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# Changing and Unchanging Secure Integer Domination in Graphs

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**Abstract:**

An Integer dominating function on a graph G is a function $f : V(G) \rightarrow W$ such that for every vertex

$v \in V(G)$, $\sum_{v \in V(G)}(N[v]) \geq k$. For any function $f : V(G) \rightarrow W$ and any pair of adjacent vertices with $f(v) = 0$ and $u > 0$, the function $g_{uv}$ is defined by $g_{uv}(l) = 1$, $g_{uv}(l) = f(u) - 1$ and $g_{uv}(l) = f(l)$ if $l \in V - \{u, v\}$. A secure integer dominating function on a graph $G$ is defined as an integer dominating function g which satisfies that for every vertex $v$ with $f(v) = 0$, a neighbour $u$ with $f(u) > 0$ such that $g_{uv}$ is an integer dominating function. The weight of $f$ is $w(f) = \sum_{v \in V(G)} f(v)$. Minimum weight among all the secure integer dominating function on $G$ is secure integer domination number on $G$. This paper is devoted to initiating the study of SIDF of a graph. In particular, we have studied the changing and unchanging behavior of the graphs.

**Objectives**: We propose a novel generalization of domination, which incorporates additional security and broader applicability. This refined framework offers new possibilities for research and practical implementation.

**Keywords**: Domination, Secure domination, Integer domination, changing and unchanging domination.

## 1. Introduction

The study of domination can be traced back to 1862, when de Jaenish attempted to determine the minimum number of queens required to cover the $n * n$ chess board. The study of domination in graphs was further developed in 1958 by C. Berge and O. Ore in 1962 [6]. One of the variation of domination is Secure domination in graphs. This was studied and introduced by E. J. Cockayne et.al [4]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. For the general concepts not mentioned, the readers may be referred to [11].

A graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the vertex set of $G$ and $E(G)$ is a set of unordered pairs $xy$ of distinct elements from $V(G)$ called the edge set of $G$. The elements of $V(G)$ are called vertices. The order of $G$ is denoted by $n = | V(G) |$ and the size of G by $m = | E(G) |$. A subset $H \in V(G)$ the subgraph induced by $H$ is the graph $G[H]$ with vertex set $H$ and edge set $\{xy \in E(G) \mid x, y \in H\}$. We write $K_n$ for complete graph of order $n$, $K_{m,n}$ for complete bipartite graph with partite sets of order $n$ and $m$, $P_n$ for the path on $n$ vertices and $C_n$ for the cycle of length $n$. A star is the graph $S_{1,r}$ where $r \leq 1$. For any vertex $v \in V(G)$ open neighbourhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood is the set $N[v] = N(v) \cup v$. The private neighbor set of a $v \in V(G)$ with respect to a set $D$, denoted by $pn[v,D]$ is $N[v]N[Sv]$ and each $u \in pn[v,D]$ is called a private neighbor of $v(G)$ with respect to $D$.

O. Ore introduced the concept of Domination theory. A set $D \in V(G)$ is a dominating set if every vertex in $V(G)$is adjacent to at least one vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. Domination theory has wide application in real life. Various kinds of domination have been studied in recent times, one of the variation is Integer domination. Domke et al.[3] introduced the concept of Integer $k$-domination and further integer domination on Vizing's conjecture was studied by Bresar et al.[2]. A function $f : V(G) \rightarrow W$ ($W$ is the whole number) is called integer $k$-dominating function if the sum of functional value over any closed neighbor is at least $k$. The weight of integer $k$-dominating function is the value of $F(v) = \sum_{v \in V(G)} f(v)$. The minimum weight of integer $\{k\}-$ dominating function is denoted as $\gamma_k(G)$ and is called integer domination number. Note that, when $k = 1$ its a general domination.

In this article, we introduce secure integer domination and study about changing and unchanging behavior of graph.

A function $g : V(G) \rightarrow \{0, 1, 2, .., k\}$ is called secure integer dominating function (SIDF) of $G$ if it satisfies the following:

1) $\forall\, y \in V(G),\ \sum_{v \in V(G)} g(N[y]) \geq k.$
2) $\forall\, y \in V_0\ \exists\, z \in N(y) - V_0$ such that $g_{yz}$ is an integer dominating function on $G$.

The secure integer dominating function has a weight equal to the value of $w(y) = \sum_{y \in V(G)} g(y)$. The minimum weight of secure integer dominating function is denoted by $\gamma_k^s(G)$ is called secure integer domination number (SIDN).

For a SIDF $g$, let $V_i^g = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2,..., k$. Since these $k$ sets determine $g$, we can equivalently write $g = (V_0^g, V_1^g, V_2^g, \ldots V_k^g)$. We examine the effects on the domination number when the graph is modified by deleting a vertex or deleting or adding an edge. Let $G$-$v$ denote the graph formed by removing vertex $v$ and $G - e$ denote the graph formed by removing edge $e$ from $G$. We use acronyms to denote these classes (V represents vertex; E: edge; R: removal; A: addition).

Note that, there are six class of subgraphs obtainted. The following are the class of graphs.

- $\gamma_k^s(G - v) \neq \gamma_k^s(G)$ *for all* $v \in V(G)$. *(CVR)*
- $\gamma_k^s(G - v) = \gamma_k^s(G)$ *for all* $v \in V(G)$. *(UVR)*
- $\gamma_k^s(G - e) \neq \gamma_k^s(G)$ *for all* $e \in E(G)$. *(CER)*
- $\gamma_k^s(G - e) = \gamma_k^s(G)$ *for all* $e \in E(G)$. *(UER)*
- $\gamma_k^s(G + e) \neq \gamma_k^s(G)$ *for all* $e \in E(G)$. *(CEA)*
- $\gamma_k^s(G - e) = \gamma_k^s(G)$ *for all* $e \in E(G)$. *(UEA)*

Here we examine two cases that is, $\gamma_k^s(G-v) \neq \gamma_k^s(G)$ for all $v \in V(G)$ . We partition the vertex set of *G* into three sets according to their removal affects $\gamma_k^s(G)$. Let $V(G) = \{V^0 \cup V^+ \cup V^-\}$ where $V^0 = \{u \in V(G): \gamma_k^s(G-v) = \gamma_k^s(G)\}$, $V^+ = \{u \in V(G): \gamma_k^s(G-v) > \gamma_k^s(G)\}$ and $V^- = \{u \in V(G): \gamma_k^s(G-v) < \gamma_k^s(G)\}$. Similarly, the edge set can be partitioned into $E^0 = \{xy \in E(G) : \gamma_k^s(G-xy) = \gamma_k^s(G)\}$ and $E^+ = \{xy \in E(G): \gamma_k^s(G-xy) > \gamma_k^s(G)\}$.

## 2. Results:

**Theorem 2.1.** For every vertex *v* in $K_n$, $\gamma_k^s(G-v) = \gamma_k^s(G)$.

*Proof.* Let $G = K_n$ be a complete graph with $n$ vertices. On contrary, assume that $\gamma_k^s(G-v) \neq \gamma_k^s(G)$. Since every vertex is of degree $n-1$. $G-v$ will be a graph with $n-1$ vertices and degree $n-2$, which will be a complete graph with $K_{n-1}$. Removal of vertex in *G*, does not affect secure domination number. Therefore, our assumption is wrong.

**Theorem 2.2.** *Let* $G = S_{1,r}$ and $v = \Delta(G) - 1$, then $\gamma_k^s(G-v) > \gamma_k^s(G)$.

*Proof.* Let $G = S_{1,r}$ be a star graph, where $r = \{r_1, r_2, \ldots r_{n-1}\}$. On contrary, assume that $\gamma_k^s(G-v) < \gamma_k^s(G)$. Let $v$ be a vertex with degree $n-1$, removal of vertex *v*, will disconnect the graph. We obtain a graph with $n-1$ vertices and $n-1$ component, which increase SIDN. Therefore, our assumption is wrong.

**Theorem 2.3.** A vertex *v* in $V^-$ iff $pn[v,D] = v$ for some $\gamma_k^s$ set *D* containing *v*.

*Proof.* Let $v \in V^-$ and *S* be a $\gamma_k^s$ set of $G - v$. Then $D = S \cup \{v\}$ is a $\gamma_k^s$ set of *G*. If *S* contains a vertex of *N(v)*, then *S* is a dominating set of *G*, which contradicts our assumption. Thus, $pn[v,D] = \{v\}$. Conversely, $D - \{v\}$ dominates $G-v$, therefore $v \in V^-$.

**Theorem 2.4.** For any tree *T* with $n \geq 2$, there exists a vertex $v \in V(G)$ such that $\gamma_k^s(T-v) = \gamma_k^s(G)$.

*Proof.* Assume that *T* has atleast one vertex $v$ with $deg(v) \geq 2$ that is adjacent to atleast one endvertex and at most one non endvertex. If *v* is adjacent to two or more endvertices $r_1$ *and* $r_2$, then *v* is in every $\gamma_k^s$ set for *T* and $\gamma_k^s(T-r_1) = \gamma_k^s(T)$. If not, then *v* is adjacent to one endvertex *r* and $deg = 2$. Let $T' = T - v - r$. For any graph *G, if* $deg(r) - 1$, then $\gamma_k^s(G-r) \leq \gamma_k^s(G)$. Hence $\gamma_k^s(T') \leq \gamma_k^s(T)$. However, $\gamma_k^s(T') \leq \gamma_k^s(T) - 1$. If $\gamma_k^s(T') = \gamma_k^s(T) - 1$, then $\gamma_k^s(T) \leq \gamma_k^s(T-v)$. Otherwise, $\gamma_k^s(T') = \gamma_k^s(T) = \gamma_k^s(T-v)$.

## 3. Conclusions

In this article, we have studies introduced the concept of secure integer domination in graphs. We have observed the changing and unchanging behavior of secure integer domination in some graphs. To generalize, the changing and unchanging secure integer domination in graph is open.

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**Conflicts of Interest**

The authors declare that none of the work reported in this study could have been influenced by any known competing financial interests or personal relationships.

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