quad i \neq 1$$

$$p(s_{3i} s_{3i+1}) = 8i - 3,$$

$$p(s_{3i-1} s_{3i}) = 8i - 5,$$

$$p(s_{3i-2} s_{3i-1}) = -7 + 8i,$$

$$p(s_1 v_1) = 1$$

$$p(s_{3i-1} v_{3i-1}) = 8i - 5,$$

$$p(s_{3i} v_{3i}) = 8i - 3,$$

$$p(s_{3i-2} v_{3i-2}) = 8i - 9, \quad i \neq 1$$

$$p(v_{3i-2} t_{3i-2}) = -7 + 8i,$$

$$p(v_{3i-1} t_{3i-1}) = -3 + 8i,$$

$$p(v_{3i} t_{3i}) = 8i - 1,$$

$$p(s_1 t_1) = 3$$

$$p(s_{3i-1} t_{3i-1}) = 8i - 3,$$

$$p(s_{3i} t_{3i}) = 8i - 3,$$

$$p(s_{3i-2} t_{3i-2}) = 8i - 7, \quad i \neq 1$$

Induced weight function for edge $\psi: E(P_n \circ K_2) \rightarrow \{3, 5, 7, \dots, 8n - 1\}$ is given by

$$\psi(s_j s_{j+1}) = 1 + 8j,$$

$$\psi(s_1 v_1) = 5$$

$$\psi(s_i v_i) = -5 + 8i, \quad i \neq 1$$

$$\psi(s_i t_i) = -1 + 8i,$$

$$\psi(v_1 t_1) = 3$$

$$\psi(v_i t_i) = -3 + 8i, \quad i \neq 1$$

Therefore, the induced edge weights of $P_n \circ K_2$ differs by 2 in the arithmetic progression.

$$\text{Thus, } (3, 2)\text{-tes}(P_n \circ K_2) \leq \left\lceil \frac{8(n+1)}{3} \right\rceil - 3.$$

$$\text{Proposition 1 exhibits that } (3, 2)\text{-tes}(P_n \circ K_2) \geq \left\lceil \frac{8(n+1)}{3} \right\rceil - 3.$$

Example 4: $(3, 2)\text{-tes}(P_8 \circ K_2)$ is shown in Fig 5.

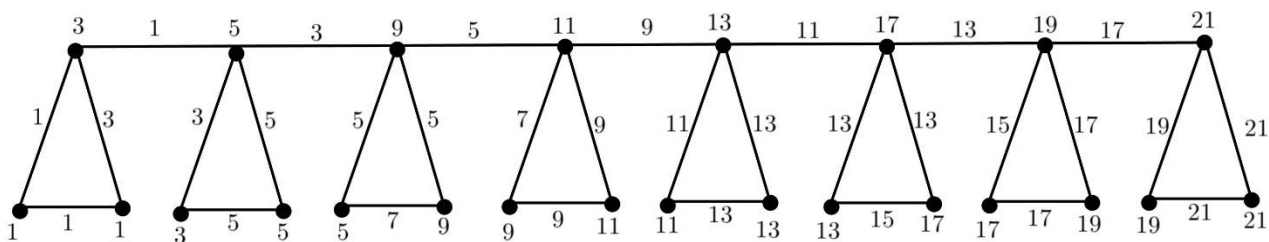


Figure 5. $(3, 2)$ - $tes(P_8 \circ K_2) = 21$.

Theorem 5: $(3, 2)$ - $tes(P_n^3) = 2n - 3$ for every positive integer n more than 3.

Proof. Consider the index set $\mathbb{I} = \{1, 2, \dots, n\}$. Let $V(P_n^3) = \{v_i\}$ and let $E(P_n^3) = \cup_{n=1}^3 E_n$ where $E_1 = \{v_i v_{i+1}\}$, $E_2 = \{v_i v_{i+2}\}$ and $E_3 = \{v_i v_{i+3}\}$ denotes the vertex set and edge set of P_n^3 for all $i \in \mathbb{I}$.

Total labeling $\rho: \mathcal{P} \rightarrow \mathcal{Q}$ where $\mathcal{P} = V(P_n^3) \cup E(P_n^3)$ and $\mathcal{Q} = \{1, 2, \dots, 2n - 3\}$ is predicted by:

$$\rho(v_1) = 1, \quad f(v_2) = 1,$$

$$\rho(v_i) = 2i - 3, \quad i \neq 1, 2.$$

$$\rho(v_i v_{i+1}) = \begin{cases} 1 & i = 1 \\ -1 + 2i & i \neq 1 \end{cases}$$

$$\rho(v_i v_{i+2}) = \begin{cases} 1 & i = 1 \\ 1 + 2i & i \neq 1 \end{cases}$$

$$\rho(v_i v_{i+3}) = \begin{cases} 3 & i = 1 \\ 2i + 3 & i \neq 1 \end{cases}$$

Induced edge weight function $\psi: E(P_n^3) \rightarrow \{3, 5, 7, \dots, 6n - 11\}$ is given by

$$\psi(v_1 v_2) = 3$$

$$\psi(v_i v_{i+1}) = -5 + 6i, \quad i \neq 1$$

$$\psi(v_i v_{i+2}) = -1 + 6i,$$

$$\psi(v_i v_{i+3}) = 3 + 6i,$$

Therefore, induced edge weights of P_n^3 differs by 2 in arithmetic progression.

Thus, $(3, 2)$ - $tes(P_n^3) \leq -3 + 2n$.

By Proposition 1, we get $(3, 2)$ - $tes(P_n^3) \geq -3 + 2n$. This concludes the proof.

Example 5: $(3, 2) - tes(P_7^3)$ is shown in Fig 6.

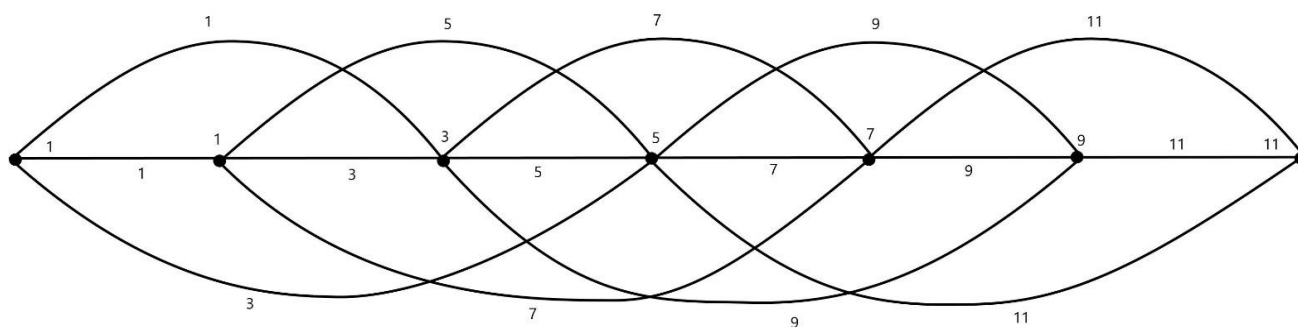


Figure 6. $(3, 2)\text{-tes}(P_7^3) = 11$.

Theorem 6: If T is a tree other than star on p vertices, then $(3, 2)\text{-tes}(T) = \left\lfloor \frac{2p-1}{3} \right\rfloor$.

Proof.

Step i: Arrange the given tree from left to right by considering one pendant as starting node.

Step ii: Label the edges of tree from left to right by completing lower branch and then upper branch using the induced edge weights $\{3, 5, 7, \dots, 2p - 1\}$.

Step iii: Each pair of vertices and its edge should be labeled by at most odd numbers which are less than or equal to either the corresponding induced edge weights or $\left\lfloor \frac{2p-1}{3} \right\rfloor$ with non-decreasing order.

Step iv: From the above labeling, $(3, 2)\text{-tes}(T) \leq \left\lfloor \frac{2p-1}{3} \right\rfloor$.

Proposition 1 exhibits that $(3, 2)\text{-tes}(T) \geq \left\lfloor \frac{2p-1}{3} \right\rfloor$ and hence the result follows.

Example 6: $(3, 2)$ total edge irregular labeling for tree with 10 vertices is shown in Fig 7.

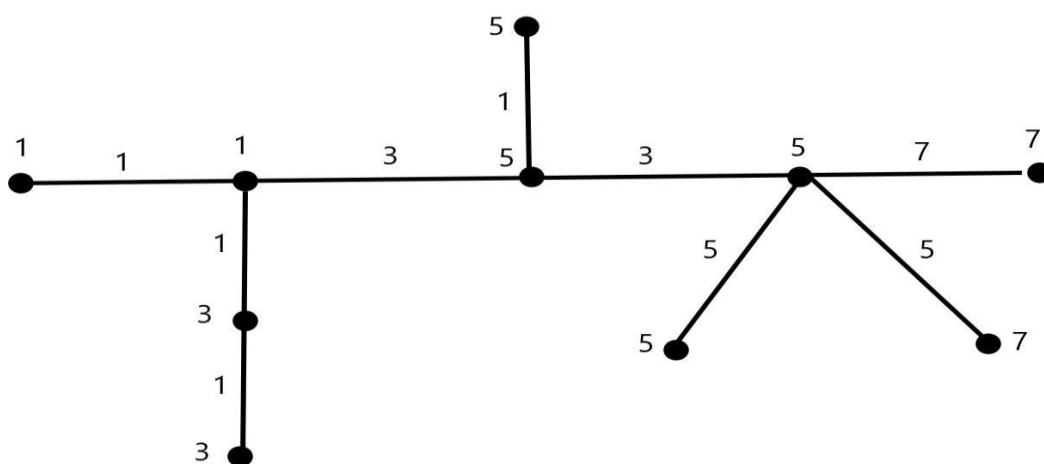


Figure 7. $(3, 2)\text{-tes}(T) = 7$

Theorem 7: $(3, 2)\text{-tes}(K_n) = n^2 - 5n + 9$ for $n \geq 4$.

Proof. Consider the index set $\mathbb{I} = \{1, 2, \dots, n\}$. Let $V(K_n) = \{u_i\}$ and let $E(K_n) = \{u_i u_{i+1}\} \cup \{u_i u_j / 1 \leq i < j \leq n\}$ denotes the vertex set and edge set of K_n for all $i \in \mathbb{I}$.

Total labeling $\rho: \mathcal{P} \rightarrow \mathcal{Q}$ where $\mathcal{P} = V(K_n) \cup E(K_n)$ and $\mathcal{Q} = \{1, 2, \dots, n^2 - 5n + 9\}$ is given below:

$$\rho(u_1) = 1$$

$$\rho(u_i) = -3 + 2i, \quad i \neq 1$$

$$\rho(u_1 u_i) = 1,$$

$$\rho(u_i u_j) = 2ni - 2n - (i^2 + 3i - 7), \quad i \neq 1, n - 1, i + 1 \leq j \leq n,$$

$$\rho(u_{n-1} u_n) = n^2 - 5n + 9$$

Induced weight function for edge $\psi: E(K_n) \rightarrow \{3, 5, 7, \dots, 1 - n + n^2\}$ is given by

$$\rho(u_1 u_i) = -1 + 2i,$$

$$\rho(u_{n-1} u_n) = 1 - n + n^2$$

$$\rho(u_i u_j) = 2ni - 2n - (3 + i - i^2) - 2 - 2i + 2j, \quad i \neq 1, n - 1, i + 1 \leq j \leq n,$$

The induced edge weights of K_n differs by 2 in arithmetic progression.

Hence, $(3, 2)\text{-tes}(K_n) \leq 9 - 5n + n^2$.

Further it is not possible to obtain $(3, 2)\text{-total edge irregular labelling}$ by assigning label fewer than $n^2 - 5n + 9$, hence $(3, 2)\text{-tes}(K_n) = n^2 - 5n + 9$.

Example 7: $(3, 2)\text{-tes}(K_6)$ is given in Fig 8.

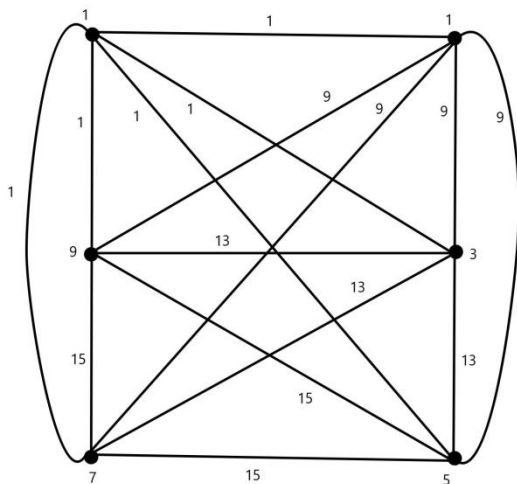


Figure 8. $(3, 2)\text{-tes}(K_6) = 15$.

Remark 2:

When $n = 3$, $(3, 2)\text{-tes}(K_3) = 3$, total edge irregular labelling for K_3 is shown in Fig 9.

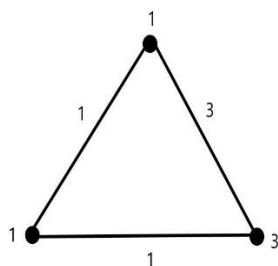


Figure 9. $(3, 2)\text{-tes}(K_3) = 3$.

Corollary 1.

If a simple connected graph G has p vertices and q edges, then

$$\left\lfloor \frac{2q+1}{3} \right\rfloor \leq (3, 2)\text{-tes}(G) \leq 9 - 5p + p^2.$$

Theorem 8: $(3, 2)\text{-tes}(\mu(P_n)) = \left\lfloor \frac{8n}{3} \right\rfloor - 1$ for $n \geq 3$.

Proof. Consider the index set $\mathbb{I} = \{1, 2, \dots, n\}$. The vertex set and edge set of P_n are \mathcal{V}_1 and \mathcal{E}_1 respectively. Let $V(\mu(P_n)) = \cup_{\varphi=1}^3 \mathcal{V}_\varphi$ and $E(\mu(P_n)) = \cup_{\sigma=1}^4 \mathcal{E}_\sigma$ denotes the vertex set and edge set of $\mu(P_n)$ where $\mathcal{V}_1 = \{x_i\}$, $\mathcal{V}_2 = \{x'_i\}$, $\mathcal{V}_3 = \{w\}$, $\mathcal{E}_1 = \{x_i x_{i+1}\}$, $\mathcal{E}_2 = \{x'_i x'_{i+1}\}$, $\mathcal{E}_3 = \{x_i x'_{i+1}\}$ and $\mathcal{E}_4 = \{x'_i w\}$ for all $i \in \mathbb{I}$.

Total labeling $p: \mathcal{X} \rightarrow \mathcal{Y}$ where $\mathcal{X} = V(\mu(P_n)) \cup E(\mu(P_n))$ and $\mathcal{Y} = \{1, 2, \dots, \left\lfloor \frac{8n}{3} \right\rfloor - 1\}$ is given below:

Case i Suppose $n \equiv 0 \pmod{3}$

$$p(x_1) = \frac{8n}{3} - 1,$$

$$p(x_{3i}) = 1 + 2i,$$

$$p(x_{3i-1}) = 2i - 1,$$

$$p(x_{3i-2}) = 2i - 1,$$

$$p(x'_{3i}) = 2n + 2i - 1,$$

$$p(x'_{3i-1}) = 2n + 2i - 3,$$

$$p(x'_{3i-2}) = 2n + 2i - 3,$$

$$p(x_{3i-1} x_{3i}) = p(x_{3i-2} x_{3i-1}) = 2i - 1,$$

$$p(x_{3i} x_{3i+1}) = 2i - 1,$$

$$p(x_{3i-2} x'_{3i-1}) = -7 + 8i,$$

$$p(x_{3i-1} x'_{3i-2}) = p(x_{3i-1} x'_{3i}) = 8i - 5,$$

$$p(x_{3i} x'_{3i+1}) = -3 + 8i,$$

$$p(x_{3i} x'_{3i-1}) = 8i - 3,$$

$$p(x_{3i+1} x'_{3i}) = -1 + 8i,$$

$$p(x'_{3i-1} w) = p(x'_{3i} w) = \frac{4n}{3} + 1 + 4(i - 1).$$

$$p(x'_{3i-2} w) = \frac{4n}{3} - 1 + 4(i - 1),$$

Case ii Consider $n \equiv 1 \pmod{3}$

$$p(x_1) = \frac{8}{3}(n - 1) + 1,$$

$$p(x_{3i}) = 2i + 1,$$

$$p(x_{3i-1}) = -1 + 2i,$$

$$p(x_{3i-2}) = 2i - 1,$$

$$p(x'_{3i}) = 2(n + i) - 1,$$

$$p(x'_{3i-1}) = 2(n + i) - 3,$$

$$p(x'_{3i-2}) = 2n + 2i - 3,$$

$$p(x_{3i-2} x_{3i-1}) = p(x_{3i-1} x_{3i}) = p(x_{3i} x_{3i+1}) = 2i - 1,$$

$$p(x_{3i-2} x'_{3i-1}) = 8i - 7,$$

$$p(x_{3i-1} x'_{3i}) = 8i - 5,$$

$$p(x_{3i} x'_{3i+1}) = 8i - 3,$$

$$p(x_{3i-1} x'_{3i-2}) = 8i - 5,$$

$$p(x_{3i} x'_{3i-1}) = 8i - 3,$$

$$p(x_{3i+1} x'_{3i}) = 8i - 1,$$

$$p(x'_{3i-1} w) = p(x'_{3i} w) = \frac{4(n - 1)}{3} + 4i - 1.$$

$$p(x'_{3i-2} w) = \frac{4n + 12i - 7}{3},$$

Case iii Suppose $n \equiv 2 \pmod{3}$

$$p(x_1) = \frac{8n - 4}{3},$$

$$p(x_{3i}) = 2i + 1,$$

$$p(x_{3i-1}) = p(x_{3i-2}) = 2i - 1,$$

$$p(x'_{3i}) = 2i - 1 + 2n,$$

$$p(x'_{3i-1}) = p(x'_{3i-2}) = -3 + 2i + 2n,$$

$$p(x_{3i-2} x_{3i-1}) = -1 + 2i,$$

$$p(x_{3i-1} x_{3i}) = p(x_{3i} x_{3i+1}) = 2i - 1,$$

$$p(x_{3i-2} x'_{3i-1}) = 8i - 7,$$

$$p(x_{3i} x'_{3i-1}) = 8i - 3,$$

$$p(x_{3i-1} x'_{3i}) = 8i - 5,$$

$$p(x_{3i+1} x'_{3i}) = -1 + 8i,$$

$$p(x_{3i} x'_{3i+1}) = 8i - 3,$$

$$p(x_{3i-1} x'_{3i-2}) = 8i - 5,$$

$$p(x'_{3i} w) = \frac{4n + 12i - 8}{3},$$

$$p(x'_{3i-1} w) = \frac{4n + 12i - 8}{3},$$

$$p(x'_{3i-2} w) = \frac{4n + 12i - 14}{3},$$

Induced weight function for edge $\psi: E(\mu(P_n)) \rightarrow \{3, 5, 7, \dots, -5 + 8n\}$ is given by

$$\psi(x_i x_{i+1}) = 2i + 1, \quad i \neq 1$$

$$\psi(x_i x'_{i+1}) = -3 + 2n + 4i, \quad i \neq 1$$

$$\psi(x_{i+1} x'_i) = -1 + 4i + 2n, \quad i \neq 1$$

$$\psi(x'_i w) = -5 + 6n + 2i.$$

Therefore, induced weights in edges of $\mu(P_n)$ generates an arithmetic progression and it differed by 2.

$$\text{Hence } (3, 2)\text{-tes}(\mu(P_n)) \leq \left\lfloor \frac{8n}{3} \right\rfloor - 1.$$

From Proposition 1, we get $(3, 2)\text{-tes}(\mu(P_n)) \geq -1 + \left\lfloor \frac{8n}{3} \right\rfloor$ and it concludes the result.

Example 8: $(3, 2)$ total edge irregular labeling of $\mu(P_6)$ is given in Fig 10.

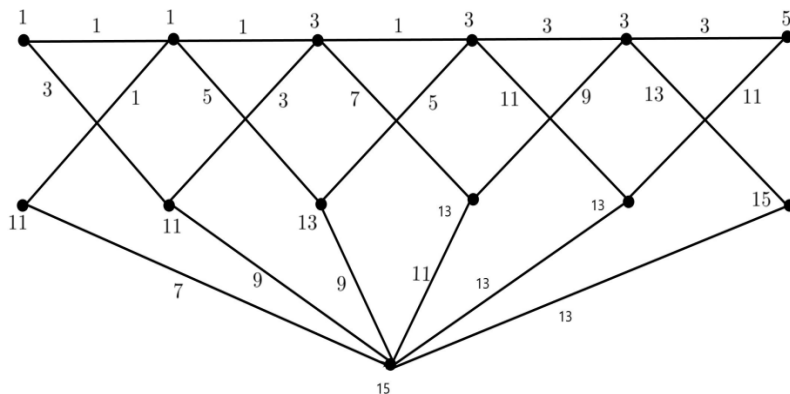


Figure 10. $(3, 2)\text{-tes}(\mu(P_6)) = 15.$

Remark 3: When $n = 2$, $\mu(P_2) \cong C_5$, and $(3, 2)\text{-tes}(C_5) = 4$, total edge irregular labelling for C_5 is shown in Fig 11.

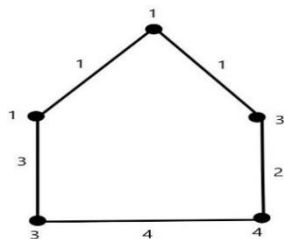


Figure 11. $(3, 2)\text{-tes}(C_5) = 4$.

3. Conclusion

The exact value of $(3, 2)\text{-tes}(G)$ is presented in this article namely: $(3, 2)\text{-tes}(S(P_n \circ S_2)) = 4n - 1$, $(3, 2)\text{-tes}(M(P_n)) = 2n - 2$, $(3, 2)\text{-tes}(K_n) = n^2 - 5n + 9$ for $n \geq 4$, $(3, 2)\text{-tes}(T(P_n)) = \lfloor \frac{8n}{3} \rfloor - 3$ for $n \geq 3$, $(3, 2)\text{-tes}(P_n \circ K_2) = \lfloor \frac{8(n+1)}{3} \rfloor - 3$, and $(3, 2)\text{-tes}(\mu(P_n)) = \lfloor \frac{8n}{3} \rfloor - 1$ for $n \geq 3$. In addition, an open problem of $(3, 2)\text{-tes}(P_n^3) = 2n - 3$ for $n > 3$ and $(3, 2)\text{-tes}(T) = \lfloor \frac{2p-1}{3} \rfloor$ where T is a tree other than star are solved positively. There is still an opportunity for an investigation towards similar findings for other graph families.

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